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Author(s):

S.M. Jafarian Amiri and H. Rostami

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FINITE GROUPS ALL OF WHOSE PROPER CENTRALIZERS ARE CYCLIC

S.M. JAFARIAN AMIRI* AND H. ROSTAMI

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Abstract. A finite group G is called a CC-group ($G \in CC$) if the centralizer of each noncentral element of G is cyclic. In this article we determine all finite CC-groups.

Keywords: Finite group, CA-group, CC-group, centralizer.
MSC(2010): Primary: 20D60; Secondary: 20E99.

1. Introduction

Throughout this paper G is a finite group. By $Z(G)$ and G' we mean the center of G and the derived subgroup of G, respectively. We will use usual notation, for example C_n, D_{2n} and Q_{2^n} denote respectively the cyclic group of order n, the dihedral group of order $2n$ and the generalized quaternion group of order 2^n; S_n and A_n stand for the symmetric and the alternating group on n letters, respectively; $SL(2,p^n)$, $GL(2,^n)$, $PSL(2,p^n)$ and $PGL(2,p^n)$ denote the special linear group, the general linear group, the projective special linear group and the projective general group of degree 2 over the field with p^n elements where n is a positive integer and p is a prime, respectively. The rest of our notation and terminology are standard for which the reader may refer to [14].

It is well-known that the centralizers of elements play an important role in the group theory. Hence many authors investigated the influence of centralizers of elements on the structure of groups. Some authors have determined the structure of some finite group G by the number of the centralizers of its element. (See [2–4, 9–11, 19] and [18].)

Another approach in classification of finite groups G is based on the structure of centralizers of the elements in G. In 1961 Suzuki [16] clarified the structure of nonsolvable groups in which the centralizer of any nonidentity element is

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*Corresponding author.

nilpotent. Schmidt [15] determined all CA-groups, consisting of groups in which all proper centralizers are abelian. In 1953, Ito [8] investigated the class of F-groups, consisting of finite groups G in which for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$. Also Ito [8] studied I-groups in which all centralizers of noncentral elements have the same order. In [6], Dolfi, Herzog and Jabara investigated all CH-groups in which noncentral commuting elements have centralizers of equal size.

In this paper we call a group G a CC-group if all noncentral centralizers are cyclic. It is clear that a finite CC-group G is abelian if and only if G is cyclic. It is well-known that $CC \subset CA \subset CH \subset F$. In Section 2 we classify all nonabelian CC-groups. (See Theorem 2.13.) In the sequel m is an odd integer, n is a positive integer and p is a prime.

2. The structure of CC-groups

Since every CC-group is a CA-group, we will use Theorem A of [6] in the proof of the main result. For convenience we bring this theorem in the following.

Theorem 2.1. *Let G be a nonabelian group and write $Z = Z(G)$. Then G is a CA-group if and only if it is of one of the following types:*

(I) *G is nonabelian and has an abelian normal subgroup of prime index.*

(II) *$\frac{G}{Z}$ is a Frobenius group with Frobenius kernel $\frac{K}{Z}$ and Frobenius complement $\frac{L}{Z}$, where K and L are abelian.*

(III) *$\frac{G}{Z}$ is a Frobenius group with Frobenius kernel $\frac{K}{Z}$ and Frobenius complement $\frac{L}{Z}$, such that $K = PZ$, where P is a normal Sylow p-subgroup of G for some $p \in \pi(G)$, P is a CA-group (F-group), $Z(P) = P \cap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*

(IV) *$\frac{G}{Z} \cong S_4$ and if $\frac{V}{Z}$ is the Klein four group in $\frac{G}{Z}$, then V is nonabelian.*

(V) *$G = P \times A$, where P is a nonabelian CA-group (F-group) of prime-power order and A is abelian.*

(VI) *$\frac{G}{Z} \cong PSL(2, p^n)$ or $PGL(2, p^n)$ and $G' \simeq SL(2, p^n)$ where p is a prime and $p^n > 3$.*

(VII) *$\frac{G}{Z} \cong PSL(2, 9)$ or $PGL(2, 9)$ and G' is isomorphic to the Schur cover of PSL(2, 9).*

First we deal with the structure of Sylow subgroups of a CC-group.

Lemma 2.2. *Let G be a CC-group. If P is a Sylow p-subgroup of G, then P is either cyclic or a generalized quaternion.*

Proof. By hypothesis, P is a CC-group. It follows that P has no noncyclic abelian subgroup and we get the result by [14, Theorem 5.3.6].

□

By Lemma 2.2, we obtain the structure of nonabelian nilpotent CC-groups. (In fact we investigate the condition (V) of Theorem 2.1.)

Corollary 2.3. *Let G be a nonabelian nilpotent group. Then G is a CC-group if and only if $G \cong Q_{2^n} \times C_m$ for some positive integers m and n.*

In what follows we determine nonabelian CC-groups whose Sylow subgroups are all cyclic.

Lemma 2.4. *Suppose that G is a nonabelian group whose Sylow 2-subgroup is not isomorphic to the generalized quaternion. Then G is a CC-group if and only if $G = \langle a, b | a^l = 1 = b^k, b^{-1}ab = a^r \rangle$ where $r^k \equiv 1 (mod\ l), l$ is odd, $0 \leq r < l$, $gcd(l, k(r-1)) = 1$ and $\frac{G}{Z(G)}$ is a Frobenius group with the cyclic kernel and complement.*

Proof. Suppose that G is a CC-group and P is a Sylow p-subgroup of G where p is an arbitrary prime divisor of $|G|$. Then P is cyclic by Lemma 2.2. Now [14, Theorem 10.1.10] yields $G = \langle a, b | a^l = 1 = b^k, b^{-1}ab = a^r \rangle$ where $r^k \equiv 1 (mod\ l), l$ is odd, $0 \leq r < l$, and l and $k(r-1)$ are coprime.

Now assume that K and H are maximal abelian subgroups of G such that $\langle a \rangle \subseteq K$ and $\langle b \rangle \subseteq H$. Then $K = C_G(a)$ and $H = C_G(b)$. By [14, Theorem 10.1.7], we have $Z(G) = Z_2(G)$ is the hypercenter of G. Since $G = C_G(a)C_G(b)$, we see that $C_G(a) \cap C_G(b) = Z(G)$. It follows that $\frac{G}{Z(G)} = \frac{C_G(a)}{Z(G)} \rtimes \frac{C_G(b)}{Z(G)}$ is a Frobenius group with cyclic kernel and complement which gives the desired result.

Conversely, let $C_G(x)$ be a proper centralizer of G. Then $C_G(x)$ is abelian by part (II) of Theorem 2.1. Hence $C_G(x)$ is the direct product of its Sylow subgroups. But Sylow subgroups of G are cyclic which gives $C_G(x)$ is cyclic. □

Remark 2.5. We notice that if all Sylow subgroups of a group G are cyclic, then we can not say that G is necessarily a CC-group. For instance, it can be checked by GAP [17], that all sylow subgroups of $G := SmallGroup(300, 6)$ are cyclic, but G contains some nonabelian centralizers of order 12.

In the following we give the structure of a CC-group G such that $\frac{G}{Z(G)}$ is Frobenius. (In fact we investigate two conditions (II) and (III) of Theorem 2.1.)

Lemma 2.6. *Suppose that G is a finite group such that $\frac{G}{Z(G)}$ is Frobenius. Then G is a CC-group if and only if it is of one of the following types:*

(I) *$G = \langle a, b | a^l = 1 = b^k, b^{-1}ab = a^r \rangle$ where $r^k \equiv 1 (mod\ l), l$ is odd, $0 \leq r < l$ and $gcd(l, k(r-1)) = 1$.*

(II) *$G \cong Q_8 \rtimes C_{3m}$ for some m and $\frac{G}{Z(G)} \cong A_4$.*

Proof. Let $\frac{K}{Z(G)}$ and $\frac{L}{Z(G)}$ be the Frobenius kernel and complement of $\frac{G}{Z(G)}$, respectively. Suppose, first, that G is a CC-group. If K is abelian, then L is abelian too by Theorem 2.1. Then the result follows by Lemma 2.4.

Now assume that K is not abelian. It is well-known that the Frobrnius kernel is always nilpotent. It follows that K is nilpotent and so $K = QC$ where

Q is the normal Sylow 2-subgroup of G and C is the normal cyclic subgroup of G of odd order by Corollary 2.3. Also, we have $Q \cong Q_{2^n}$ for some $n \geq 3$ and $Z(G) = Z(K) = IC$ where $|I| = 2$. It follows from Theorem 2.1 that $L = HZ(G)$ for some abelian $2'$-subgroup H of G and so L is abelian. Since G is a CC-group, L is cyclic. Therefore $G = QL = Q \rtimes T$ for some nonnormal cyclic subgroup T of G. It follows that $Q \cong Q_8$. Since $Aut(Q_8) \cong S_4$, we conclude that $|\frac{T}{T \cap Z(G)}| = 3$ and then $\frac{G}{Z(G)} \cong A_4$.

Conversely, if G satisfies (I) of our lemma, then G is a CC-group by Lemma 2.4. Now assume that G satisfies (II). Then we have $Z(G) \cong C_2 \times C_m$, $K \cong Q_8 \times C_m$ and $L \cong C_2 \times C_{3m}$.

Suppose that $g \in G \setminus Z(G)$. Then $g \in K$ or $g \in L^k$ for some $k \in K$ because $\overline{G} = \frac{G}{Z(G)} = \frac{K}{Z(G)} \rtimes \frac{L}{Z(G)}$ is Frobenius. Then either $C_G(g) \leq K$ or $C_G(g) \leq L^k$. If $C_G(g) \leq K$, then $C_G(g) \neq K$ because G is a CA-group. But every proper subgroup of K is cyclic and so $C_G(g)$ is cyclic. If $C_G(g) \leq L^k$, then $C_G(g)$ is cyclic because L is cyclic. Therefore G is CC-group and the proof is complete. □

For an element g in the group G and a subgroup H of G, we shall write $\overline{g}$ and $\overline{H}$ to denote the images of g and H in the quotient group $\frac{G}{Z(G)}$, respectively. In the following we give a necessary and sufficient condition for a CC-group G to satisfy (I) of Theorem 2.1.

Lemma 2.7. *Let G be a nonabelian group and A be an abelian normal subgroup of index 2 in G. Then G is a CC-group if and only if G satisfies one of the following conditions.*

- (I) *$G \cong C_m \rtimes Q_{2^n}$ for some m and $\frac{G}{Z(G)} \cong D_{2t}$ such that $t = 2^{n-2}k$ and k is odd;*
- (II) *$G \cong C_m \rtimes C_{2^n}$ for some m and n and $\frac{G}{Z(G)} \cong D_{2t}$ such that t is odd.*

Proof. Suppose that G is a CC-group of order $2^n m$ such that $n \geq 1$ and m is an odd integer. Then A is cyclic by hypothesis and so G is supersolvable. Therefore all elements of odd order in G form a characteristic subgroup, say B. Since $B \leq A$, we have B is cyclic. If Q is a Sylow 2-subgroup of G, then Q is cyclic or a generalized quaternion of order 2^n by Lemma 2.2 and $G = B \rtimes Q$. It follows that $\overline{G} = \frac{G}{Z(G)} = \overline{B} \rtimes \overline{Q}$ where $\overline{B} = \frac{BZ(G)}{Z(G)}$ and $\overline{Q} = \frac{QZ(G)}{Z(G)}$. So $\overline{B} = \langle \overline{b} \rangle$ for some $b \in G$. Now we consider the following two cases.

Case 1. Suppose that Q is the generalized quaternion group. Then $Z(Q) \subseteq Z(G)$ and so $\overline{Q} = \langle \overline{r}, \overline{s} | \overline{r}^{2^{n-2}} = \overline{s}^2 = 1, \overline{r}^{\overline{s}} = \overline{r}^{-1} \rangle$ is the dihedral group of order 2^{n-1}. Now if $N_{\overline{G}}(\overline{Q}) = \frac{L}{Z(G)}$ for some subgroup L of G, then $Z(G) = Z(L)$, since G is CC-group. Therefore L is nilpotent by the normality of all Sylow subgroups of $\frac{L}{Z(L)}$. It follows from Corollary 2.3 that $\frac{L}{Z(L)}$ is a 2-group and so $N_{\overline{G}}(\overline{Q}) = \overline{Q}$. Note that $\overline{A} \cap \overline{Q} = \langle \overline{r} \rangle$ and so $\overline{r}\overline{b} = \overline{b}\overline{r}$. Now we claim that $\overline{G} = \frac{G}{Z(G)}$ is a dihedral group.

Since $\overline{s}$ is an involution in $\overline{G}$ and $\overline{Q} = N_{\overline{G}}(\overline{Q})$, the induced inner automorphism by $\overline{s}$ on $\overline{B}$ is a fixed-point free automorphism of order 2. Therefore $\overline{s}\overline{b}\overline{s} = (\overline{b})^{-1}$. Since $\overline{G} = \overline{A} \rtimes \langle \overline{s} \rangle = (\langle \overline{b} \rangle \langle \overline{r} \rangle) \rtimes \langle \overline{s} \rangle$, $\overline{G} = \frac{G}{Z(G)}$ is a dihedral group.

Case 2. Suppose that Q is cyclic. Then $Z(G) \cap Q = A \cap Q$ is cyclic of order 2^{n-1} and so $|\overline{Q}| = 2$. A similar argument as in Case 1 gives the result.

Conversely, suppose that G satisfies one of the conditions (I) or (II) of our lemma. Then $C_G(x)$ is abelian for each $x \in G \setminus Z(G)$ by Theorem 2.1 (I). It follows that $C_G(x)$ is cyclic by the structure of G. This completes the proof. $\square$

Among all groups of order 24, the only CC-groups are $C_3 \rtimes C_8 = \langle a, b | a^8 = b^3 = 1, a^{-1}ba = b^{-1} \rangle, SL(2,3), C_3 \times Q_8$ and $C_3 \rtimes Q_8 = \langle a, b | a^6 = b^2 = (ab)^2 \rangle$. In the following we investigate condition (IV) of Theorem 2.1.

Lemma 2.8. *Let G be a nonabelian group such that $\frac{G}{Z(G)} \cong S_4$. Then G is a CC-group if and only if $G = G'Z(G)\langle a \rangle$ where $a^2 \in Z(G)$, $G' \cong SL(2,3)$, $G'Z(G) \cong SL(2,3) \times C_m$, $gcd(m,6) = 1$ and a Sylow 2-subgroup of G is isomorphic to Q_{16}.*

Proof. Suppose that G is a CC-group. It follows from [14, Theorem 11.4.18] that the central extension $Z(G) \rightarrowtail G \twoheadrightarrow \frac{G}{Z(G)}$ determines a homomorphism $\delta : M(\frac{G}{Z(G)}) \to Z(G)$ where $M(\frac{G}{Z(G)})$ is the Schur multiplier of $\frac{G}{Z(G)}$. We conclude with the aid of [14, Exercise 11.4.10] that $Im(\delta) = G' \cap Z(G)$. This implies that $|Z(G) \cap G'| = 1$ or 2 since Shur multiplier of S_4 is isomorphic to C_2. Now $(\frac{G}{Z(G)})' = \frac{G'Z(G)}{Z(G)} \cong \frac{G'}{G' \cap Z(G)} \cong A_4$ and since A_4 is not a CC-group, $|G' \cap Z(G)| = 2$. Next, let $\frac{V}{Z(G)}$ be the normal subgroup of $\frac{G}{Z(G)}$ of order 4. Then V is nonabelian nilpotent and so $V = QC$ where Q and C are normal subgroups of G such that $Q \cong Q_8$, $C \cong C_m$ and m is odd. Also, we have $Z(G) = Z(V) = Z(Q)C$. Since G' is a nonnilpotent CC-group of order 24 with the normal Sylow 2-subgroup, we conclude that $G' \cong SL(2,3)$. Therefore

$$G'Z(G) = G'Z(Q)C = G'C \cong SL(2,3) \times C_m.$$

Also, since every abelian subgroup of G is cyclic, we have $gcd(m, |SL(2,3)|) = 1$, as required. Now let $aZ(G) \in \frac{G}{Z(G)} \setminus \frac{G'Z(G)}{Z(G)}$ be an element of order 2. Then $G = G'Z(G)\langle a \rangle$ where $a^2 \in Z(G)$. Finally, if P is a Sylow 2-subgroup of G, then P is a nonabelian CC-group and so $P \cong Q_{16}$ by Lemma 2.2.

Conversely, first we prove that G is a CA-group. If $\frac{V}{Z(G)}$ is the normal subgroup of order 4 of $\frac{G}{Z(G)}$, then $\frac{V}{Z(G)} \leq \frac{PZ(G)}{Z(G)}$ for some Sylow 2-subgroup P of G. Therefore $V = Z(G)(P \cap V)$. Since $P \cong Q_{16}$, every abelian subgroup of P is cyclic and so $V = Q_8 \times A$ where A is a cyclic subgroup of $Z(G)$ of index 2. Therefore G is a CA-group by Theorem 2.1(IV).

If $g \in G \setminus Z(G)$, then $C_G(g)$ is abelian and so $C_G(g) \cap P$ is cyclic. Also, it is clear that any Sylow p-subgroup of G is cyclic for each odd prime p. Therefore $C_G(g)$ is cyclic and the proof is complete. □

Remark 2.9. Note that the last sentence of the above lemma can not be removed. For example if $G = GL(2,3)$ is the general linear group, then we have $\frac{G}{Z(G)} \cong S_4$ and $G'Z(G) = SL(2,3)$, but G is not a CC-group.

In view of Theorem 2.1 and the previous results, it remains to investigate two conditions (VI) and (VII) of Theorem 2.1.

Lemma 2.10. *Let G be a finite group such that $\frac{G}{Z(G)} \cong PSL(2,p^k)$, $p^k > 3$ and $G' \cong SL(2,p^k)$. Then G is a CC-group if and only if $k = 1$ and $G \cong C_m \times SL(2,p)$ such that $gcd(m, |SL(2,p)|) = 1$.*

Proof. Suppose that G is a CC-group. If all Sylow subgroups of G are cyclic, then G is solvable by [14, Theorem 10.1.10], which is a contradiction. So a Sylow 2-subgroup Q of G is isomorphic to Q_{2^n} for some positive integer n by Lemma 2.2. From [7, Satz 8.2] we know that Sylow p-subgroups of $PSL(2,p^k)$ are elementary abelian. Hence $k = 1$. Since $\frac{G'Z(G)}{Z(G)} \cong PSL(2,p)$, we must have $G = G'Z(G)$. On the other hand since $Z(G) \cap Q \leq Z(Q)$, we have $|Z(G) \cap Q| = 2$. So $Z(G) = IA$ where $|I| = 2$ and A is cyclic of odd order m. Now since $G' \cong SL(2,p)$, we see that $|G' \cap Z(G)| = 2$. It follows that $G = G'Z(G) = G'A \cong SL(2,p) \times C_m$. Since all Sylow r-subgroups of G are cyclic for any odd prime r by Lemma 2.2, we must have $gcd(m, |SL(2,p)|) = 1$.

Conversely, assume that G has the stated property. Then we show that G is a CC-group. By hypothesis, $Z(G) = AI$ where A is cyclic of odd order m and $I = Z(G')$ has order 2. If $a \in G \setminus Z(G)$, then $C_G(a) = Z(G)C_{G'}(a) = AC_{G'}(a)$, since $G = Z(G)G'$. So it is enough to show that $C_{G'}(a)$ is cyclic.

It follows from Theorem 2.1 (VI) that $C_G(a)$ is abelian. Now we claim that every Sylow r-subgroup R of $C_{G'}(a)$ is cyclic for each prime r.

Suppose that $r = 2$. Since a Sylow 2-subgroup of $SL(2,p)$ is isomorphic to a generalized quaternion and $C_G(a)$ is abelian, R is cyclic. If $r \neq 2$, then every Sylow r-subgroup of $SL(2,p)$ is cyclic and so R is cyclic. This completes the proof.

□

Lemma 2.11. *Let G be a finite group such that $\frac{G}{Z(G)} \cong PGL(2,p^k)$, $p^k > 3$ and $G' \cong SL(2,p^k)$. Then G is a CC-group if and only if $k = 1$, $G'Z(G) \cong SL(2,p) \times C_m$ such that $gcd(m, |SL(2,p)|) = 1$ and if $\frac{L}{Z(G)}$ is a dihedral subgroup of $\frac{G}{Z(G)}$ of order $2(p \pm 1)$, then $L \cong C_{ml} \rtimes Q_{2^n}$ for a positive integer n and an odd integer l.*

Proof. Suppose that G is a CC-group. By an argument similar to the proof of the Lemma 2.10, we have $k = 1$, a Sylow 2-subgroup Q of G is a (generalized) quaternion, $Z(G) \cong C_2 \times C_m$ for some odd integer m and $|G' \cap Z(G)| = 2$. Therefore $G'Z(G) \cong SL(2,p) \times C_m$ such that $gcd(m, |SL(2,p)|) = 1$.

Now let $\frac{L}{Z(G)} \cong D_{2(p\pm1)}$. Since L is a CC-group and $p \pm 1$ is even, we have $L \cong C_{ml} \rtimes Q_{2^n}$ for some integer n by Lemma 2.7.

To establish the converse we have G is a CA-group by Theorem 2.1(VI). It follows that $\Pi = \{\frac{C_G(x)}{Z(G)} : x \in G \setminus Z(G)\}$ is a partition of $\frac{G}{Z(G)}$. Since $\frac{G}{Z(G)} \cong PGL(2,p)$, we have $|\frac{C_G(x)}{Z(G)}| \in \{p-1, p, p+1\}$ for every $x \in G \setminus Z(G)$. (See [7, pp. 185–187].)

If $|\frac{C_G(x)}{Z(G)}| = p$, then $C_G(x)$ is cyclic. But if $|\frac{C_G(x)}{Z(G)}| \in \{p-1, p+1\}$ for some $x \in G \setminus Z(G)\}$, then $\frac{C_G(x)}{Z(G)}$ is contained in some subgroups of dihedral type of order $2(p \pm 1)$ of $\frac{G}{Z(G)}$, say $\frac{L}{Z(G)}$. Since G is a CA-group, we will have $Z(G) = Z(L)$ and so L is a CC-group by assumption and Lemma 2.7. Therefore $C_G(x)$ is cyclic, as required. □

Remark 2.12. Note that we can not delete the last part of the above lemma. For example if $G = GL(2,7)$, then we have $\frac{G}{Z(G)} \cong PGL(2,7)$ and $G' = SL(2,7)$, but G is not a CC-group.

Now we are ready to characterize nonabelian CC-groups.

Theorem 2.13. *Let G be a nonabelian group. Then G is a CC-group if and only if it is of one of the following structures.*

(1) $G = \langle a, b | a^l = 1 = b^k, b^{-1}ab = a^r \rangle$ *where* $r^k \equiv 1 (mod\ l), l$ *is odd,* $0 \le r < l$, $gcd(l, k(r-1)) = 1$ *and also* $\frac{G}{Z(G)}$ *is a Frobenius group with the cyclic kernel and complement.*
(2) $G \cong C_m \rtimes C_{2^n}$ *for some* m, n *and* $\frac{G}{Z(G)} \cong D_{2t}$ *for some odd* t.
(3) $G \cong Q_8 \rtimes C_{3m}$ *for some integer* m *and* $\frac{G}{Z(G)} \cong A_4$.
(4) $G \cong Q_{2^n} \times C_m$ *for some* m, n *and* $\frac{G}{Z(G)} \cong D_{2^{n-1}}$*;*
(5) $G \cong C_m \rtimes Q_{2^n}$ *for some intgers* m, n *and* $\frac{G}{Z(G)} \cong D_{2t}$ *where* $t = 2^{n-2}k$.
(6) $G = G'Z(G)\langle a \rangle$, $a^2 \in Z(G)$, $G'Z(G) \cong SL(2,3) \times C_m$, $gcd(m,6) = 1$, *a Sylow 2-subgroup of* G *is isomorphic to* Q_{16} *and* $\frac{G}{Z(G)} \cong S_4$.
(7) $G \cong SL(2,p) \times C_m$ *where* $p > 3$ *is a prime and* $gcd(m, |SL(2,p)|) = 1$.
(8) $G = G'Z(G)\langle a \rangle$, $a^2 \in Z(G)$ *and* $G'Z(G) \cong SL(2,p) \times C_m$ *where* p *is prime,* $gcd(m, |SL(2,p)|) = 1$ *and if* $\frac{L}{Z(G)}$ *is a dihedral subgroup of* $\frac{G}{Z(G)}$ *of order* $2(p \pm 1)$*, then* $L \cong C_{ml} \rtimes Q_{2^n}$ *for some positive integer* n *and odd number* l .

Proof. The result follows from Theorem 2.1, Corollary 2.3, and Lemmas 2.6, 2.7, 2.8, 2.10, and 2.11. □

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(Seyyed Majid Jafarian Amiri) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran.

E-mail address: sm_jafarian@znu.ac.ir

(Hojjat Rostami) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran.

E-mail address: h.rostami5991@gmail.com