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CRITICAL FIXED POINT THEOREMS IN BANACH ALGEBRAS UNDER WEAK TOPOLOGY FEATURES

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ABSTRACT. In this paper, we establish some new critical fixed point theorems for the sum $AB + C$ in a Banach algebra relative to the weak topology, where $\frac{I-C}{A}$ is allowed to be non invertible. In addition, a special class of Banach algebras will be considered. As an application, our results are used to prove the existence of solutions for a nonlinear integral equation.
Keywords: Banach algebras, sequentially weakly continuous operators, weakly compact operators, fixed point theorems.
MSC(2010): Primary: 47H10; Secondary: 47J05, 47J10.

1. Introduction

Fixed point theory is an important topic and the core part of nonlinear functional analysis and is a useful tool to investigate the existence theorems for nonlinear differential and integral equations. Some of these equations can be formulated into the following nonlinear equations:

$$x = AxBx + Cx. \tag{1.1}$$

The equations (1.1) arise in many fields of science as mechanics, physics and economics. To study these equations in Banach algebras, many authors concerned fixed point techniques, see, e.g., [12–14] and the references therein.

In the lack of local compactness in infinite dimensional Banach spaces (algebras), weak topology is the best environment to investigate the existence of solutions of nonlinear integral equations and nonlinear differential equations in this setting. So in recent years, many papers have been devoted to the existence of fixed points for mappings acting on a Banach algebra equipped with its weak

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topology, see e.g., [2–4, 7, 8, 20, 21]. In most of the above quoted works, the assumptions that $\frac{I-C}{A}$ is sequentially weakly continuous and invertible, and that A, B and C are weakly sequentially continuous, play a fundamental role in the arguments. So, it would be interesting to investigate the problem when $\frac{I-C}{A}$ is not injective and the operators A and C are not sequentially weakly continuous. The object of this paper is to explore this kind of extensions by looking for the resulting mapping $(\frac{I-C}{A})^{-1}B$ achieving a fixed point in its domain.

The paper is organized as follows. Section 2 is devoted to give some definitions which are needed in the sequel. In Section 3, we will establish the existence of solutions of equation (1.1) in a Banach algebra or in a Banach algebra satisfying condition $(\mathcal{P})$ (see Definition 2.4 below), where $\frac{I-C}{A}$ is not necessarily invertible. The obtained results significantly extend and improve the above mentioned works. In Section 4, we use our fixed point results to present a new existence theory for solutions to the following nonlinear integral equation in the Banach algebra $C(J, X)$

$$x(t) = a(t) + (Tx)(t)[(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds).u], \quad 0 < \lambda < 1, \tag{1.2}$$

where $J = [0, 1]$ and X is a Banach algebra. Let the functions a, q and σ be continuous on J and let T, p be nonlinear functions. Furthermore, we assume that u is a non vanishing vector.

2. **Preliminaries**

In this section, we give some definitions and preliminary results which are useful for our analysis.

Definition 2.1. An algebra E is a vector space endowed with an internal composition law denoted by (.), i.e.,

$$\begin{cases} (.) : E \times E \to E, \\ (x, y) \mapsto x.y, \end{cases}$$

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying the following property

$$\|xy\| \leq \|x\|\|y\| \text{ for all } x, y \in E.$$

A complete normed algebra is called a Banach algebra.

Definition 2.2. Let X be a Banach space. An operator $T : X \to X$ is said to be weakly compact if $T(B)$ is relatively weakly compact for every bounded subset $B \subset X$.

Definition 2.3. Let X be a Banach space. An operator $T : X \to X$ is said to be sequentially weakly continuous on X if for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$ (here $\rightharpoonup$ denotes weak convergence).

In general, the product of two weakly sequentially continuous mappings on a Banach algebra X is not necessarily weakly sequentially continuous.

Definition 2.4 ([3])**.** We will say that a Banach algebra X satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P})\begin{cases}\text{For any sequences } (x_n)_n \text{ and } (y_n)_n \text{ in } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y,\\ \text{we have } x_n y_n \rightharpoonup xy.\end{cases}$$

Note that, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. If X satisfies condition $(\mathcal{P})$ then $\mathcal{C}(K, X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$, where K is a compact Hausdorff space. The proof is based on Dobrokov's theorem:

Theorem 2.5 (Dobrakov, [16, p. 36])**.** *Let K be a compact Hausdorff space and X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $\mathcal{C}(K, X)$ and $f \in \mathcal{C}(K, X)$. Then $(f_n)_n$ is weakly convergent to f if and only if $(f_n(t))_n$ is weakly convergent to $f(t)$ for each $t \in K$.*

Definition 2.6 ([5])**.** Let X be a Banach space. An operator $F : X \longrightarrow X$ is said to be strongly continuous on X if for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Fx_n \longrightarrow Fx$, here $\longrightarrow$ denotes convergence in X.

Definition 2.7. A Banach space X has the Dunford-Pettis property (for short DP property) if for each Banach space Y every weakly compact linear operator $F : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.

The Dunford-Pettis property as defined above was explicitly defined by A. Grothendieck [19] who undertook an extensive study of this and related properties (see also [15]). It is well known that any L_1 space has the DP property [17]. It was proved in [2] that any Banach algebra with the Dunford-Pettis property satisfies condition $(\mathcal{P})$.

Proposition 2.8. *Let X be a Dunford-Pettis space and T a weakly compact linear operator on X. Then T is strongly continuous.*

Definition 2.9. Let X be a Banach space and let Ω be a nonempty subset of X. If T maps Ω into X, we say that T is β-condensing if T is bounded and $\beta(T(D)) < \beta(D)$ for all bounded subsets D of Ω with $\beta(D) > 0$. Here β is the De Blasi [11] measure of weak non-compactness defined by

$$\beta(X) = \inf\{r > 0 : \text{ there exits } Y \in \mathcal{K}^w \text{ such that } X \subset Y + B_r\}.$$

Definition 2.10. Let X be a Banach space. A mapping $\mathcal{G} : X \to X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_{\mathcal{G}} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \phi_{\mathcal{G}}\|x - y\|$$

for all $x, y \in X$, with $\phi_{\mathcal{G}}(0) = 0$. If $\phi_{\mathcal{G}}(r) = kr$ for some $k > 0$, then $\mathcal{G}$ is called a Lipschitzian function on X with the Lipschitz constant k. Furthermore, if $k < 1$, then $\mathcal{G}$ is called a contraction on X with contraction k.

Remark 2.11. Every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse is not true in general. For example [2], take $\mathcal{G}(x) = \sqrt{| x |}, x \in \mathbb{R}$, and consider $\phi_{\mathcal{G}}(r) = \sqrt{r}, r \geq 0$. Then $\mathcal{G}$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\phi_{\mathcal{G}}$, but $\mathcal{G}$ is not Lipschitzian.

Remark 2.12. If $\phi_{\mathcal{G}}$ is not necessarily nondecreasing and satisfies $\phi_{\mathcal{G}}(r) < r$, for $r > 0$, the mapping $\mathcal{G}$ is called a nonlinear contraction with a contraction function $\phi_{\mathcal{G}}$.

Theorem 2.13 ([6, Theorem 2.5]). *Let X be a Banach space, Ω a nonempty closed convex subset of X and $F : \Omega \to \Omega$ a sequentially weakly continuous map. If $F(\Omega)$ is relatively weakly compact, then F has a fixed point in Ω.*

Remark 2.14. One of the advantages of the weak topology of a Banach space X is the fact that if a set Ω is weakly compact, then every sequentially weakly continuous mapping $F : \Omega \to X$ is weakly continuous. This is an immediate consequence of the Eberlein-Smulian theorem (see [18, Theorem 8.12.4, p. 549]).

Theorem 2.15 ([5, Theorem 3.5]). *Let Ω be a nonempty closed and convex subset of a Banach space X. Suppose that A and B are sequentially weakly continuous mappings from Ω into X such that*

(i) *$A(\Omega) \subset (I - B)(X)$ and $[x = Bx + Ay,\ y \in \Omega] \Rightarrow x \in \Omega$ (or $A(\Omega) \subset (I - B)(\Omega)$).*
(ii) *$A(\Omega)$ is relatively weakly compact.*
(iii) *If $(I - B)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$.*
(iv) *For every y in the range of $(I-B)$, $D_y = \{x \in \Omega$ such that $(I-B)x = y\}$ is a convex set.*

Then there exists $x \in \Omega$ such that $x = Ax + Bx$.

Remark 2.16. Suppose $B : X \to X$ is a β-condensing, $B(X)$ is a bounded subset of X and $I - B$ is invertible. Then the assumptions (iii) and (iv) of Theorem 2.15 are satisfied. Indeed, suppose that $(I - B)x_n \rightharpoonup y$, for some $(x_n) \subset X$ and $y \in X$. Rewriting x_n as $x_n = (I - B)x_n + Bx_n$ and using the subadditivity of the De Blasi measure of weak noncompactness, we get

$$\beta(\{x_n\}) \leq \beta(\{(I - B)x_n\}) + \beta(\{Bx_n\}).$$

Since $\overline{\{(I - B)x_n\}}^w$ is weakly compact, we obtain $\beta(\{x_n\}) \leq \beta(\{Bx_n\})$. Now we show that $\beta(\{x_n\}) = 0$. If we suppose the contrary, then using the fact that B is β-condensing, we obtain

$$\beta(\{x_n\}) \leq \beta(\{Bx_n\}) < \beta(\{x_n\}),$$

which is absurd. Therefore, we get $\beta(\{x_n\}) = 0$. Consequently, $\overline{\{x_n\}}^w$ is weakly compact and using the Eberlein-Smulian theorem guarantees that there exists a weakly convergent subsequence of $(x_n)_n$. Hence, the assumption (iii) is satisfied. On the other hand, since $I - B$ is invertible, we get that the set $D_y = \{x \in \Omega : (I-B)x = y\}$ is reduced to the convex set $\{(I-B)^{-1}y\}$, where y is in the range of $I - B$.

Definition 2.17. Let E be a Banach algebra. An operator $T : E \to E$ is called regular on E if T maps E into the set of all invertible elements of E.

Definition 2.18. Let U be a subset of a Banach space X.

(i) U is said to be absorbing if for every $x \in X$, there exists $\alpha > 0$, such that $\forall\ \lambda \in \mathbb{K}$; $(\mid \lambda \mid \leq \alpha \Rightarrow \lambda x \in U)$.

(ii) U is called balanced if for each α with $\mid \alpha \mid \leq 1$; $\alpha U \subseteq U$.

We next introduce the concept of multi-valued mappings.

Definition 2.19. Let Ω be a nonempty subset of a Banach space X. Every application $H : \Omega \to \mathcal{P}(X)$, where

$$\mathcal{P}(X) := \{M : sM \subset X \text{ and } M \neq \emptyset\}.$$

is called a multi-valued mapping (or multi-function) defined on Ω

Remark 2.20. (i) Every single valued mapping (or mapping) $F : \Omega \to X$ can be identified with a multi-valued mapping $H : \Omega \to \mathcal{P}(X)$, (see [24, p. 447]) by setting

$$Hx = \{Fx\} \text{ for all } x \in \Omega.$$

(ii) We denote $Gr(H) = \{(x, y) \in \Omega \times X : x \in \Omega \text{ and } y \in Hx\}$ the graph of H.

Definition 2.21. Let Ω be a nonempty subset of a Banach space X and $H : \Omega \to \mathcal{P}(X)$ a multi-valued mapping. H is said to have a weakly sequentially closed graph if $Gr(H)$ is weakly sequentially closed, i.e., for every sequence $(x_n)_n \subset \Omega$ with $x_n \rightharpoonup x$ and for every sequence $(y_n)_n$ with $y_n \in Hx_n$, forall $n \in \mathbb{N}$, $y_n \rightharpoonup y$ implies that $y \in Hx$.

Theorem 2.22 ([9, Theorem 2.2])**.** *Let Ω be a nonempty closed and convex subset of a Banach space X, and let $H : \Omega \to \mathcal{P}(\Omega)$ be a multi-valued mapping such that*

(i) *$H(\Omega)$ is relatively weakly compact;*

(ii) *H has a weakly sequentially closed graph;*

(iii) *the set Hx is a nonempty closed and convex set for all $x \in \Omega$.*

Then there exists $x \in \Omega$ such that $x \in Hx$.

Remark 2.23. Since every sequentially weakly continuous single valued mapping can be identified with a multi-valued mapping having a weakly sequentially closed graph, Theorem 2.15 is the multi-valued analogue of Theorem 2.5.

The next theorem extends a result of H. Schaefer [22] to the case of multi-valued mappings in the context of weak topology, dealing with the method of a priori estimate in the Leray-Schauder theory.

Theorem 2.24 ([5, Theorem 3.7])**.** *Let X be a Banach space and $H : X \to \mathcal{P}(X)$ a multi-valued mapping such that*

(i) *H has a weakly sequentially closed graph;*
(ii) *there exists a closed convex, balanced and absorbing weak neighborhood U of θ such that the set $H(mU)$ is relatively weakly compact for all $m \in \mathbb{N}$;*
(iii) *the set Hx is closed, convex and nonempty for all $x \in X$.*

Then either for any $\lambda \in [0, 1]$ there exists an $x \in X$ such that

$$x \in \lambda Hx,$$

or the set $\{x \in X : \exists \lambda \in]0, 1[, \quad x \in \lambda Hx\}$ is an unbounded.

3. Main results

We will establish some new existence results for the equation (1.1) in Banach algebras under weak topology features.

Theorem 3.1. *Let Ω be a nonempty closed and convex subset of a Banach algebra E. Let $A, C : E \to E$ and $B : \Omega \to E$ be three operators which satisfy the following conditions:*

(i) *A is regular on Ω.*
(ii) *$B(\Omega)$ is a relatively weakly compact subset of E;*
(iii) *If $(\frac{I-C}{A})x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$;*
(iv) *$(\frac{I-C}{A})^{-1}Bx$ is convex for all $x \in \Omega$.*
(v) *B and $\frac{I-C}{A}$ are sequentially weakly continuous on Ω;*
(vi) *$B(\Omega) \subset (\frac{I-C}{A})(E)$ and $[x = AxBy + Cx, \ y \in \Omega] \Rightarrow x \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = AxBx + Cx$.

Proof. First, we assume that $\frac{I-C}{A}$ is invertible. Choose fixed y in Ω and define the mapping

$$\begin{cases} F : \Omega \to \Omega \\ y \mapsto Fy = (\frac{I-C}{A})^{-1}By. \end{cases}$$

Note that F is well defined by the assumption (vi).

Step 1: $F(\Omega)$ is relatively weakly compact. Let $(y_n)_n \subset F(\Omega)$, we choose $(x_n)_n \subset \Omega$ such that $y_n = Fx_n = (\frac{I-C}{A})^{-1}Bx_n$. Taking into account assumption (ii), together with the Eberlein-Smulian theorem, we get a subsequence $(y_{\varphi_{1(n)}})_n$ of $(y_n)_n$ such that $(\frac{I-C}{A})y_{\varphi_{1(n)}} \rightharpoonup z$, for some $z \in \Omega$. Thus by assumption (iii), there exists a subsequence $y_{\varphi_1(\varphi_{2(n)})}$ converging weakly to $y_0 \in \Omega$. Hence by the Eberlein-Smulian theorem $F(\Omega)$ is relatively weakly compact.

Step 2: F is sequentially weakly continuous. Let $(x_n)_n \subset \Omega$ such that $x_n \rightharpoonup x \in \Omega$. Because $F(\Omega)$ is relatively weakly compact, it follows that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $Fx_{n_k} \rightharpoonup y$ for each $y \in \Omega$. The weak sequential continuity of $(\frac{C-I}{A}+I)$ leads to $(\frac{C-I}{A}+I)Fx_{n_k} \rightharpoonup (\frac{C-I}{A}+I)y$. Also, from the equality

$$(\tfrac{C-I}{A}+I)F = -B + F,$$

it result that

$$-Bx_n + Fx_n \rightharpoonup -Bx + y.$$

So $y = Fx$. We claim that $Fx_n \rightharpoonup Fx$. Indeed, suppose that this not the case, then there exists a subsequence $(x_{\varphi_{1(n)}})_n$ and a weak neighborhood V^w of $(\frac{I-C}{A})^{-1}Bx$ such that $(\frac{I-C}{A})^{-1}B(x_{\varphi_{1(n)}}) \notin V^w$ for all $n \in \mathbb{N}$. On the other hand, we have $x_{\varphi_{1(n)}} \rightharpoonup x$, then arguing as before, we find a subsequence $(x_{\varphi_1(\varphi_{2(n)})})_n$ such that $(\frac{I-C}{A})^{-1}B(x_{\varphi_1(\varphi_{2(n)})})$ converges weakly to $(\frac{I-C}{A})^{-1}Bx$, which is a contradiction and hence F is sequentially weakly continuous. So an application of Theorem 2.5 yields that the mapping F has a fixed point in Ω.

Second, if $\frac{I-C}{A}$ is not invertible, then $(\frac{I-C}{A})^{-1}$ could be seen as a multi-valued mapping. Define $H : \Omega \to \mathcal{P}(\Omega)$ by $Hy = (\frac{I-C}{A})^{-1}By$. H is well defined by assumption (vi). We prove that H fulfills the hypotheses of Theorem 2.15.

Step 1: Hx is a convex set for each $x \in \Omega$. This is an immediate consequence of assumption (iv).

Step 2: H has a weakly sequentially closed graph. Let $x \in \Omega$ and $(x_n)_n \subset \Omega$ such that $x_n \rightharpoonup x$ and $y_n \in Hx_n$ such that $y_n \rightharpoonup y$. By the definition of H, we have $(\frac{I-C}{A})y_n = Bx_n$. Since $\frac{I-C}{A}$ and B are sequentially weakly continuous, we obtain $(\frac{I-C}{A})y = Bx$. Thus $y \in (\frac{I-C}{A})^{-1}Bx$.

Step 3: $H(\Omega)$ is relatively weakly compact. This assertion is proved by using the same reasoning as the one in Step 1 in the first part of the proof.

Step 4: Hx is a nonempty closed set for each $x \in \Omega$. This assertion follows from Step 1 and Step 2 in the first part of the proof.
In view of Theorem 2.15, we get $x \in Hx$ for some $x \in \Omega$. Thus, there exists $x \in \Omega$ such that $x = AxBx + Cx$. □

Remark 3.2. Hypothesis (iv) of Theorem 3.1 makes sense if $\frac{I-C}{A}$ is not invertible on $B(\Omega)$.

Theorem 3.3. *Let Ω be a nonempty closed convex subset of a Banach algebra E. Let $A, C : E \to E$ and $B : \Omega \to E$ be three operators such that*

(i) *A is regular on E;*
(ii) *$B(\Omega)$ is a relatively weakly compact subset of E;*
(iii) *if $(\frac{I-C}{A})x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$;*

(iv) *A and C are nonlinear contractions on E with contraction functions φ_A and φ_C, respectively;*
(v) *B and $\frac{I-C}{A}$ are sequentially weakly continuous on Ω;*
(vi) *$B(\Omega) \subset (\frac{I-C}{A})(E)$ and $[x = AxBy + Cx,\ y \in \Omega] \Rightarrow x \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = AxBx + Cx$, as soon as $M\varphi_A(r) + \varphi_C(r) < r$, for $r > 0$, where $M = \|B(\Omega)\|$.

Proof. Let $y \in \Omega$ be fixed and define the mapping

$$\begin{cases} \phi_y : E \to E \\ x \mapsto \phi_y(x) = AxBy + Cx. \end{cases}$$

Let $x_1, x_2 \in E$; $x_1 \neq x_2$, the use of the assumption (vi) leads to

$$\begin{aligned} \|\phi_y(x_1) - \phi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|By\|\|Ax_1 - Ax_2\| + \|Cx_1 - Cx_2\| \\ &\leq M\varphi_A(\|x_1 - x_2\|) + \varphi_C(\|x_1 - x_2\|) < r. \end{aligned}$$

This shows that ϕ_y is a nonlinear contraction on E, where $r = \|x_1 - x_2\| > 0$. Therefore, an application of a fixed point theorem of Boyd and Wong [10] yields that there is a unique element $x_y \in E$ such that

$$\phi_y(x_y) = x_y.$$

Or equivalently,

$$x_y = Ax_yBy + Cx_y.$$

Since the hypothesis (vi) holds, then $x_y \in \Omega$. So, by virtue of the hypothesis (i), x_y verifies $(\frac{I-C}{A})x_y = By$. Thus, the mapping $(\frac{I-C}{A})^{-1}$ is well defined on $B(\Omega)$. Define a mapping

$$\begin{cases} N : \Omega \to \Omega \\ y \mapsto Ny = (\frac{I-C}{A})^{-1}By. \end{cases}$$

By an application of Theorem 3.1, we obtain $N(\Omega)$ is relatively weakly compact and N is sequentially weakly continuous. Hence N has a fixed point in Ω. □

Remark 3.4. Theorem 3.3 extends and improves [3, Theorem 3.2].

Corollary 3.5. *Let Ω be a closed convex set of a Banach algebra E. Suppose that B and C satisfy the conditions of Theorem 3.3 and $A : \Omega \to E,\quad x \mapsto e$, where "$e$" is the neutral element of E. Then there exists $x \in \Omega$ such that $Bx + Cx = x$.*

Theorem 3.6. *Let E be a Banach algebra and Ω a nonempty closed convex and bounded subset of E. Let $A, C : \Omega \to E$ and $B : \Omega \to \Omega$ be three operators such that*

(i) *A is regular on Ω;*

(ii) *for every* $y \in B(\Omega)$, $D_y = \{x \in \Omega,$ *such that* $(\frac{I-C}{A})(x) = y\}$ *is a nonempty convex set of* E;
(iii) $(\frac{I-C}{A})^{-1}B$ *has a weakly sequentially closed graph;*
(iv) $(\frac{I-C}{A})^{-1}B$ *is a weakly compact operator;*
(v) $x = AxBy + Cx \Rightarrow x \in \Omega$ *for all* $y \in \Omega$.

Then there exists $x \in \Omega$ *such that* $x = AxBx + Cx$.

Proof. Define the multi-valued mapping $H : \Omega \to \mathcal{P}(\Omega)$ by $Hy := (\frac{I-C}{A})^{-1}By$. Then H is well defined by assumption (v).

Hx is a convex set for each $x \in \Omega$. This is an immediate consequence of assumption (ii). Let $K = \overline{\text{conv}}(H(\Omega))$ be the closed convex hull of $H(\Omega)$. Clearly K is closed convex and bounded and $H(K) \subset K \subset \Omega$. So, by hypothesis K is weakly compact. Therefore $H(\Omega)$ is relatively weakly compact.

Hx is closed for each $x \in \Omega$ by setting $(x_n)_n \equiv x$. This assertion follows from the assumption (iii) since $H(\Omega)$ is relatively compact.

Thus H verifies the assumptions of Theorem 3.3. Hence there exists $x \in \Omega$ such that $x \in Hx$. □

Remark 3.7. Theorem 3.6 extends and improves [3, Theorem 3.1], in the case where $\frac{I-C}{A}$ is not invertible.

An interesting corollary of Theorem 3.6 is

Corollary 3.8. *Let* E *be a Banach algebra and let* Ω *be a nonempty closed convex and bounded subset of* E. *Let* $A, C : E \to E$ *and* $B : \Omega \to E$ *be three operators such that*

(i) A *is regular on* E;
(ii) $B(\Omega) \subset (\frac{I-C}{A})(E)$ *and* $[x = AxBy + Cx \Rightarrow x \in \Omega]$ *for all* $y \in \Omega$;
(iii) $(\frac{I-C}{A})^{-1}B(\Omega)$ *is relatively weakly compact on* E;
(iv) A *and* C *are nonlinear contractions on* E *with contraction functions* φ_A *and* φ_C, *respectively;*
(v) $\frac{I-C}{A}$ *and* B *are sequentially weakly continuous.*

Then there exists $x \in \Omega$ *such that* $x = AxBx + Cx$, *as soon as* $M\varphi_A(r) + \varphi_C(r) < r$, *for all* $r > 0$, *where* $M = \|B(\Omega)\|$.

Proof. By using the hypotheses (i), (iv) and Theorem 3.3, we prove that $(\frac{I-C}{A})^{-1}$ exists on $B(\Omega)$. By virtue of assumption (ii), we obtain

$$(\frac{I-C}{A})^{-1}B(\Omega) \subset \Omega.$$

Therefore, we can define

$$\begin{cases} F : \Omega \to \Omega \\ y \mapsto Fy = (\frac{I-C}{A})^{-1}By. \end{cases}$$

Using Theorem 3.3, it suffices to prove that the operator $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous. To see this, let $(x_n)_n \subset \Omega$ be any sequence in Ω such that $x_n \rightharpoonup x$ in Ω. A similar reasoning as in step 2 of Theorem 3.1, proves that the operator $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous. Finally we conclude that F has a fixed point y in Ω. Hence y verifies equation (1.1), i.e.,

$$y = AyBy + Cy.$$

□

The next result extends [5, Theorem 3.7] in a Banach algebra relative to the weak topology.

Theorem 3.9. *Let E be a Banach algebra, and let A, B and $C : E \to E$ be three operators such that*

(i) *A is regular on E;*

(ii) *There exists a closed convex, balanced and absorbing weak neighborhood U of θ such that the set $B(mU)$ is relatively weakly compact for all $m \in \mathbb{N}$;*

(iii) *If $(\frac{I-C}{A})(x_n) \rightharpoonup y$, then there exists a weakly convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$;*

(iv) *B and $\frac{I-C}{A}$ are sequentially weakly continuous;*

(v) *for every y in the range of B, $D_y = \{x \in E$ such that $(\frac{I-C}{A})(x) = y\}$ is a nonempty convex set.*

Then either for any $\lambda \in [0,1]$, there exists $x \in E$ such that

$$x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}),$$

or the set $\{x \in E : \exists\, \lambda \in]0,1[$ such that $x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda})\}$ is unbounded.

Proof. First, we assume that $\frac{I-C}{A}$ is invertible on $B(E)$. Define $F : E \to E$ by $Fy := (\frac{I-C}{A})^{-1}By$. Then F is well defined.

Step1: $F(mU)$ is relatively weakly compact. Let $(y_n)_n \subset F(mU)$, we choose $(x_n)_n \subset mU$ such that $y_n = Fx_n$. Or, equivalently $(\frac{I-C}{A})y_n = Bx_n$. Taking into account assumption (ii) and by the Eberlein-Smulian theorem, we get a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $(\frac{I-C}{A})y_{n_k} \rightharpoonup z$. By item (iii), $(y_{n_k})_k$ has a weakly convergent subsequence. Thus $F(mU)$ is relatively weakly compact.

Step 2: F is sequentially weakly continuous. Let $(x_n)_n \subset E$ such that $x_n \rightharpoonup x$, and let $y_n = (\frac{I-C}{A})^{-1}Bx_n$. Since $\frac{I-C}{A}$ is sequentially weakly continuous and by virtue of assumption (iii) there exists a weak convergent subsequence $(x_{n_k})_k \subset mU$ of $(x_n)_n$. Since $F(mU)$ is relatively weakly compact, there exists a subsequence $(x_{n_{k_j}})_j$ such that $x_{n_{k_j}} \rightharpoonup x$. Hence, $Fx_{n_{k_j}} \rightharpoonup y$ for each $y \in E$. The weak sequential continuity of $(\frac{C-I}{A} + I)$ leads to

$$(\frac{C-I}{A} + I)F(x_{n_{k_j}}) \rightharpoonup (\frac{C-I}{A} + I)(y).$$

Also, from the equality

$$(\frac{C-I}{A}+I)F=-B+F,$$

it results that

$$-Bx_{n_{k_j}}+Fx_{n_{k_j}}\rightharpoonup -Bx+y.$$

So $y=Fx$. We claim that $Fx_n \rightharpoonup Fx$. Indeed, suppose that this is not the case, then there exists a subsequence $(x_{\varphi_1(n)})_n$ and a weak neighborhood V^w of $(\frac{I-C}{A})^{-1}Bx$ such that $(\frac{I-C}{A})^{-1}B(x_{\varphi_1(n)}) \notin V^w$, for all $n\in\mathbb{N}$. On the other hand, we have $x_{\varphi_1(n)} \rightharpoonup x$, then arguing as before, there is a renamed subsequence $(x_{\varphi_1(\varphi_{2(n)})})_n$ such that $(\frac{I-C}{A})^{-1}B(x_{\varphi_1(\varphi_{2(n)})})$ converges weakly to $(\frac{I-C}{A})^{-1}Bx$, which is a contradiction. This yields that F is weakly sequentially continuous. Consequently, using Theorem 2.5, we get the desired result.

Second, if $\frac{I-C}{A}$ is not invertible, then $(\frac{I-C}{A})^{-1}$ could be seen a multi-valued mapping. Define $H:\Omega\to\mathcal{P}(\Omega)$ by $Hy:=(\frac{I-C}{A})^{-1}By$. Then H is well defined by assumption (ii).

Step 1: Hx is a convex set for each $x\in\Omega$. This is an immediate consequence of assumption (v). Also, Hx is closed for each $x\in\Omega$. This assertion follows from Step 1 and Step 2 in the first part of the proof by setting $(x_n)_n\equiv x$.

Step 2: H has a weakly sequentially closed graph. Let $(x_n)\subset E$ such that $x_n\rightharpoonup x$ and $y_n\in Hx_n$ such that $y_n\rightharpoonup y$. By definition of H, we have $(\frac{I-C}{A})y_n=Bx_n$. Since B and $\frac{I-C}{A}$ are weakly sequentially continuous, we obtain $(\frac{I-C}{A})y=By$. Thus $y\in(\frac{I-C}{A})^{-1}Bx=Hx$. So H has a weakly sequentially closed graph.

Step 3: $H(mU)$ is relatively weakly compact for all $m\in\mathbb{N}$. Let $(y_n)_n\subset F(mU)$, we choose $(x_n)_n\subset mU$ such that $y_n=Fx_n$. Taking into account assumption (ii), together with the Eberlein-Smulian theorem, we get a subsequence $(y_{\varphi_{1(n)}})_n$ of $(y_n)_n$ such that $(\frac{I-C}{A})y_{\varphi_{1(n)}}\rightharpoonup z$, for some $z\in\Omega$. Thus, by assumption (iii), there exists a subsequence $y_{\varphi_1(\varphi_{2(n)})}$ converging weakly to $y_0\in mU$. Hence $H(mU)$ is relatively weakly compact for all $m\in\mathbb{N}$. An application of Theorem 2.22 gives the result. □

As a consequence of Theorem 3.9 we obtain the following corollary.

Corollary 3.10. *Let E be a Banach algebra and let A,B and $C:E\to E$ satisfy the following conditions.*

(i) *A is regular on E;*

(ii) *there exists a closed convex, balanced and absorbing weak neighborhood U of θ such that $B(mU)$ is relatively weakly compact, for all $n\in\mathbb{N}$;*

(iii) *if $(\frac{I-C}{A})(x_n)\rightharpoonup y$, then there exists a weakly convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$;*

(iv) *$(\frac{I-C}{A})^{-1}B$ has a weakly sequentially closed graph;*

(v) *$(\frac{I-C}{A})^{-1}Bx\in\mathcal{P}_{cv}(E)$ for all $x\in E$.*

Then there exists $x\in E$ such that $x=AxBx+Cx$.

Proof. If $\frac{I-C}{A}$ is not invertible, $(\frac{I-C}{A})^{-1}$ could be seen as a multi-valued mapping. For every $y \in E$, define $H : E \to \mathcal{P}(E)$ by $Hy = (\frac{I-C}{A})^{-1}By$. Then H is well defined by assumption (ii). As an immediate consequence of assumption (v), the set Hx is convex for each $x \in E$. Also, H has a weakly sequentially closed graph by assumption (iv). By assumption (ii) there exists a closed convex, balanced and absorbing weak neighborhood U of θ such that $H(mU)$ is relatively weakly compact. It suffices to prove that Hx is a closed set. Indeed, for $x_n \equiv x$ and $y_n \in Hx_n$ such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, because H has a weakly sequentially closed graph we have $y \in Hx$. Furthermore $H(\Omega)$ is relatively weakly compact, so Hx is closed for each $x \in E$. Now, an application of Theorem 3.9 gives the result. □

Now, we give some new existence results for the equation $x = AxBx + Cx$ in Banach algebras satisfying condition $(\mathcal{P})$.

Theorem 3.11. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$, and let A, B and $C : E \to E$ be three sequentially weakly continuous operators such that*

- (i) *A is regular on E;*
- (ii) *$B(E) \subset (\frac{I-C}{A})(E)$ and $B(E)$ is bounded;*
- (iii) *A and C are nonlinear contractions on E with contractions functions φ_A and φ_C, respectively;*
- (iv) *There exists a closed convex, balanced and absorbing weak neighborhood U of θ such that $B(mU)$ is relatively weakly compact, for all $m \in \mathbb{N}$;*
- (v) *If $(\frac{I-C}{A})x_n \rightharpoonup y$, then there exists a weakly convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$.*

Then either for any $\lambda \in [0, 1]$, there exists $x \in E$ such that

$$x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda})$$

as soon as $M\varphi_A(r) + \varphi_C(r) < r$, for $r > 0$, where $M = \|B(E)\|$ or the set $\{x \in E \ / \ \exists \lambda \in]0, 1[$ such that $x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda})\}$ is an unbounded.

Proof. By a similar argument as in the proof of Theorem 3.3, we obtain $(\frac{I-C}{A})^{-1}$ exists on E. Define the mapping $F : E \to E$ by $Fy = (\frac{I-C}{A})^{-1}By$. By assumptions (ii), (iv) and (v), we deduce that $F(mU)$ is relatively weakly compact. It suffices to establish that $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous. To see this, let $(x_n)_n$ be a sequence in E which is weakly convergent to a point x in E. Now, define the sequence y_n in Ω by

$$y_n = (\frac{I-C}{A})^{-1}Bx_n.$$

Since $(\frac{I-C}{A})^{-1}B(mU)$ is relatively weakly compact, there is a renamed subsequence such that

$$y_n = (\frac{I-C}{A})^{-1}Bx_n \rightharpoonup y.$$

But, on the other hand, $(y_n)_n$ verifies

$$y_n = A(y_n)Bx_n + Cy_n.$$

Therefore, from hypothesis and in view of condition $(\mathcal{P})$, we deduce that y verifies the following equation

$$y = AyBx + Cy,$$

or equivalently,

$$y = (\frac{I-C}{A})^{-1}Bx.$$

Thus,

$$(\frac{I-C}{A})^{-1}Bx_n \rightharpoonup (\frac{I-C}{A})^{-1}Bx.$$

Indeed, suppose that this not the case, so there is V^w, a weak neighborhood of y, such that for all $n \in \mathbb{N}$, there exists an $N \geq n$ with $y_N \notin V^w$. Hence, there is a renamed subsequence $(y_n)_n$ verifying the property

$$\forall n \in \mathbb{N}, \quad y_n \notin V^w. \tag{3.1}$$

However,

$$\forall n \in \mathbb{N}, \quad y_n \notin (\frac{I-C}{A})^{-1}B(E).$$

Again, there is a renamed subsequence such that

$$y_n \rightharpoonup y'.$$

According to the preceding, we have

$$y' = (\frac{I-C}{A})^{-1}Bx,$$

and consequently, $y' = y$, which is a contradiction with the property (3.1). This shows that $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous. $\square$

Theorem 3.12. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and let Ω be a nonempty closed convex subset of E. Let $A, C : E \to E$ and $B : \Omega \to E$ be three operators such that*

(i) *A is regular in E;*
(ii) *A, B and C are sequentially weakly continuous on Ω;*
(iii) *$(\frac{I-C}{A})^{-1}$ is weakly compact on $B(\Omega)$ and $B(\Omega)$ is bounded;*
(iv) *$(\frac{I-C}{A})^{-1}Bx$ is convex for all $x \in \Omega$;*
(v) *$x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = AxBx + Cx$.

Proof. Let $y \in \Omega$, we define the mapping $H : \Omega \to \mathcal{P}_{cv}(\Omega)$ by $Hy = (\frac{I-C}{A})^{-1}By$. Then H is well defined and $(\frac{I-C}{A})^{-1}B(\Omega)$ is relatively weakly compact. In view of Theorem 3.9, it suffices to establish that $(\frac{I-C}{A})^{-1}B$ has a weakly sequentially closed graph. To see this, let $(x_n)_n$ be a weakly convergent sequence of Ω to a point $x \in \Omega$ and let $y_n \in (\frac{I-C}{A})^{-1}Bx_n$ with $y_n \rightharpoonup y$. Then y_n satisfies the following equation

$$y_n = Ay_nBx_n + Cy_n.$$

Therefore from assumption (ii) and in view of condition $(\mathcal{P})$, we deduce that y satisfies the following equation

$$y = A(y)Bx + Cy.$$

Thus, $y \in (\frac{I-C}{A})^{-1}Bx$. □

We also have:

Theorem 3.13. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and let Ω be a nonempty closed convex subset of E. Let $A, C : E \to E$ and $B : \Omega \to E$ be three operators such that*

(i) *A is regular in E;*
(ii) *A, B and C are sequentially weakly continuous on Ω;*
(iii) *$A(\Omega)$, $B(\Omega)$ and $C(\Omega)$ are relatively weakly compact;.*
(iv) *$(\frac{I-C}{A})^{-1}Bx$ is convex for all $x \in \Omega$;*
(v) *$x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = AxBx + Cx$.

Proof. In view of Theorem 3.12, it is enough to prove that $(\frac{I-C}{A})^{-1}B(\Omega)$ is relatively weakly compact. To do this, let $(x_n)_n$ be a sequence in Ω and let

$$y_n \in (\frac{I-C}{A})^{-1}Bx_n.$$

Since $B(\Omega)$ is relatively weakly compact, there is a renamed subsequence $(Bx_n)_n$ weakly converging to an element $z \in E$. On the other hand, we have

$$y_n = A(y_n)Bx_n + C(y_n). \tag{3.2}$$

Since $(y_n)_n$ is a sequence in Ω, by assumption (iii) there is a renamed subsequence such that $A(y_n) \rightharpoonup x \in E$ and $C(x_n) \rightharpoonup y \in E$. Hence, in view of condition $(\mathcal{P})$ and the equation (3.2), we obtain

$$y_n \rightharpoonup xz + y.$$

This shows that $(\frac{I-C}{A})^{-1}B(\Omega)$ is sequentially relatively weakly compact. Now, an application of the Eberlein-Smulian theorem yields that $(\frac{I-C}{A})^{-1}B(\Omega)$ is relatively weakly compact. □

Remark 3.14. Theorem 3.13 extends [3, Corollary 3.1].

4. **Functional integral equations**

In this section we illustrate the applicability of our Theorem 3.13 to prove the existence of solutions of FIE (1.2). Let $(X, \|.\|)$ be a Banach algebra satisfying condition $\mathcal{P}$. Let $J = [0,1]$, and let $E = C(J,X)$ be the Banach algebra of all continuous functions from $[0,1]$ to X, endowed with the supremum norm $\|.\|_\infty$, defined by

$$\|x\|_\infty = \sup\{\|x(t)\|, t \in [0,1]\},$$

for each $x \in C(J,X)$. Now to discuss the existence of solutions of FIE (1.2), we list the following assumptions:

(I) $a : J \to X$ is a continuous function.

(II) $\sigma : J \to J$ is a continuous and nondecreasing function.

(III) $q : J \to \mathbb{R}$ is a continuous function.

(IV) The operator $T : C(J,X) \to C(J,X)$ is such that
- (a) T is regular on $C(J,X)$,
- (b) T is weakly sequentially continuous and weakly compact on $C(J,X)$,
- (c) T is affine on $C(J,X)$(i.e. for all $x_1, x_2 \in C(J,X)$ and $t \in [0,1]$, $T(tx_1 + (1-t)x_2) = tT(x_1) + (1-t)T(x_2)$).

(V) The function $p : J \times J \times X \times X \to \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \mapsto p(t,s,x,y)$ is continuous uniformly for $(s,x,y) \in J \times X \times X$.

(VI) There exists $r_0 > 0$ such that
- (a) $|\ p(t,s,x,y)\ | \leq r_0 - \|q\|_\infty$ for each $t, s \in J$; $x, y \in X$ such that $\|x\|_\infty \leq r_0$ and $\|y\|_\infty \leq r_0$,
- (b) $\|Tx\|_\infty \leq (1 - \frac{\|a\|_\infty}{r_0})\frac{1}{\|u\|}$ for each $x \in C(J,X)$.

Theorem 4.1. *Under assumptions (I)–(VI), equation (1.2) has at least one solution $x = x(t)$ which belongs to the space $C(J,X)$.*

Proof. We shall use some ideas from [3]. Note that $C(J,X)$ verifies the condition $(\mathcal{P})$. Let us define the subset Ω of $C(J,X)$ by

$$\Omega = \{y \in C(J,X)\ : \|y\|_\infty \leq r_0\} = B_{r_0}.$$

Obviously Ω is a nonempty closed convex and bounded subset of $E = C(J,X)$. Let us consider three operators A, B and C defined on Ω by

$$(Ax)(t) = (Tx)(t),$$

$$(Bx)(t) = q(t) + \int_0^{\sigma(t)} p(t,s,x(x),x(\lambda s))ds, \quad 0 < \lambda < 1,$$

$$(Cx)(t) = a(t).$$

We shall prove that the operators A, B and C satisfy all the conditions of Theorem 3.13.

(i) From assumption (IVa), it follows that the operator A is regular on $C(J,X)$ and by assumption (IVb), A is weakly sequentially continuous on Ω.

(ii) Since C is constant, it is weakly sequentially continuous on Ω. Now, by [1, Proposition 3.1] the multivalued operator $(\frac{I-C}{A})^{-1}$ exists in $B(\Omega)$.

Next, we show that the operator B is weakly sequentially continuous on Ω. Firstly, we verify that if $x \in \Omega$, then $Bx \in C(J,X)$. Let $(t_n)_n$ be any sequence in J converging to $t \in J$. Then

$$\begin{aligned}
&\|(Bx)(t_n) - (Bx)(t)\| \\
= &\left\| \left[\int_0^{\sigma(t_n)} p(t_n, s, x(s), x(\lambda s)) - \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) ds \right] .u \right\| \\
\leq &\left[\int_0^{\sigma(t_n)} |p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))| ds \right] \|u\| \\
&+ \left| \int_{\sigma(t)}^{\sigma(t_n)} |p(t, s, x(s), x(\lambda s))| ds \right| \|u\| \\
\leq &\left[\int_0^{1} |p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))| ds \right] \|u\| \\
&+ (r_0 - \|q\|_\infty) |\sigma(t_n) - \sigma(t)| \ \|u\|.
\end{aligned}$$

Since $t_n \to t$, we have $(t_n, s, x(s),\ x(\lambda s) \to (t, s, x(s), x(\lambda s)$, for all $s \in J$. Taking into account the hypothesis (V), we obtain

$$p(t_n, s, x(s), x(\lambda s)) \to p(t, s, x(s), x(\lambda s)) \text{ in } \mathbb{R}.$$

Moreover, the use of assumption (VI) leads to

$$| \, p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s)) \, | \leq 2(r_0 - \|q\|_\infty), \text{ for all } t, s \in J, \lambda \in (0,1).$$

Consider

$$\begin{cases} \phi : J \to \mathbb{R} \\ \quad s \mapsto \phi(s) = 2(r_0 - \| \|_\infty). \end{cases}$$

Clearly $\phi \in L^1(J)$. Therefore, from the dominated convergence theorem and assumption (II), we obtain

$$(Bx)(t_n) \to (Bx)(t)$$

in X. It follows that

$$Bx \in C(J, X).$$

Next, we prove B is weakly sequentially continuous on Ω. Let $(x_n)_n$ be any sequence in Ω weakly converging to a point $x \in \Omega$. So, from assumptions (V)-(VI) and the dominated convergence theorem, we get

$$\lim_{n\to+\infty} \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds = \int_0^1 p(t, s, x(s), x(\lambda s)),$$

which implies

$$\lim_{n\to+\infty} \Big(q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds\Big).u = \Big(q(t) + \int_0^1 p(t, s, x(s), x(\lambda s))ds\Big).u.$$

Hence,

$$(Bx_n)(t) \to (Bx)(t) \text{ in } X.$$

Since $(Bx_n)_n$ is bounded by $r_0\|u\|$, then

$$Bx_n \rightharpoonup Bx.$$

We conclude that B is weakly sequentially continuous on Ω.

(iii) We prove now that $A(\Omega), B(\Omega)$ and $C(\Omega)$ are relatively weakly compact. Since Ω is bounded by r_0 and taking into account the hypothesis (IV), it follows that $A(\Omega)$ is relatively weakly compact. As $C(\Omega) = \{a\}$, hence $C(\Omega)$ is relatively weakly compact. Now, we prove that $B(\Omega)$ is relatively weakly compact.

Step 1: By definition,

$$B(\Omega) = \{Bx : \ \|x\|_\infty \le r_0\}.$$

For all $t \in J$, we have

$$B(\Omega)(t) = \{Bx(t) : \ \|x\|_\infty \le r_0\}.$$

We claim that $B(\Omega)(t)$ is sequentially weakly relatively compact in X. To see this, let $(x_n)_n$ be any sequence in Ω, we have $(Bx_n)(t) = r_n(t).u$, where $r_n(t) = q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds$. Since

$$| (r_n)(t) | = | q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds |$$

$$\le | q(t) | + \int_0^1 | p(t, s, x_n(s), x_n(\lambda s)) | ds$$

$$\le \|q\|_\infty + r_0 - \|q\|_\infty = r_0,$$

and $r_n(t)$ is a real sequence, then there is a renamed subsequence such that

$$r_n(t) \to r(t) \text{ in } \mathbb{R},$$

which implies

$$(Bx_n)(t) \to (Bx)(t) \text{ in } X.$$

We conclude that $B(\Omega)(t)$ is relatively sequentially compact in X, then $B(\Omega)(t)$ is relatively weakly sequentially compact in X.

Step 2: We prove that $B(\Omega)$ is weakly equicontinuous on J. If we take $\epsilon > 0,\ x \in \Omega,\ \Gamma \in X^*, and\ t, t' \in J$ such that $t \leq t'$ with $t' - t \leq \epsilon$, then

$$\begin{aligned}
\mid \Gamma((Bx)(t) - (Bx)(t')) \mid = & \mid \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \\
& - \int_0^{\sigma(t')} p(t', s, x(s), x(\lambda s)) ds \mid \\
& \leq \int_0^{\sigma(t)} \mid p(t, s, x(s), x(\lambda s)) - p(t', s, x(s), x(\lambda s)) \mid ds \\
& + \int_{\sigma(t)}^{\sigma(t')} \mid p(t', s, x(s), x(\lambda s)) \mid ds \\
& \leq \tau(p, \epsilon) + (r_0 - \|q\|_\infty)\tau(\sigma(t), \epsilon),
\end{aligned}$$

where

$$\tau(p, \epsilon) = \sup\{\mid p(t, s, x, y) - p(t', s, x, y) \mid;\ t, t', s \in J; \mid t - t' \mid \leq \epsilon;\ x, y \in B_{r_0}\},$$

and

$$\tau(\sigma, \epsilon) = \sup\{\mid \sigma(t) - \sigma(t') \mid;\ t, t', s \in J\}.$$

Taking into account of hypothesis (V) and in view of the uniform continuity of the function σ on the set J, it follows that $\tau(p, \epsilon) \to 0$ and $\tau(\sigma, \epsilon) \to 0$ as $\epsilon \to 0$. The use of the Arzelà-Ascoli theorem [23] guarantees that $B(\Omega)$ is relatively weakly sequentially compact in X. Again an application of Eberlein-Smulian theorem [24] yields that $B(\Omega)$ is relatively weakly compact.

(iv) Now we show that for each $x \in \Omega$, $(\frac{I-C}{A})^{-1}Bx$ is a convex subset of Ω. Let $x \in \Omega$ be arbitrary and let $u, v \in (\frac{I-C}{A})^{-1}Bx$, then

$$(\tfrac{I-C}{A})u = Bx,$$

and

$$(\tfrac{I-C}{A})v = Bx$$

or equivalently,

$$u = AuBx + Cu,$$

and

$$v = AvBx + Cv.$$

Now for any $\lambda \in [0,1]$,

$$\begin{aligned}\lambda u + (1-\lambda)v &= \lambda AuBx + \lambda Cu + (1-\lambda)AvBx + (1-\lambda)Cv \\ &= [\lambda Au + (1-\lambda)Av]Bx + [\lambda Cu + (1-\lambda)Cv].\end{aligned}$$

Since, T is affine and C is constant, it follows that

$$\lambda u + (1-\lambda)v = [A(\lambda u + (1-\lambda)v)]Bx + [C(\lambda u + (1-\lambda)v)].$$

Thus

$$\lambda u + (1-\lambda)v \in \left(\frac{I-C}{A}\right)^{-1} Bx.$$

(v) Finally, we prove the assertion (v) of Theorem 3.13. To see this, let $x \in C(J,X)$ and $y \in \Omega$ such that

$$x = AxBy + Cx,$$

or equivalently, for all $t \in J$,

$$x(t) = a(t) + (Tx)(t)(By)(t).$$

However, for all $t \in J$ we have,

$$\|x(t)\| \leq \|x(t) - a(t)\| + \|a(t)\|.$$

Therefore,

$$\begin{aligned}\|x(t)\| &\leq \|Tx\|_\infty \|u\| r_0 + \|a\|_\infty \\ &\leq \left(1 - \frac{\|a\|_\infty}{r_0}\right) r_0 + \|a\|_\infty = r_0.\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|x\|_\infty \leq r_0,$$

and consequently $x \in \Omega$. Now, invoking Theorem 3.13, we infer that there exists $x \in C(J,X)$ with $x = AxBx + Cx$ i.e., x is a solution to equation (1.2) in $C(J,X)$, and the proof is complete.

$\square$

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