ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

# Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 5, pp. 975–994

Title:

Benson's algorithm for nonconvex multiobjective problems via nonsmooth Wolfe duality

Author(s):

N. Shafiei

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 5, pp. 975–994
Online ISSN: 1735-8515

# BENSON'S ALGORITHM FOR NONCONVEX MULTIOBJECTIVE PROBLEMS VIA NONSMOOTH WOLFE DUALITY

N. SHAFIEI

(Communicated by Maziar Salahi)

Abstract. In this paper, we propose an algorithm to obtain an approximation set of the (weakly) nondominated points of nonsmooth multiobjective optimization problems with equality and inequality constraints. We use an extension of the Wolfe duality to construct the separating hyperplane in Benson's outer algorithm for multiobjective programming problems with subdifferentiable functions. We also formulate an infinitive approximation set of the (weakly) nondominated points of biobjective optimization problems. Moreover, we provide some numerical examples to illustrate the advantage of our algorithm.

**Keywords:** Multiobjective optimization, approximation algorithm, efficient solution, nondominated point.
**MSC(2010):** Primary: 90C29; Secondary: 49J52, 90C30.

## 1. Introduction

In multiobjective optimization, several objective functions have to be optimized simultaneously. The problem occurs in many applications, such as finance, scheduling, engineering design, medical treatment, etc (see [7, 9, 17]). A multiobjective programming problem (MOP) often has no optimal solution that could optimize all objectives simultaneously. Therefore, the concept of optimality has to be replaced by the concept of efficiency. An efficient solution is a feasible point of MOP if there are not any other feasible points with the same or smaller objective function values, such that there is a decrease in at least one objective function value. The image of an efficient point in variable space is a nondominated point in the objective space.

The fact that an improvement in one objective consequences a loss in another is known as the trade-off between the solutions. The set of all efficient points

Article electronically published on 31 October, 2017.
Received: 20 April 2015, Accepted: 23 February 2016.

and all nondominated points are called the efficient set and the nondominated set, respectively.

The primary goal of multiobjective programming is to seek the efficient set and/or the nondominated set of MOPs. Unfortunately, it is not easy to obtain an exact description of the efficient set and the nondominated set that include usually an infinite number of points. Therefore, an approximated description (or a finite number) of the solution set is considered.

In the literature, there are a variety of approaches to obtain or approximate the solution set of MOPs such as weighted sum method, $\epsilon$-constraint method, goal attainment method, etc (see [6, 10, 12, 14, 18]). Since in many practical cases, it is not possible for a decision maker to choose a preferred solution from the set of all efficient solutions, we aim to propose an algorithm to generate the nondominated set of nonconvex multiobjective optimization problems (NMOPs). Since working in the lower dimensional space may require less computational effort, it is more natural to compare the objective values rather than the decision values.

Benson's "outer approximation algorithm" [2] proposes an algorithm for approximating the nondominated set of a multiobjective linear programming problem. Ehrgott et al. [4] employ results from geometric duality theory for multiobjective linear programmes (MOLPs) to derive a dual variant of Benson's outer approximation algorithm. In [16] a modification of Benson's algorithm is proposed to solve MOLPs by an inner and an outer approximations of the nondominated set. An extension of Benson's outer approximation algorithm provides a set of weakly $\varepsilon$-nondominated points of convex multiobjective problems, see [5]. Recently, an extension of Benson's outer approximation algorithm and a dual variant of it are presented for convex MOPs in [11].

We modify the idea of Benson's algorithm in order to solve a class of NMOPs with equality and inequality constraints. In fact, our method is an extension of [5] for MOPs with convexlike and nonsmooth objective functions. The separating hyperplane in Benson's algorithm is constructed by an extension of the Wolfe duality to solve some MOPs with subdifferentiable functions. This method, similar to [11], often provides more (weakly) nondominated points than the method explained in [5] without requiring any more computation. In comparison with [11], our method is suitable for a class of nonconvex MOPs with additional equality constraints. It is worth to mention that we do not require any compactness assumptions on the feasible set. The other advantage of our approach is that one can calculate the maximum error.

It is obvious that our method is also applicable for biobjective problems (BOPs). There are some methods to obtain the efficient set of these problems, see, e.g., [8]. We investigate that every weakly efficient solution generated by the algorithm is an efficient solution of BOPs. However, this fact is not true for three or more objective functions, see Section 7 for a counterexample.

Moreover, a set of $\varepsilon$-nondominated points and a set of weakly $\varepsilon$-nondominated points of BOPs are formulated.

The article is organized as follows. In Section 2, some definitions and notations are collected. We present some Wolfe duality Theorems in Section 4. An algorithm for NMOPs and some details of the algorithm are proposed in Section 5. In Section 6, we show that every weakly nondominated point generated by the algorithm is a nondominated point in biobjective problems. We arrange these points to construct a set of (weakly) $\varepsilon$-nondominated points. Finally, we provide numerical examples in Section 7.

## 2. **Preliminaries**

Throughout this paper we use the following notations. A vector with only 1's as components is denoted by $e$. Let $x, y \in \mathbb{R}^p$ and $p \in \mathbb{N}\backslash\{1\}$. The notations $\leqq, \leq$ and $<$ are used for the vectors $x$ and $y$ as follows

$$x \leqq y \iff x_i \leqslant y_i, \quad \forall i = 1, ..., p,$$

$$x \leq y \iff x \leqq y \quad \text{and} \quad x \neq y,$$

$$x < y \iff x_i < y_i, \quad \forall i = 1, ..., p.$$

Therefore, $\mathbb{R}^p_{\geqq}$ is given by

$$\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y \geqq 0\}.$$

Let $\mathcal{C}$ be a nonempty subset of $\mathbb{R}^n$ where $n \in \mathbb{N}$. Clarke's tangent cone (resp. Clarke's normal cone) to $\mathcal{C}$ at $x \in \mathcal{C}$ is defined as follows (see [3])

$$T_{\mathcal{C}}(x) = \{v \in \mathbb{R}^n : \forall t_n \downarrow 0, \forall x_n \to x \text{ with } x_n \in \mathcal{C}, \exists v_n \to v x_n + t_n v_n \in \mathcal{C}, \forall n\}$$

$$(\text{resp. } N_{\mathcal{C}}(x) = \{v \in \mathbb{R}^n : \langle v, w \rangle \leqslant 0, \forall w \in T_{\mathcal{C}}(x)\}).$$

Let $\mathcal{C}$ be a convex set. A vertex of $\mathcal{C}$ is an element of $\mathcal{C}$ that cannot be expressed in terms of a strictly convex combination of two distinct points of $\mathcal{C}$. The convex hull of $\mathcal{C}$ is the smallest convex set containing $\mathcal{C}$. It consists exactly of all convex combination of elements of $\mathcal{C}$. The notations conv $\mathcal{C}$, int $\mathcal{C}$, vert $\mathcal{C}$, bd $\mathcal{C}$ and cl $\mathcal{C}$ denote the convex hull of $\mathcal{C}$, the interior of $\mathcal{C}$, the set of vertices of $\mathcal{C}$, the boundary of $\mathcal{C}$ and the closure of $\mathcal{C}$, respectively.

It is well known that Clarke's directional derivative is usually defined for lipschitzian functions. Here we recall an extension of it for non-lipschitzian functions from [15].

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) be an arbitrary function. The extended Clarke directional derivative of $f$ at point $x \in \mathbb{R}^n$ in direction $v \in \mathbb{R}^n$, denoted by $f^\circ(x;v)$, is defined as follows

$$\limsup_{y\to x, t\downarrow 0} \frac{f(y+tv)-f(y)}{t} \qquad (\infty-\infty=0).$$

The set

$$\partial f(x) = \{\xi \in \mathbb{R}^n :< \xi, v > \leqslant f^\circ(x;v), \forall v \in \mathbb{R}^n\},$$

is called the subdifferential of $f$ at $x$. If $\partial f(x) \neq \emptyset$, the function $f$ is called subdifferentiable at $x$. The function $f$ is called subdifferentiable on $\mathcal{C} \subseteq \mathbb{R}^n$ if it is subdifferentiable at all points of $\mathcal{C}$. A vector function is subdifferentiable if its components are subdifferentiable.

Let us recall the definition of convexlike functions from [1]. The function $f : \mathbb{R}^n \to \mathbb{R}$ is called convexlike on $\mathcal{C}$ if for every $x_1, x_2 \in \mathcal{C}$ and $\lambda \in (0,1)$ there exists $x \in \mathcal{C}$ such that

$$f(x) \leqslant \lambda f(x_1) + (1-\lambda) f(x_2).$$

A vector function is convexlike on $\mathcal{C}$ if its components are convexlike on $\mathcal{C}$.

Consider the following multiobjective programming problem:

$$\text{(P)} \qquad \begin{aligned} &\min\ f(x) = (f_1(x), ..., f_p(x)) \\ &s.t.\ x \in \mathcal{X} = \{x \in C : g(x) = (g_1(x), ..., g_m(x)) \leqq 0,\ m \in \mathbb{N} \cup \{0\}, \\ &\qquad\qquad h(x) = (h_1(x), ..., h_q(x)) = 0,\ q \in \mathbb{N} \cup \{0\}\}, \end{aligned}$$

where $f(x) : A \to \mathbb{R}^p$, $g(x) : A \to \mathbb{R}^m$ and $h(x) : A \to \mathbb{R}^q$ are subdifferentiable functions on $A$. Here $A$ is a nonempty open subset of $\mathbb{R}^n$ and $C \subseteq A$ is a nonempty set. The feasible set $\mathcal{Y}$ in the objective space $\mathbb{R}^p$ is defined by

$$\mathcal{Y} = \{f(x) : x \in \mathcal{X}\}.$$

The point $z^{ideal} \in \mathbb{R}^p$ is the ideal point of the problem (P) if it's $i$-th component is the optimal value of

$$\begin{aligned} \min\ & f_i(x). \\ \text{s.t. } & x \in \mathcal{X} \end{aligned}$$

Assume that $z^{ideal}$ exists and $\mathcal{P} := \mathcal{Y} + \mathbb{R}^p_{\geqq}$ is a closed set.

A point $y \in \mathcal{Y}$ is called weakly nondominated if $(\{y\} - \text{int } \mathbb{R}^p_{\geqq}) \cap \mathcal{Y} = \emptyset$ and it is called nondominated if $(\{y\} - \mathbb{R}^p_{\geqq} \setminus \{0\}) \cap \mathcal{Y} = \emptyset$. The set of (weakly) nondominated points of $\mathcal{Y}$ is given by

$$(\mathcal{Y}_{WN} := \{y \in \mathcal{Y} : (\{y\} - \text{int } \mathbb{R}^p_{\geqq}) \cap \mathcal{Y} = \emptyset\})$$

$$\mathcal{Y}_N := \{y \in \mathcal{Y} : (\{y\} - \mathbb{R}^p_{\geqq} \setminus \{0\}) \cap \mathcal{Y} = \emptyset\}.$$

A point $x \in \mathcal{X}$ is called (weakly) efficient solution, if $f(x)$ is a (weakly) nondominated point of $f(\mathcal{X})$. Let $\varepsilon \in \mathbb{R}^p_{\geqq}$. A point $y \in \mathcal{Y}$ is called (weakly) $\varepsilon$-nondominated if there is no $\hat{y} \in \mathcal{Y}$ such that $\hat{y}(<) \leqq y - \varepsilon$.

## 3. Benson's algorithm

In this section, a summery of Benson's algorithm is presented from [2] for multiobjective linear problems. Benson has proposed an outer approximation algorithm to solve the problem (P) with $f(x) = Dx$ and $\mathcal{X} = \{x \in \mathbb{R}^n : \ Ax = b, \ x \geqq 0\}$, named as (MOLP), where $D$ is an $p \times n$ matrix, $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$. The set $X$ is assumed to be a nonempty and compact polyhedron.

Let $\hat{y} \in \mathbb{R}^p$ satisfy $\hat{y}_i < \max_{y \in \mathcal{Y}} y_i$ for $i = 1, 2, ..., p$ and

$$Y = \{y \in \mathbb{R}^p : \hat{y} \leqslant y \leqslant Dx \quad \text{for some } x \in X\}.$$

Assume that $v^0 = \hat{y}$ and define $v^j \in \mathbb{R}^p$ for each $j = 1, 2, ..., p$, such that

$$v_i^j = \begin{cases} \hat{y}_i & \text{if} \quad i \neq j, \\ \beta + \hat{y}_i - e^T \hat{y} & \text{if} \quad i = j, \end{cases} \qquad i = 1, 2, ..., p,$$

where $\beta := \min\{e^T y : y \in Y\}$ is a finite number. In the outer approximation algorithm of MOLP, an initial simplex $S$ containing $Y$ is defined by $S = \text{conv}V(S)$ where $V(S) := \{v^j : j = 0, 1, ..., p\}$.

Suppose $Y$ is a $p$-dimensional polyhedron with a finite number of faces. The following proposition from [2] provides a meaning for finding a face of $Y$.

**Proposition 3.1.** *Assume that $w \in \mathbb{R}^p$ is a weakly nondominated point of MOLP and let $(u^{*T}, v^{*T})$ denote any optimal solution for the dual linear program of the following problem*

$$\text{(Q(w))} \qquad \begin{aligned} &\min\ z \\ &\text{s.t. } Dx - ez \leqq w, \\ &\qquad Ax = b, \\ &\qquad (x, z) \geqq 0. \end{aligned}$$

*Here $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^m$ correspond to the first and second constraints of problem (Q(w)), respectively. Then $\bar{H} := \{y \in Y : y^T u^* = b^T v^*\}$ is a face of $Y$ containing $w$.*

The outer approximation algorithm identifies the set of all efficient extreme points in $\mathcal{Y}$.

Starting with the simplex $S$, the outer approximation algorithm iteratively generates a finite number of nonempty and compact polyhedra $S^k$, for $k = 0, 1, 2, ..., K$, such that $S = S^0 \supseteq S^1 \supseteq \ldots \supseteq S^{K-1} \supseteq S^K = Y$. For each $k \geqslant 0$, the inequality $y^T u^k \leqslant b^T v^k$, which is appended to $S^k$ in the step **(k3)**, is called an inequality cut. This kind of inequality allows $S^{k+1}$ to cut off a portion of $S^k$ containing $s^k$. In a typical iteration $k$, a vertex $s^k$ of $S^k$ will be identified such that $s^k \notin Y$ . Subsequently, the unique point $w^k$ in the boundary of $Y$ that lies on the line segment with end points $\bar{p} \in \text{int}\, Y$ and $s^k$, will be identified. The boundary point $w^k$ is a weakly nondominated point of

**Algorithm 1** Outer approximation algorithm for MOLP

| | |
|---|---|
| **Initialization.** | |
| **(i1)** | Compute $\bar{p} \in \text{int } Y$. Let $S^0 := S$ and $k = 0$. |
| **Iteration steps.** | |
| **(k1)** | If $V(S^k) \subseteq Y$, stop and put $Y = S^k$. Otherwise, choose any $s^k \in V(S^k) \setminus Y$ and continue. |
| **(k2)** | Find the unique value $\lambda^k \in (0, 1)$ such that $\lambda^k s^k + (1 - \lambda^k)\hat{p} \in$ bd $Y$ and set $w^k = \lambda^k s^k + (1 - \lambda^k)\hat{p}$. |
| **(k3)** | Set $S^{k+1} = S^k \cap \{y \in \mathbb{R}^p : y^T u^k \leqslant b^T v^k\}$, where $({u^k}^T, {v^k}^T)$ is any dual optimal solution to the linear program $(\mathrm{Q}(w^k))$. |
| **(k4)** | Using $V(S^k)$ and the definition of $S^{k+1}$ given in the step **(k3)**, determine $V(S^{k+1})$. Set $k = k + 1$ and go to the step **(k1)**. |

MOLP. Benson [2] proves that the algorithm is finite and it terminates with finding all the nondominated extreme points of $Y$.

## 4. Nonsmooth Wolfe duality

In this section, we state the Wolfe duality theorems for the single objective problem (P), where $p = 1$. For $x^0 \in \mathcal{X}$, let

$$I^0 = \{i : g_i(x^0) = 0\} \quad \text{and} \quad J^0 = \{I, ..., m\} \setminus I^0.$$

Relative to the problem (P), the following constraint qualification is used

$$(\mathcal{R}) \begin{cases} \exists v \in \mathbb{R}^n : g^\circ_{I^0}(x^0; v) \leq 0, h^\circ(x^0; v) = 0, \\ \exists \delta > 0, U_{x^0}, \forall x \in U_{x^0}, 0 < \lambda < \delta : g_{J^0}(x + \lambda v) \leqq 0, h(x + \lambda v) = 0, \end{cases}$$

where $U_{x^0}$ is a neighborhoods of $x^0$. Here $g^\circ_{I^0}(x^0; v)$ is the vector of components $g^\circ_i(x^0; v), \forall i \in I^0$, and $h^\circ(x^0; v) = (h^\circ_1(x^0; v), ..., h^\circ_q(x^0; v))$.

Mititelu [15] extended the known theorems of Wolfe's duality for nonsmooth problems. This section summarizes these extensions relating to the program (P). Let us consider the function $\phi : A \times \mathbb{R}^m \times \mathbb{R}^q \to \bar{\mathbb{R}}$, defined by

$$\phi(x, u, v) = f(x) + \Sigma_{i=1}^m u_i g_i(x) + \Sigma_{j=1}^q v_j h_j(x).$$

The Wolfe type dual of the problem (P) is

$$(\mathrm{DW}) \qquad \max_{(x,u,v)\in\Omega} \phi(x, u, v)$$

where $\Omega \subseteq C \times \mathbb{R}^m \times \mathbb{R}^q$ is defined by

$$\Omega = \{(x, u, v) : 0 \in \partial f(x) + \Sigma_{i=1}^m u_i \partial g_i(x) + \Sigma_{j=1}^q v_j \partial h_j(x) + N_C(x), u \geqq 0\}.$$

Let us recall the following theorems from [15], which are called weak duality and direct duality theorems, respectively.

**Theorem 4.1.** *Let the domains* $\mathcal{X}$ *and* $\Omega$ *of problems* (P) *and* (DW) *be nonempty and*

$$\phi(t,u,v)=\min_{x\in C}\phi(x,u,v),$$

*for every* $(t,u,v)\in\Omega$. *Then*

$$\inf(P)\geqslant\sup(DW),$$

*where*

$$\inf(P):=\inf\{f(x):x\in\mathcal{X}\}$$

*and*

$$\sup(DW):=\sup\{\phi(x,u,v):(x,u,v)\in\Omega\}.$$

**Theorem 4.2.** *Let* $x^0$ *be a local solution to the problem (P) and* $h^\circ(x^0,.)$ *be finite on* $\mathbb{R}^n$. *Suppose that*

(w1) *the domain* $\mathcal{X}$ *satisfies the constraint qualification (*$\mathcal{R}$*),*
(w2) $(t,u,v)\in\Omega\Longrightarrow\phi(t,u,v)=\min_{x\in C}\phi(x,u,v)$.

*Then there are vectors* $u^0\geqq 0$ *and* $v^0\in\mathbb{R}^q$ *(as the optimal Lagrange multipliers), such that the point* $(x^0,u^0,v^0)$ *is a global solution to the dual program* (DW) *and, in addition, one has*

$$f(x^0)=\phi(x^0,u^0,v^0).$$

## 5. An approximation algorithm

In this section, we propose an algorithm to obtain the (weakly) nondominated set of $\mathcal{Y}$. We use the Wolfe duality to extend Benson's algorithm for the problem (P). First, we propose an assumption on objective functions which is equivalent to the convexity of $\mathcal{P}$.

**Proposition 5.1.** *The objective function* $f$ *of the problem* (P) *is convexlike on* $\mathcal{X}$ *if and only if* $\mathcal{P}$ *is convex.*

*Proof.* Suppose $y_1,\ y_2\in\mathcal{P}$ and $f$ is convexlike on $\mathcal{X}$. This means that there are $x_1,x_2\in\mathcal{X}$ such that $f(x_1)\leqq y_1$ and $f(x_2)\leqq y_2$. Due to the convexlikeness of $f$ there is $x\in\mathcal{X}$ such that $f(x)\leqq\lambda y_1+(1-\lambda)y_2$, for all $\lambda\in(0,1)$. Thus $\mathcal{P}$ is convex. Conversely, let $x_1,x_2\in\mathcal{X}$ and $\lambda\in(0,1)$. Due to the convexity of $\mathcal{P}$ we have $\lambda f(x_1)+(1-\lambda)f(x_2)\in\mathcal{P}$. It means that there is $x\in\mathcal{X}$ such that $f(x)\leqq\lambda f(x_1)+(1-\lambda)f(x_2)$. Therefore the objective function $f$ of the problem (P) is convexlike on $\mathcal{X}$. □

Now we present some properties of $\mathcal{P}$ and the connections between two sets $\mathcal{P}$ and $\mathcal{Y}$.

**Proposition 5.2.** *Let the objective function* $f$ *of the problem* (P) *be convexlike on* $\mathcal{X}$ *and let* $\mathcal{P}$ *be a closed set. The following statements hold.*

1. *The set* $\mathcal{P}\subseteq\mathbb{R}^p$ *is nonempty and* $\mathbb{R}^p_{\geqq}$*-bounded from below.*

2. $\mathcal{Y}_N = \mathcal{P}_N$.
3. $\mathcal{P}_{WN} = \text{bd } \mathcal{P}$.

*Proof.* According to Proposition 5.1, $\mathcal{P}$ is convex. Therefore the proofs of assertions 1 and 3 are similar to those in [5, Proposition 4.1]. To prove 2, first we show that $\mathcal{Y}_N \subseteq \mathcal{P}_N$. Suppose, to the contrary, that there is $y \in \mathcal{Y}_N$ such that $y \notin \mathcal{P}_N$. We conclude that there are $\bar{y} \in \mathcal{P}$ and $\hat{y} \in \mathcal{Y}$ such that $\hat{y} \leqq \bar{y} \leq y$. This is a contradiction which leads to $\mathcal{Y}_N \subseteq \mathcal{P}_N$.

To prove the opposite side, let $y \in \mathcal{P}_N$, i.e., there is no $\bar{y} \in \mathcal{P}$ such that $\bar{y} \leq y$. Since $\mathcal{Y} \subseteq \mathcal{P}$, there is no $\bar{y} \in \mathcal{Y}$ such that $\bar{y} \leq y$. To justify $\mathcal{P}_N \subseteq \mathcal{Y}_N$, it remains to show that $y \in \mathcal{Y}$. On the contrary, Let $y \in \mathcal{P} \setminus \mathcal{Y}$. Hence there is $\tilde{y} \in \mathcal{Y}$ such that $\tilde{y} \leq y$. Therefore, we get $\tilde{y} \in \mathcal{P}$ and $\tilde{y} \leq y$ which contradicts $y \in \mathcal{P}_N$. □

5.1. **Constructing a boundary point.** According to Proposition 5.2, every (weakly) nondominated point of the problem (P) is a boundary point of $\mathcal{P}$. Hence we aim to obtain a boundary point by solving the following problem. Assume that $s \in \mathbb{R}^p \setminus \text{int } \mathcal{P}$ and $d \in \mathbb{R}^p$ such that $s + d \in \text{int } \mathcal{P}$. If $\mathcal{P}$ is convex and closed, the problem

$$\min\{z \in \mathbb{R} : s + zd \in \mathcal{P}\}. \tag{5.1}$$

has an optimal solution. Now let us consider the following problem

$$z^* = \min\{z \in \mathbb{R} : x \in \mathcal{X}, f(x) \leqq s + ze\}, \tag{P(s)}$$

where $s \in \mathbb{R}^p \setminus \text{int } \mathcal{P}$. Note that $e$ can be replaced by any $d \in \text{int } \mathbb{R}^p_{\geqq}$. Since $\mathcal{P} = \mathcal{Y} + \mathbb{R}^p_{\geqq}$, there exists $z_0 \in \mathbb{R}$ such that $s + z_0 e \in \mathcal{P}$. Therefore $(\text{P}(s))$ is equivalent to problem (5.1) provided that $d = z_0 e$. Assume that $f$ is convexlike on $\mathcal{X}$ and $\mathcal{P}$ is closed. Then, according to Proposition 5.1, an optimal solution to $(\text{P}(s))$ exists for every $s \in \mathbb{R}^p \setminus \text{int } \mathcal{P}$.

The following proposition shows that every optimal solution to $(\text{P}(s))$ yields a weakly efficient solution to the problem (P) and a weakly nondominated point of $\mathcal{P}$.

**Proposition 5.3.** *Let the objective function $f$ of the problem* (P) *be convexlike on $\mathcal{X}$ and $\mathcal{P}$ be a closed set. Assume that $s^* \in \mathbb{R}^p \setminus \text{int } \mathcal{P}$. Then an optimal solution $(x^*, z^*)$ to problem $(\text{P}(s^*))$ exists and $x^*$ is a weakly efficient solution to the problem* (P)*. Furthermore, it results $y^* = s^* + z^* e \in \mathcal{P}_{WN}$.*

*Proof.* The existence of an optimal solution $(x^*, z^*)$ to $(\text{P}(s^*))$ follows from above discussion. In the case of $s^* \notin \mathcal{P}$, according to [5, Proposition 4.4], $x^*$ is a weakly efficient solution to the problem (P) and $y^* = s^* + z^* e \in \mathcal{P}_{WN}$. If $s^* \in \mathcal{P} \setminus \text{int } \mathcal{P}$, then $s^* \in \text{bd } \mathcal{P}$. We conclude that there is $x^* \in \mathcal{X}$ such that $f(x^*) \leqq s^*$ and there is no $x \in \mathcal{X}$ such that $f(x) < s^*$. Therefore, $(x^*, z^*)$ is an optimal solution to the problem (P) when $z^* = 0$. According to Proposition 5.2, this implies that $y^* = s^*$ is a weakly nondominated point of $\mathcal{P}$. □

5.2. **Constructing a separating hyperplane.** In this part, a new method is presented to construct a hyperplane separating the point $s$ from $\mathcal{P}$ at a boundary point of $\mathcal{P}$. We introduce the following primal and dual pair (P($s$)) and (DW($s$)) depending on $s \in \mathbb{R}^p \setminus \operatorname{int} \mathcal{P}$ to obtain the required supporting hyperplane. Let

$$(\mathrm{P}(s)) \qquad \begin{array}{ll} \min\limits_{(x,z)} & z \\ \text{s.t.} & f(x) - ze - s \leqq 0, \\ & g(x) \leqq 0, \\ & h(x) = 0, \\ & x \in C, \end{array}$$

and

$$(\mathrm{DW}(s)) \quad \begin{array}{ll} \max & z + \lambda_1^T g(x) + \lambda_2^T (f(x) - ze - s) + \mu^T h(x) \\ \text{s.t.} & 0 \in \Sigma_{i=1}^m \lambda_1^i \partial g_i(x) + \Sigma_{i=1}^p \lambda_2^i \partial f_i(x) + \Sigma_{i=1}^k \mu_i \partial h_i(x) + N_C(x), \\ & 1 - \lambda_2^T e = 0, \\ & \lambda = (\lambda_1, \lambda_2) \geqq 0. \end{array}$$

In the following theorem we show that an optimal solution to the dual problem exists and the hyperplane $H$, separating the point $s^*$ from the convex set $\mathcal{P}$ at a boundary point of $\mathcal{P}$, is obtained by the optimal Lagrange multipliers of (P($s^*$)).

**Theorem 5.4.** *Suppose that $(x^*, z^*)$ is an optimal solution to (P($s^*$)) and $y^* = s^* + z^* e$ under the assumptions of Proposition 5.3. Furthermore, suppose that $h^\circ(x^*; .)$ is finite on $\mathbb{R}^n$, and assumptions (w1) and (w2) in Theorem 4.2, hold for (P($s^*$)). Then there are $\lambda^* \geqq 0$ and $\mu^* \in \mathbb{R}^q$ such that $(x^*, z^*, \lambda^*, \mu^*)$ is an optimal solution to (DW($s^*$)) and*

(i) ${y^*}^T \lambda_2^* = {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T f(x^*)$;
(ii) $y^T \lambda_2^* \geqslant {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T f(x^*), \forall y \in \mathcal{P}$.

*This means that the set $H := \{y \in \mathbb{R}^p : y^T \lambda_2^* = {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T f(x^*)\}$ is a supporting hyperplane of $\mathcal{P}$ at $y^*$.*

*Proof.* According to Theorem 4.2, an optimal solution $(x^*, z^*, \lambda^*, \mu^*)$ to the dual problem (DW($s^*$)) exists.

(i). It follows from Theorem 4.2 that

$$z^* = z^* + {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T (f(x^*) - z^* e - s^*) + {\mu^*}^T h(x^*).$$

Since $(x^*, z^*, \lambda^*, \mu^*)$ is a feasible solution to (DW($s^*$)), we have

$$\begin{aligned} {y^*}^T \lambda_2^* &= (s^* + z^* e)^T \lambda_2^* \\ &= {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T f(x^*). \end{aligned}$$

(ii). Let $s$ be a boundary point of $\mathcal{P}$. Therefore there exists $x_s \in \mathcal{X}$ such that $f(x_s) \leqq y$ and $(x_s, 0)$ is a feasible solution to (P($s$)) with the objective value $z_s = 0$. On the other hand, the point $(x^*, z^*, \lambda^*, \mu^*)$ is feasible for (DW($s$)). According to Theorem 4.1, the objective value of (DW($s$)) at $(x^*, z^*, \lambda^*, \mu^*)$ is nonpositive, i.e.,

$$z^* + {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T (f(x^*) - z^* e - y) + {\mu^*}^T h(x^*) \leqslant 0,$$

and therefore we get

$$y^T \lambda_2^* \geqslant {\lambda_1^*}^T g(x^*) + {\lambda_2^*}^T f(x^*).$$

□

5.3. **Approximation algorithm for the problem (P).** In this section we introduce our algorithm. Let $\epsilon > 0$ be an approximation error and $\hat{p}$ be a point in $\mathbb{R}^p$. Decision makers should choose $\hat{p}$ as an upper bound according to their need. The algorithm starts with polyhedron $S^0 = z^{ideal} + \mathbb{R}^p_{\geqq}$ containing $\mathcal{P}$. In the $k$-th iteration, solving (P($s^k$)) yields a boundary point $y^k$ of the closed set $\mathcal{P}$, where $s^k$ is a vertex of $S^k$ and $s^k \leqq \hat{p}$. If $s^k \nleqq \hat{p}$, the point $s^k$ is added to $\mathcal{O}$ and another vertex is selected. Since, using Proposition 5.3, the boundary point $y^k$ is a weakly nondominated point of $\mathcal{P}$, the algorithm obtains weakly efficient solutions to the problem (P). The set of these points is denoted by $\mathcal{X}_{WE}$. If $d(s^k, y^k) \leqslant \epsilon$, we add the point $s^k$ to the outer approximation $\mathcal{O}$. Otherwise, we determine a hyperplane separating $s^k$ from $\mathcal{P}$ at $y^k$ using the optimal solution of (P($s^k$)) and it's optimal Lagrange multipliers. Then we update $S^k$ by using this hyperplane to obtain $S^{k+1}$ and the procedure is repeated. Let $\mathcal{I}$ denote all boundary points which are obtained by the algorithm. The best approximation error is the maximum distance between every vertex $s^k$ of total $S^k$ with $s^k \leqq \hat{p}$ and it's corresponding boundary point $y^k$.

This algorithm constructs an inner approximation ($\mathcal{P}^i$) and an outer approximation ($\mathcal{P}^o$) of $\mathcal{P}$ such that $\mathcal{P}^i \subseteq \mathcal{P} \subseteq \mathcal{P}^o$. Both $\mathcal{P}^i$ and $\mathcal{P}^o$ are convex polyhedrons and the distance between corresponding vertices are at most $\epsilon$, i.e., for every $s \in \text{vert } \mathcal{P}^o$ with $s \leqq \hat{p}$ there exists a vertex $v \in \text{vert } \mathcal{P}^i$ such that $d(v, s) \leqslant \epsilon$. Another property is that the number of common points of $\mathcal{P}^i$ and $\mathcal{P}^o$ are more than or equal to the number of hyperplanes obtained in Algorithm 2. It shows that every point of $\mathcal{I}_N \cap (\hat{p} - \mathbb{R}^p_{\geqq})$ is a weakly $\epsilon e$-nondominated point of $\mathcal{P}$. We summarize the result in the following theorem.

**Theorem 5.5.** *Let $\mathcal{P}^i$ be an inner approximation of $\mathcal{P}$ in Algorithm 2 and $\varepsilon = \epsilon e$. Then $\mathcal{P}^i_N \cap (\hat{p} - \mathbb{R}^p_{\geqq})$ is a set of weakly $\varepsilon$-nondominated points of $\mathcal{P}$.*

*Proof.* It is easy to check that $\mathcal{P}^i$, $\mathcal{P}$ and $\mathcal{P}^o$ are convex and the points of (vert $\mathcal{P}^o$) $\cap (\hat{p} - \mathbb{R}^P_{\geqq})$ has a one to one corresponding relationship with some

**Algorithm 2** Approximation algorithm for NMOP

| | |
|---|---|
| **Initialization.** | |
| **(i1)** | Let $S^0 := z^{ideal} + \mathbb{R}^p_{\geqq}$. Then vert $S^0 = \{z^{ideal}\}$. |
| **(i2)** | Let $\epsilon > 0$ be an approximation error and $\hat{p} \in \mathbb{R}^p$ be an upper bound. |
| **(i3)** | Let $\mathcal{O} := \emptyset$, $\mathcal{I} := \emptyset$, $\mathcal{X}_{WE} := \emptyset$, $\epsilon_0 := 0$ and $k := 0$. |
| **Iteration steps.** | |
| **(k1)** | If vert $S^k \subseteq \mathcal{O}$ stop. Otherwise, choose $s^k \in$ vert $S^k \setminus \mathcal{O}$. |
| **(k2)** | If $s^k \nleqq \hat{p}$ add $s^k$ to $\mathcal{O}$ and go to (**k1**). |
| **(k3)** | Solve $(\mathrm{P}(s^k))$. Set $\mathcal{X}_{WE} = \mathcal{X}_{WE} \cup \{x^k : (x^k, z^k)$ is an optimal solution to $(\mathrm{P}(s^k))\}$ and $y^k = s^k + z^k e$. Add $y^k$ to $\mathcal{I}$. |
| **(k4)** | If $d(s^k, y^k) \leqslant \epsilon$, add $s^k$ to $\mathcal{O}$. Set $\epsilon_0 = \max\,\{\epsilon_0,\, d(s^k, y^k)\}$ and go to (**k1**). |
| **(k5)** | If $d(s^k, y^k) > \epsilon$, determine a separating hyperplane $H := \{y \in \mathbb{R}^p : y^T \lambda_2^k = {\lambda_1^k}^T g(x^k) + {\lambda_2^k}^T f(x^k)\}$, where $(\lambda^k, \mu^k)$ are the optimal Lagrange multipliers to $(\mathrm{P}(s^k))$ and set $S^{k+1} := S^k \cap \{y \in \mathbb{R}^p : y^T \lambda_2^k \geq {\lambda_1^k}^T g(x^k) + {\lambda_2^k}^T f(x^k)\}$. |
| **(k6)** | Determine vert $S^{k+1}$ and set $k = k + 1$. go to (**k1**). |
| **Results** | |
| **(r1)** | $\mathcal{X}_{WE}$ is a set of weakly efficient solution and $f(\mathcal{X}_{WE})$ is a set of weakly nondominated points for the main problem (P). |
| **(r2)** | $\mathcal{I}$ is a set of finite number of weakly non-dominated points of $\mathcal{P}$. |
| **(r3)** | $\mathcal{P}^i = \mathrm{conv}\, \mathcal{I} + \mathbb{R}^p_{\geqq}$ is an inner approximation for $\mathcal{P}$. |
| **(r4)** | $\mathcal{P}^o = S^k$ is an outer approximation for $\mathcal{P}$. |
| **(r5)** | $\epsilon_0$ is the maximum error in this approximation. |

vertices of polyhedron $\mathcal{P}^i$. Note also that the most distances between all corresponding vertices is $\epsilon$. Therefore the proof is similar to [5, Theorem 4.3] and [16, Theorem 5]. □

Defining the approximation error $\epsilon_0$ in the algorithm has some advantages. The value $\epsilon_0$ shows minimum approximations error. It means that we have the same results for every approximation error between $\epsilon_0$ and $\epsilon$ when $\epsilon_0 < \epsilon$. Therefore, using an approximation error less than $\epsilon_0$ gives results that are more accurate. Moreover, we have $\mathcal{P}^i \cap (\hat{p} - \mathbb{R}^p_{\geqq}) = \mathcal{P} \cap (\hat{p} - \mathbb{R}^p_{\geqq}) = \mathcal{P}^o \cap (\hat{p} - \mathbb{R}^p_{\geqq})$ where $\epsilon_0 = 0$. It follows that $\mathcal{P}^i_N \cap (\hat{p} - \mathbb{R}^p_{\geqq}) = \mathcal{P}_N \cap (\hat{p} - \mathbb{R}^p_{\geqq}) = \mathcal{Y}_N \cap (\hat{p} - \mathbb{R}^p_{\geqq})$.

As discussed above, according to Theorem 5.5, Algorithm 2 constructs $\mathcal{P}^i_N \cap (\hat{p} - \mathbb{R}^p_{\geqq})$ as a set of weakly $\bar{\varepsilon}$-nondominated points of the problem (P) for every $\bar{\varepsilon} \in \mathbb{R}^p$ with $\varepsilon_0 \leqq \bar{\varepsilon} \leqq \varepsilon$.

The following theorem proves the convergence of Algorithm 2.

**Theorem 5.6.** *Assume that $z^{ideal}$ exists. Under the assumptions of Theorem 5.4 Algorithm 2 works correctly: It terminates finitely and returns the results* $(r1) - (r5)$.

*Proof.* It is obvious that $S^0$ exists. According to Proposition 5.3, the optimal solution to $(\mathrm{P}(s^k))$ exists for all $s^k \in \text{vert } S^k \setminus \mathcal{O}$ and hence the step $(k3)$, Result $(r1)$ and Result $(r2)$ are true. Theorem 5.4 proves that the hyperplane $H$ presented in the step $(k5)$ exists and separates $s^k$ from $\mathcal{P}$ at $y^k$. Therefore, we have $\mathcal{P} \subseteq S^k$ in each iteration which yields Result $(r4)$. Since every $S^k$ is a convex polyhedron (the intersection of a finite number of half space), then the number of all vertices of $S^k$ is finite. On the other hand, in each iteration where $d(s^k, y^k) > \epsilon$, a set of points containing $\{y \in \mathbb{R}^p : s^k \leq y \leqq y^k\}$ is removed from $S^k$ to obtain $S^{k+1}$. Therefore Algorithm 2 stops at the finite number of iterations, by the existence of upper bound $\hat{p}$. It is easy to see that the convexity of $\mathcal{P}$ yields Result $(r3)$. The last Result $(r5)$ is true because we have $d(s^k, y^k) \leqslant \epsilon_0$ for all vertices $s^k$ of the total $S^k$. □

It should be noticed that we have more boundary points in $\mathcal{I}$, the same as [11], in the comparison with the algorithm presented in [5]. These additional boundary points are those $y^k$ which their distance with their corresponding $s^k$ is at most $\epsilon$. Therefore we often have a better inner approximation. For instance in [5, Example 5.7], we add 7 boundary points to $\mathcal{I}$ (instead of 3 boundary points obtained by [5, Algorithm 4.2]) with the same approximation error at no additional cost. Moreover, at the end of algorithm we have three additional results namely (r1, r2 and r5).

Algorithm 1 proposed in [11] is an extension of Benson's algorithm for convex MOPs. It is worth mentioning that problem (P(s)) in our proposed algorithm has the same form as the primal problem $\mathrm{P}_2(\mathrm{v})$ in [11, Algorithm 1]. Therefore, the cost of two algorithms is the same. But Algorithm 2 is also suitable for a class of nonconvex problems. We require the convexlikeness of the objective funtion $f$ on $\mathcal{X}$ which is weaker than the convexity of the problem (P).

Note that [11, Algorithm 1] use Slater's condition (which requires the convexity of functions and a point $x_0 \in \mathcal{X}$ such that $g(x_0) < 0$) to prove direct duality. Thus equality constrains cannot be treated with [11] by writing them as two inequality constrains. Algorithm 2 is applicable for a class of NMOPs with equality and inequality constraints via Wolfe duality. Another advantage is that we don't require any compactness assumptions on the feasible set $\mathcal{X}$ of the problem (P). In fact, we suppose the point $\hat{p}$ in the algorithm to stop the algorithm at a finite number of iterations. Calculating the minimum approximation error is another preference of our algorithm.

## 6. (Weakly) $\varepsilon$-nondominated set in BOPs

In this section, we consider weakly nondominated points for BOPs. We sort all points of $\mathcal{I}$ in Algorithm 2 to construct an approximation set of (weakly) nondominated points for BOPs. In the following theorem we prove that all $y^k \in \mathcal{I}$ are nondominated points when $p = 2$.

**Theorem 6.1.** *Let $p = 2$ in the problem* (P)*, then every $y^k$ is a nondominated point of $\mathcal{Y}$ in Algorithm* 2.

*Proof.* Suppose on the contrary that there exist $k \in \mathbb{N} \cup \{0\}$ and $s^k \in \text{vert } S^k$ such that $y^k = s^k + z^k e$ and $y^k \notin \mathcal{P}_N$. We conclude that there exists $\bar{y} \in \mathcal{P}$ such that $\bar{y} \leq y^k$. By Theorem 5.2 we get $y^k \in \mathcal{P}_{WN}$ and hence $y^k \not> \bar{y}$. Without loss of generality, assume that $y^k = [y_1^k, y_2^k]^T$, $\bar{y} = [\bar{y}_1, \bar{y}_2]^T, \alpha := y_1^k = \bar{y}_1$ and $y_2^k > \bar{y}_2$. Since $y^k \in \text{bd } \mathcal{P}$ and $\bar{y} \leq y^k$, due to the convexity of $\mathcal{P} = \mathcal{Y} + \mathbb{R}^2_{\geqq}$, we have $z_1^{ideal} = \alpha$ and therefore $s^k = y^k$. Thus $s^k$ is a linear combination of two points of $S^k$ (e.g. $\bar{y}$ and $y^k + \bar{y}$). This contradicts the fact that $s^k \in \text{vert } S^k$. $\square$

Note that above theorem is not true for the problem (P) with $p > 2$. We present a counterexample showing that Algorithm 2 obtains a weakly nondominated point which is not nondominated point when $p = 3$.

In the following, we arrange the points of $\mathcal{I}$ to approximate a set of nondominated points and a set of weakly nondominated points of a BOP, for the case of $p = 2$.

Let $y^1, y^2 \in \mathbb{R}^2_{\geqq}$. Define $y^1 \prec y^2$ when $y_1^1 < y_1^2$. Therefore we have a relation "$\prec$" for every two points of $\mathcal{I}$, according to Theorem 6.1. Let $\mathcal{I} = \{y^1, ..., y^r\}$ where $r \in \mathbb{N}$, is sorted by $\mathcal{I} = \{\bar{y}^1, ..., \bar{y}^r\}$ such that $\bar{y}^i \prec \bar{y}^{i+1}, i = 1, ..., r-1$ and let $\bar{y}^0 := [\bar{y}_1^0, \bar{y}_2^0]^T$ such that $\bar{y}_1^0 = z_1^{ideal}$ and $\bar{y}_2^0 = \min\{f_2(x) : x \in \mathcal{X}, f_1(x) = z_1\}$. Assume that $\bar{y}^{r+1} = [\bar{y}_1^{r+1}, \bar{y}_2^{r+1}]$ such that $\bar{y}_2^{r+1} = z_2^{ideal}$ and $\bar{y}_1^{r+1} = \min\{f_1(x) : x \in \mathcal{X}, f_2(x) = \bar{y}_2^{r+1}\}$. The set of all nondominated points of inner approximation $\mathcal{P}^i$ is given by

$$\mathcal{P}_N^i = \{\lambda \bar{y}^i + (1 - \lambda)\bar{y}^{i+1} : \lambda \in [0, 1], i = 0, 1, ..., r\}.$$

It is easy to see that for every $y \in \mathcal{Y}_N$, there exists a point $p \in \mathcal{P}_N^i$ such that $d(y, p) \leqslant \epsilon$. Thus $\mathcal{P}_N^i$ is an approximation for $\mathcal{Y}_N$, i.e., every point of $\mathcal{P}_N^i$ is an $\varepsilon$-nondominated point to the problem (P). We define the following points to obtain $\mathcal{P}_{WN}^i$. Let $t := \max\{f_2(x) : x \in \mathcal{X}, f_1(x) = z_1\}$ and $r := 1$, provided that $\max\{f_2(x) : x \in \mathcal{X}, f_1(x) = z_1\}$ exists; otherwise let $t := \bar{y}^0 + [0, 1]^T$ and $r := \infty$. Similarly let $t' := \max\{f_1(x) : x \in \mathcal{X}, f_2(x) = \bar{y}_2^{r+1}\}$ and $r' := 1$, provided that $\max\{f_1(x) : x \in \mathcal{X}, f_2(x) = \bar{y}_2^{r+1}\}$ exists; otherwise let $t' := \bar{y}^{r+1} + [1, 0]^T$ and $r' := \infty$. Thus the set of weakly nondominated point of $\mathcal{P}^i$ is formulated as follows.

$$\mathcal{P}^i_{WN} = \mathcal{P}^i_N \cup \mathrm{cl}\{\begin{bmatrix} \bar{y}^0_1 \\ \lambda\bar{y}^0_2 + (1-\lambda)t \end{bmatrix}, \begin{bmatrix} \lambda'\bar{y}^{r+1}_1 + (1-\lambda')t' \\ \bar{y}^{r+1}_2 \end{bmatrix} : \lambda \in (0,r), \lambda' \in (0,r')\}.$$

It is easy to check that every point of $\mathcal{P}^i_{WN}$ is a weakly $\varepsilon$-nondominated point of $\mathcal{Y}$.

## 7. Examples

Now we present some examples to illustrate the obtained results. The algorithm was implemented in Matlab 8.0 (R2012b) using Optimization Toolbox to compute optimal solutions and the tests were run on a processor Intel(R) Core(TM) i5 CPU with 2.67GHz and 3GB RAM.

We start with a counterexample for Theorem 6.1 for the case of $p = 3$. This example also illustrates that all nondominated points of a problem sometimes can be found by Algorithm 2 for small approximation errors.

**Example 7.1.** Consider the following multiobjective problem:

$$\begin{aligned} \min\ & f_1(x) = x_1, \\ & f_2(x) = 2 - x_1, \\ & f_3(x) = x_2 \\ \text{s.t. } & g_1(x) = x_1 - 2 \leqslant 0, \\ & g_2(x) = -x_1 \leqslant 0, \\ & g_3(x) = -x_2 \leqslant 0. \end{aligned}$$

Let $\epsilon \in (0, \sqrt{2})$ and $\hat{p} \in \mathbb{R}^p$ be arbitrary. Algorithm 2 starts with $s^0 = z^{ideal} = [0,0,0]^T$ and construct the hyperplane $H = \{(y_1, y_2, y_3) : y_1 + y_2 = 2\}$ separating $s^0$ from $\mathcal{P}$ at $y^0 = s^0 + z^0 e = [1,1,1]^T$, where $x^0 = [1, 0.5]^T$ and $z^0 = 1$ is an optimal solution to $(\mathrm{P}(s^0))$. As shown in Figure 1, the weakly nondominated point $y^0$ is not a nondominated point because $[1,0] \in \mathcal{X}$ and $f([1,0]) = [1,1,0]^T \leq [1,1,1]^T$. the algorithm returns $\epsilon_0 = 0$ and $\mathcal{P}^i = \mathcal{P} = \mathcal{P}^o$. Therefore $\mathcal{P}^i_N = \mathcal{P}_N = \mathcal{Y}_N$.

In the next example we show that Algorithm 2 is applicable for nonsmooth multiobjective problems. This example is taken from [5] and it was used as an example which can not be solved by the algorithm provided in [5]. Since we do not require differentiability in our algorithm, this example can be solved.

**Example 7.2.** Consider the following nonsmooth problem:

$$\begin{aligned} \min\ & f_1(x) = |x_1| + |x_2|, \\ & f_2(x) = |x_1 - 2| + |x_2| \\ \text{s.t. } & g(x) = x_1^2 + x_2^2 - 100 \leqslant 0. \end{aligned}$$

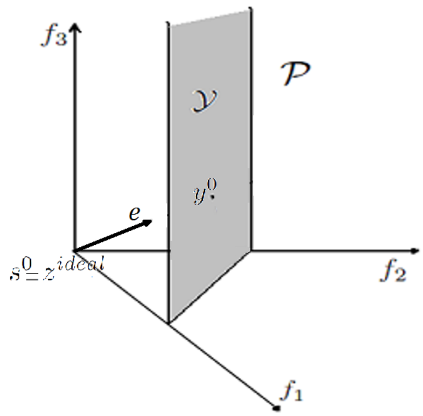


FIGURE 1. Objective space, $\mathcal{P}$, ideal point, interior point and boundary point in Example 7.1

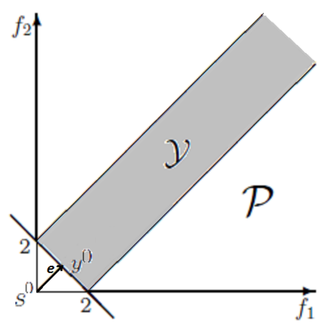


FIGURE 2. First cut in the objective space in Example 7.2

We know that $S^0 = \mathbb{R}^2_{\geqq}$ and hence $s^0 = (0,0)$. Algorithm 2 computes the optimal solution $x^0 = (1,0)$ and $z^0 = 1$ to problem $(\mathrm{P}(s^0))$ and the boundary point $y^0 = s^0 + z^0 e = (1,1)$ of $\mathcal{P}$ at the first iteration. According to Theorem 5.4, there are $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0) \geqq 0$ such that

$$\begin{cases} 0 \in \lambda_1^0 \partial f_1(x^0) + \lambda_2^0 \partial f_2(x^0) + \lambda_3^0 \partial g(x^0) \\ \lambda_1^0 + \lambda_2^0 = 1 \\ \lambda_3^0 g(x^0) + (\lambda_1^0, \lambda_2^0)^T (f(x^0) - z^0 e - s^0) = 0 \end{cases}$$

The structure of $\partial f$ and $\partial g$ is as follows.

$$\partial f_1(x^0) = \{(1, \alpha) : -1 \leqslant \alpha \leqslant 1\},$$

$$\partial f_2(x^0) = \{(-1, \beta) : -1 \leqslant \beta \leqslant 1\},$$
$$\partial g(x^0) = \{(2, 0)\}.$$

Therefore we have $\lambda^0 = (0.5, 0.5, 0)$ which yeilds that $H = \{(y_1, y_2)|y_1+y_2 = 2\}$ separates the point $s^0$ from $\mathcal{P}$ at $y^0 = (1, 1)$ for every $\epsilon \in (0, 1)$. At the end of the first iteration, we have $\mathcal{O} = \emptyset$ and $\mathcal{I} = \{(1, 1)\}$. Some details of this example and the first cut are shown in Figure 2.

In the second iteration, $S^1$ is equal to $\mathcal{P}$. Hence vert $S^1 = \{(2, 0), (0, 2)\}$. If $\hat{p} \in (2, 2) + \mathbb{R}^2_{\geqq}$, since $y^1 = s^1 = (2, 0)$ and $d(s^1, y^1) = 0 < \epsilon$, the algorithm adds the point $(2, 0)$ to $\mathcal{O}$ and $\mathcal{I}$. Similarly, the point $(0, 2)$ is added to $\mathcal{O}$ and $\mathcal{I}$ at the third iteration. Therefore $\mathcal{X}_{WE} = \{(1.0), (0, 1), (0, 0)\}$, $\epsilon_0 = 0$ and $\mathcal{P}^i = \mathcal{P} = \mathcal{P}^o$. Thus $\mathcal{P}^i_N$ is exactly equal to all nondominated points of this example.

In the following example we consider a convexlike multiobjective problem.

**Example 7.3.** Consider the following nonconvex problem:

$$\begin{aligned} \min\ & f_1(x) = x - \pi, \\ & f_2(x) = \sin x + 1 \\ \text{s.t.}\ \ & g(x) = \pi - x \leqslant 0. \end{aligned}$$

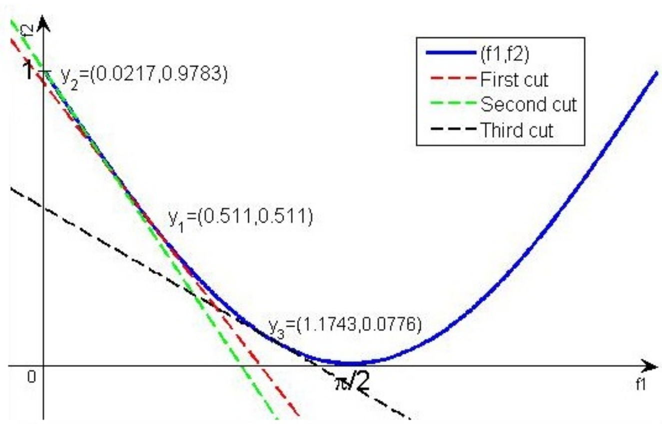


FIGURE 3. First three cutting planes in Example 7.3

We set the approximation error $\epsilon = 0.025$ and $\hat{p} = 10^5$. It is clear that the ideal point is $s^0 = z^{ideal} = (0, 0)$. In iteration $k$-th, solving the primal problem $(\mathrm{P}(s^k))$ depending on a vertex $s^k$ of $S^k$ yields the optimal solution $(x^k, z^k)$.

TABLE 1. Some details of Algorithm 2 for Example 7.3

| k | $s^k$ | $x^k$ | $y^k$ | Cutting planes |
|---|---|---|---|---|
| 0 | (0, 0) | 3.6526 | (0.5110, 0.5110) | $(0.4659, 0.5341, 0.5110)$ |
| 1 | (0, 0.9567) | 3.1633 | (0.0217, 0.9783) | $(0.5000, 0.5000, 0.5000)$ |
| 2 | (1.0968, 0) | 4.3159 | (1.1743, 0.0776) | $(0.2786, 0.7214, 0.3831)$ |
| 3 | (0, 1.0000) | 3.1416 | (0.0000, 1.0000) | $d(s^k, y^k) \leqslant \epsilon$ |
| 4 | (0.3397, 0.6604) | 3.4846 | (0.3430, 0.6637) | $d(s^k, y^k) \leqslant \epsilon$ |
| 5 | (1.3752, 0) | 4.5329 | (1.3913, 0.0161) | $d(s^k, y^k) \leqslant \epsilon$ |
| 6 | (0.8756, 0.1929) | 4.0412 | (0.8996, 0.2169) | $(0.3835, 0.6165, 0.4787)$ |
| 7 | (0.7201, 0.3286) | 3.8685 | (0.7270, 0.3354) | $d(s^k, y^k) \leqslant \epsilon$ |
| 8 | (1.0407, 0.1292) | 4.1876 | (1.0461, 0.1345) | $d(s^k, y^k) \leqslant \epsilon$ |

Then the boundary point of $\mathcal{P}$ is generated. If $d(s^k, y^k) > \epsilon$, the algorithm construct the hyperplane $H$. Figure 3 shows the first three cutting planes and the computational data can be seen in Table 1. Note that the last column is the parameters of the hyperplanes, i.e., $\lambda_2^k$ is the first two components and the last component is equal to ${\lambda_1^k}^T g(x^k) + {\lambda_2^k}^T f(x^k)$. Finally, the algorithm returns $\epsilon_0 = 0.0227$, which means that we should select $\epsilon < 0.0227$ to have more cutting planes and more vertices.

The following example is a problem where the algorithms proposed in [5,11] does not terminate for some approximation error.

**Example 7.4.** Consider the following problem:

$$
\begin{aligned}
&\min\ f(x) = (x_1, x_2, x_3) \\
&s.t.\ g(x) = \frac{1}{9.0601 - (x_1-3)^2 - (x_2-3)^2} - \frac{1}{9.0601} - x_3 \leqslant 0, \\
&\qquad x_1, x_2 \geqslant 0, \\
&\qquad x \in A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1-3)^2 + (x_2-3)^2 - 9.0601 < 0\}.
\end{aligned}
$$

This example has an unbounded feasible set and an additional constraint $x \in A$ where $A$ is an open unbounded set in $\mathbb{R}^3$. Therefore the assumptions of the algorithm proposed in [5,11] is not true for this example. Defining $\hat{p}$ helps Algorithm 2 to terminates at the finite number of iterations.

We set the approximation error $\epsilon = 0.9$ and $\hat{p} = 10^2$. Algorithm 2 starts with the ideal point $s^0 = z^{ideal} = (0,0,0)$. Table 2 shows some computational data of the algorithm and Figure 4 shows the first three cutting planes formulated as follows.

$$
\begin{aligned}
H_1 &= \{(y_1, y_2, y_3) : 0.4530y_1 + 0.4530y_2 + 0.0941y_3 = 0.9820\}, \\
H_2 &= \{(y_1, y_2, y_3) : 0.4995y_1 + 0.4995y_2 + 0.0009y_3 = 0.8913\}, \\
H_3 &= \{(y_1, y_2, y_3) : 0.5125y_1 + 0.0877y_2 + 0.3998y_3 = 0.5748\}.
\end{aligned}
$$

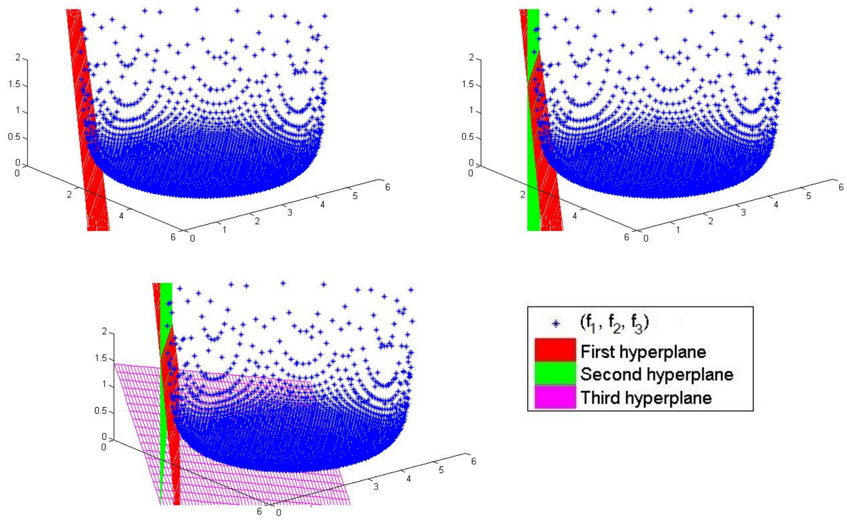


FIGURE 4. First three cutting planes in Example 7.4

TABLE 2. Some details of Algorithm 2 for Example 7.4

| k | $s^k$ | $x^k = y^k$ | Cutting planes |
|---|---|---|---|
| 0 | (0, 0, 0) | (0.9820, 0.9820, 0.9820) | $H_1$ |
| 1 | (0, 0, 10.4407) | (0.8819, 0.8819, 11.3223) | $H_2$ |
| 2 | (0, 2.1679, 0) | (0.3847, 2.5526, 0.3847) | $d(s^k, y^k) \leqslant \epsilon$ |
| 3 | (2.1679, 0, 0) | (2.5526, 0.3847, 0.3847) | $d(s^k, y^k) \leqslant \epsilon$ |
| 4 | (0, 1.7809, 1.8638) | (0.2358, 2.0167, 2.0996) | $d(s^k, y^k) \leqslant \epsilon$ |
| 5 | (1.7809, 0, 1.8638) | (2.0167, 0.2358, 2.0996) | $d(s^k, y^k) \leqslant \epsilon$ |

The last example is a multiobjective problem which is neither smooth nor convex.

**Example 7.5.** Consider the following convexlike, nonsmooth problem:

$$\min\ f_1(x) = x_1,$$

$$f_2(x) = \begin{cases} -3x + 2 & \text{if } x \leqslant 0.5, \\ \dfrac{2}{9}(x-2)^2 & \text{if } 0.5 < x \leqslant 2, \\ [x-2] & \text{if } 2 < x \end{cases}$$

$$\text{s.t. } g(x) = -x \leqslant 0.$$

Table 3. Some details of Algorithm 2 for Example 7.5

| k | $s^k$ | $x^k$ | $y^k$ | Cutting planes |
|---|---|---|---|---|
| 0 | (0, 0) | 0.5000 | (0.5000, 0.5000) | $(0.4000, 0.6000, 0.5110)$ |
| 1 | (0, 0.8333) | 0.2917 | (0.2917, 1.1250) | $(0.7500, 0.2500, 0.5000)$ |
| 2 | (1.2500, 0) | 1.3453 | (1.3453, 0.0953) | $(0.2254, 0.7746, 0.3770)$ |
| 3 | (0, 2.0000) | 0 | (0.0000, 2.0000) | $d(s^k, y^k) \leqslant \epsilon$ |
| 4 | (0.5000, 0.5000) | 0.5000 | (0.5000, 0.5000) | $d(s^k, y^k) \leqslant \epsilon$ |
| 5 | (1.6726, 0) | 1.6935 | ( 1.6935, 0.0209) | $d(s^k, y^k) \leqslant \epsilon$ |
| 6 | (0.9226, 0.2182) | 0.9496 | (0.9496, 0.2452) | $d(s^k, y^k) \leqslant \epsilon$ |

Some computational data of Algorithm 2 with $\epsilon = 0.1$ and $\hat{p} = 10^7$ is reported in Table 3. Although the function $f_2(x)$ isn't differetiable at the first weakly efficient solution $x^0 = (0.5, 0.5)$, Algorithm 2 generates the separating hyperplane at this iteration.

Summarizing information comparing the results of Algorithm 2 with various values of $\epsilon$ is given in Table 4. The number of points of the set $(\text{vert } \mathcal{P}^o) \cap (\hat{p} - \mathbb{R}^p_{\geqq})$ (# Opt) and the number of cutting planes (# H) are shown in the third and the fifth columns. The fourth column (# Out) shows the number of points of the set $(\text{vert } \mathcal{P}^o) \setminus (\hat{p} - \mathbb{R}^p_{\geqq})$.

Table 4. Different values of $\epsilon$ for three examples

| Example | $\epsilon$ | # Opt | # Out | # H | $\epsilon_0$ | Time(s) |
|---|---|---|---|---|---|---|
| 7.3 | 0.1 | 5 | 0 | 2 | 0.0339 | 0.173559 |
| | 0.01 | 11 | 0 | 5 | 0.0097 | 0.383144 |
| | 0.001 | 41 | 0 | 20 | 6.8538e-04 | 1.449901 |
| 7.4 | 0.9 | 6 | 1 | 2 | 0.6663 | 0.387581 |
| | 0.1 | 80 | 15 | 30 | 0.0995 | 5.206289 |
| | 0.02 | 182 | 48 | 66 | 0.0496 | 21.267091 |
| 7.5 | 0.2 | 5 | 0 | 2 | 0.1347 | 0.114457 |
| | 0.02 | 11 | 0 | 5 | 0.0101 | 0.334783 |
| | 0.002 | 29 | 0 | 14 | 0.0019 | 0.969401 |

## References

[1] Q.H. Ansari and J.C. Yao (eds.) Recent Developments in Vector Optimization, Springer, Berlin, 2012.

[2] H.P. Benson, An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem, *J. Global Optim.* **13** (1998), no. 1, 1–24.

[3] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, Nonsmooth Analysis and Control Theory, Grad. Texts in Math. 178, Springer-Verlag, New York, 1998.

[4] M. Ehrgott, A. Löhne and L. Shao, A dual variant of Benson's outer approximation algorithm, *J. Global Optim.* **52** (2012), no. 4, 757–778.

[5] M. Ehrgott, L. Shao and A. Schöbel, An approximation algorithm for convex multiobjective programming problems, *J. Global Optim.* **50** (2011), no. 3, 397–416.

[6] G. Eichfelder, Scalarizations for adaptively solving multi-objective optimization problems, *Comput. Optim. Appl.* **44** (2009), no. 2, 249–273.

[7] H. Eschenauer, J. Koski and A. Osyczka, Multicriteria Design Optimization Procedures and Applications, Springer-Verlag, Berlin, 1990.

[8] J. Fernández and B. Tóth, Obtaining the efficient set of nonlinear biobjective optimization problems via interval branch-and-bound methods, *Comput. Optim. Appl.* **42** (2009), no. 3, 393–419.

[9] P. Lindroth, M. Patriksson and A.B. Stromberg, Approximating the Pareto optimal set using a reduced set of objective functions, *European J. Oper. Res.* **207** (2010), no. 3, 1519–1534.

[10] G.P. Liu, J.B. Yang and J.F. Whidborne, Multiobjective Optimization and Control, Research Studies Press, Bristol, 2002.

[11] A. Löhne, B. Rudloff and F. Ulus, Primal and dual approximation algorithms for convex vector optimization problems, *J. Global Optim.* **60** (2014), no. 4, 713–736.

[12] D. T. Luc, Theory of Vector Optimization, Lecture Notes in Econom. and Math. Systems 319, Springer-Verlag, Berlin, 1989.

[13] O.L. Mangasarian, Nonlinear Programming, McGraw-Hill, New York, 1969.

[14] K.M. Miettinen, Nonlinear Multiobjective Optimizatoin, Kluwer Academic Publ. Boston, 1999.

[15] S. Mititelu, A survey on optimality and duality in nonsmooth programming, in: Generalized Convexity, pp. 211–225, Lecture Notes in Econom. and Math. Systems 405, Springer, Berlin, 1994.

[16] L. Shao and M. Ehrgott, Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning, *Math. Methods Oper. Res.* **68** (2008), no. 2, 257–276.

[17] R.B. Statnikov and J.B. Matusov, Multicriteria Optimization and Engineering, Chapman & Hall, New York, 1995.

[18] S. Ruzika and M.M. Wiecek, Approximation methods in multiobjective programming, *J. Optim. Theory Appl.* **126** (2005), no. 3, 473–501.

(Narjes Shafiei) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran.

*E-mail address*: `shafiei.narjes@sci.ui.ac.ir`