ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 807–816

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 807–816
Online ISSN: 1735-8515

ALMOST VALUATION RINGS

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(Communicated by Mohammad-Taghi Dibaei)

Abstract. The aim of this paper is to generalize the notion of almost valuation domains to arbitrary commutative rings. Also, we consider relations between almost valuation rings and pseudo-almost valuation rings. We prove that the class of almost valuation rings is properly contained in the class of pseudo-almost valuation rings. Among the properties of almost valuation rings, we show that a quasilocal ring R with regular maximal ideal M is a pseudo-almost valuation ring if and only if $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$. Furthermore, we show that pseudo-almost valuation rings are precisely the pullbacks of almost valuation rings.

Keywords: Strongly prime ideal, almost valuation domain, Pseudo-almost valuation ring.
MSC(2010): Primary: 13A18; Secondary: 13G05, 13F30, 13F05, 13A15.

1. **Introduction**

All rings considered in this paper are commutative with identity elements and all modules are unital. In [12], Hedstrom and Houston introduced a class of integral domains which is closely related to the class of valuation domains. An integral domain R with quotient field K is called a *pseudo-valuation domain* (PVD) when each prime ideal P of R is a *strongly prime ideal*, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is presented by [8].

In [10], the study of pseudo-valuation domains was generalized to arbitrary commutative rings (possibly with nonzero zero-divisors) in the following way: A prime ideal P of a ring R is said to be *strongly prime* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring* (PVR) if each prime ideal of R is strongly prime. A pseudo-valuation ring is necessarily quasi-local ([10, Lemma 1(b)]). Also, an integral

Article electronically published on 30 June, 2017.
Received: 16 April 2015, Accepted: 26 February 2016.
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domain is a pseudo-valuation ring if and only if it is a pseudo-valuation domain, see [1, Proposition 3.1], [2, Proposition 4.2] and [6, Proposition 3]. For additional characterizations of PVRs, see [3] and [7]. Badawi and Houston in [11] gave another generalization of PVDs. Anderson and Zaffrullah [5], introduced and studied the notion of an almost valuation domain which is another generalization of valuation domain. An integral domain R is called an *almost valuation domain* (AVD) if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in R$ or $x^{-n} \in R$. In [9], Badawi introduced a new class of integral domains closely related to AVD's, that is, the class of pseudo-almost valuation domains. A prime ideal P of an integral domain R is called a *pseudo-strongly prime ideal* if whenever $x, y \in K$ and $xyP \subseteq P$, there is an integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If each prime ideal of R is a pseudo-strongly prime ideal, then R is called a *pseudo-almost valuation domain* (PAVD). The same author showed that the class of AVD's is properly contained in the class of PAVD's and that PAVD's are precisely the pullbacks of AVD's. In [14], the generalization of the pseudo-almost valuation domains to arbitrary commutative rings (possibly with nonzero zero-divisors) is considered as follows. A prime ideal P of a ring R is said to be a *pseudo-strongly prime ideal* if for every $a, b \in R$, there is an integer $m \geq 1$ such that either $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$. A ring R is called a *pseudo-almost valuation ring* (PAVR) if each maximal ideal of R is pseudo-strongly prime. A pseudo-almost valuation ring is necessarily quasi-local. Also, an integral domain R is a pseudo-almost valuation ring if and only if R is a pseudo-almost valuation domain ([14, Proposition 2.7]).

In this paper, we generalize the notion of almost valuation domains as defined in [5] to arbitrary commutative rings. We show that if R is an almost valuation ring, then R is a pseudo-almost valuation ring. Also, we consider relations between almost valuation rings and pseudo-almost valuation rings.

This paper is organized as follows. In the second section, we introduce the class of almost valuation rings. We prove that every almost valuation ring is a quasi-local ring with linearly ordered prime ideals. Also, we show that every almost valuation ring is a pseudo-almost valuation ring. Furthermore, if R is an almost valuation ring, then the localization R at each prime ideal P is an almost valuation ring.

In the third section, we study almost valuation overrings of PAVRs. We show that if a ring R contains a regular principal pseudo-strongly prime ideal P, then R is an almost valuation ring with maximal ideal P. Also, we show that pseudo-almost valuation rings are precisely the pullbacks in the category of commutative rings (with 1) of diagrams of the form

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ H & \to & F. \end{array}$$

Here V is an almost valuation ring having maximal ideal $Rad_V(M)$ for some ideal M of V, $F = V/M$, the vertical map is the canonical surjection, H is a field contained in F, and the horizontal map is inclusion. Also, we prove that if P is a regular pseudo-strongly prime ideal of R, then $(P : P)$ is an almost valuation ring. Therefore, using [14] and the results of this paper, we have the following implications, none of which is reversible:

$$\begin{array}{ccc}
 & \text{valuation ring} & \\
\swarrow & & \searrow \\
\text{almost valuation ring} & & \text{pseudo-valuation ring} \\
\searrow & & \swarrow \\
 & \text{pseudo-almost valuation ring} & \\
 & \downarrow & \\
 & \text{quasi-local ring with linearly ordered prime ideals} &
\end{array}$$

We close the introduction by the following result on pseudo-almost valuation rings.

Proposition 1.1. *Let R be a PAVR with maximal ideal M. Then the nilradical N of R is a pseudo-strongly prime ideal.*

Proof. Let $a, b \in R$ such that $a^nR \not\subseteq b^nR$ for every $n \geq 1$. Since M is a pseudo-strongly prime ideal, there exists an $n \geq 1$ such that $b^nM \subseteq a^nM$. Thus, we get $b^nN \subseteq b^nM \subseteq a^nM$. Hence, for every $c \in N$, there is $d \in M$ such that $b^nc = a^nd$. Since N is a prime ideal of R, by [14, Corollary 3.3] we get, $a \in N$ or $d \in N$. If $a \in N$, then there is an $m \geq 1$ such that $a^m = 0$, and so $a^mR \subseteq b^mR$, which is a contradiction. Thus $d \in N$, and consequently, we have $b^nN \subseteq a^nN$. Therefore N is a pseudo-strongly prime ideal. $\square$

Corollary 1.2. *Let R be a ring with nonzero zero-divisors such that $dimR = 1$. Then R is a PAVR if and only if every prime ideal of R is pseudo-strongly prime.*

2. Almost valuation rings

We recall from [5] that an integral domain R with quotient field K is said to be an *almost valuation domain* if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that $x^n \in R$ or $x^{-n} \in R$. In this section, we generalize this concept to arbitrary commutative rings. Also, a commutative ring R is called a *valuation ring* if for every $a, b \in R$, $aR \subseteq bR$ or $bR \subseteq aR$. The concept of an almost valuation ring was introduced by F. Khoshayand in her Ph.D thesis as follows.

Definition 2.1. A commutative ring R is called an *almost valuation ring* if for every $a, b \in R$, there is an integer $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$.

The definition of almost valuation ring was also given independently by Mahdou et al., [15, Definition 1.1(1)]. It is clear that every valuation ring is an almost valuation ring.

Proposition 2.2. *Let R be an almost valuation ring. Then the prime ideals of R are linearly ordered. In particular, R is quasi-local.*

Proof. Suppose that P and Q are two distinct prime ideals of R such that $P \nsubseteq Q$, then there is a $p \in P \backslash Q$. Let $q \in Q$. Since R is an almost valuation ring, there is an $n \geq 1$ such that $p^nR \subseteq q^nR$ or $q^nR \subseteq p^nR$. If $p^nR \subseteq q^nR$, then $p \in Q$ because $q \in Q$, which is a contradiction. Hence $q^nR \subseteq p^nR$, and so $q \in P$. Therefore $Q \subseteq P$. □

Let R be an almost valuation ring and I be an ideal of R. It is easily shown that R/I is an almost valuation ring. In particular, if D is an almost valuation domain and I is a non-prime ideal of D then R/I is an almost valuation ring with nonzero zero-divisors.

Example 2.3. Let R be a ring in which every element is either a unit or nilpotent. Then R is an almost valuation ring. Also, every Artinian quasi-local ring is an almost valuation ring. In particular, for each prime integer p and for every $n \geq 1$, $\mathbf{Z}_{p^n}$ is an almost valuation ring.

Proposition 2.4. *Every almost valuation ring is a PAVR.*

Proof. Let R be an almost valuation ring and M be the maximal ideal of R. Suppose that $a, b \in R$ and $b^nR \nsubseteq a^nR$ for every $n \geq 1$. Since R is an almost valuation ring, there is an $n \geq 1$ such that $a^nR \subseteq b^nR$. Thus for every $d \in M$, there is an $r \in R$ such that $a^nd = b^nr$. If $r \notin M$, then r is a unit, and so $b^nR \subseteq a^nR$, which is a contradiction. Hence $r \in M$, and so $a^nM \subseteq b^nM$. Therefore, using [14, Proposition 3.2], M is a pseudo-strongly prime ideal of R, and consequently R is a PAVR. □

Example 2.5. Let F be a field, and $R_\infty = F[X_1, ..., X_n, ...]$ and let I denote the ideal of R_∞ generated by $\{X_i^i \mid i \in \mathbf{N}\}$. If $P = (X_1, X_2, ...)/(X_1, X_2^2, ...)$, then every element of P is nilpotent. Also, if $g \in R_\infty/I$ such that $g \notin P$, then $g = a_0 + f$ for some $a_0 \in F$ and $f \in P$. Since f is nilpotent and a_0 is a unit, g is a unit. Thus R_∞/I is a ring in which every element is either a unit or nilpotent, and so R_∞/I is an almost valuation ring by Example 2.3, and it follows from Proposition 2.4 that R_∞/I is a PAVR with unique prime ideal P. Let $x = (1 + X_1) + I$. Then $P \nsubseteq x(R_\infty/I)$ and $x(R_\infty/I) \nsubseteq P$, and so R_∞/I is not a pseudo-valuation ring. Therefore, R_∞/I is a PAVR with unique prime ideal $(X_1, X_2, ...)/(X_1, X_2^2, ...)$ that is not a pseudo valuation ring. Furthermore, R_∞/I is an almost valuation ring which is not a valuation ring.

The following example gives a PAVR that is not an almost valuation ring. This example uses the idealization construction $R(+)B$ arising from a ring R

and an R-module B as in [13, Chapter VI]. We recall that if R is an integral domain and B is an R-module, then B is said to be *divisible* if for every nonzero element $r \in R$ and $b \in B$, there exists an $f \in B$ such that $rf = b$.

Example 2.6. Let F be a finite field and $K = F(X)$ be the quotient field of $F[X]$. Set $D = F + KY^2 + Y^4K[[Y]]$. Then D is a pseudo-almost valuation domain that is not an almost valuation domain by [9, Example 3.5]. If B is a divisible R-module, then $R = D(+)B$ is a PAVR by [14, Proposition 3.15]. Since D is not AVD, R is not an almost valuation ring by [15, Theorem 2.1(1)].

Theorem 2.7. *Let R be an almost valuation ring with maximal ideal M and S be a multiplicatively closed subset of R such that $S \cap M \neq \emptyset$. Then R_S is an almost valuation ring.*

Proof. Let $x, y \in R_S$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and $s, t \in S$. Suppose that $x^nR_S \nsubseteq y^nR_S$ for every $n \geq 1$. If $(at)^nR \subseteq (bs)^nR$ for some $n \geq 1$, then there exists $r \in R$ such that $(at)^n = (bs)^nr$. It follows that $(a/s)^n = (r/1)(b/t)^n$ in R_S. Hence $x^nR_S \subseteq y^nR_S$, which is a contradiction. Since R is an AVR, R is a PAVR by Proposition 2.4, and so M is a pseudo-strongly prime ideal of R. Therefore, there is an integer $n \geq 1$ such that $(bs)^nM \subseteq (at)^nM$. Since $S \cap M \neq \emptyset$, there exists $u \in S \cap M$. Consequently, there is $c \in M$ such that $(bs)^nu = (at)^nc$. Hence $(b/t)^n = (a/s)^n(c/u)$ in R_S, and so $y^nR_S \subseteq x^nR_S$. Therefore, R_S is an almost valuation ring. □

Corollary 2.8. *Let R be an almost valuation ring and P be a prime ideal of R. Then R_P is an almost valuation ring.*

Proof. If P is the maximal ideal of R, then $R_P = R$ is an AVR. Suppose that P is a non-maximal prime ideal of R. Then we get, $M \cap (R \setminus P) \neq \emptyset$, and so R is an AVR by Theorem 2.7. □

The following result is an analog of Hedstrom and Houston [12, Proposition 2.6], and Badawi [9, Corollary 4.2].

Corollary 2.9. *Let R be a pseudo-almost valuation ring and P be a non-maximal prime ideal of R. Then R_P is an almost valuation ring.*

Proposition 2.10. *Let R be a ring and P be a pseudo-strongly prime ideal of R. Then for each prime ideal Q of R such that $Q \subset P$, R_Q is an almost valuation ring.*

Proof. Since P is pseudo-strongly prime ideal, R_P is a PAVR by [14, Proposition 3.9]. We now assume that Q is a prime ideal of R such that $Q \subset P$. Then QR_P is a non-maximal prime ideal of R_P, and so $R_Q = (R_P)_{QR_P}$ is an AVR by Corollary 2.9. □

3. **Almost valuation overrings**

Let R be a ring and S be the set of all regular elements of R. Then the ring of fractions $T = R_S$ is called *the total quotient ring* of R. As usual, we say that a ring A is an *overring* of R if $R \subseteq A \subseteq T$. For every ideal I of R, the subset $(I : I) = \{ x \in T \mid xI \subseteq I \}$ is an overring of R.

Proposition 3.1. *Let R be an almost valuation ring and A be an overring of R. Then A is an almost valuation ring.*

Proof. Let $x, y \in A$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and regular elements $s, t \in R$. Since R is an AVR, there exists an integer $n \geq 1$ such that $(at)^nR \subseteq (bs)^nR$ or $(bs)^nR \subseteq (at)^nR$. It follows that there is an $r \in R$ such that $(at)^n = (bs)^nr$ or $(bs)^n = (at)^nr$, and so $x^n = (a/s)^n = (b/t)^n(r/1) = y^n(r/1)$ or $y^n = x^n(r/1)$. Therefore, $x^nA \subseteq y^nA$ or $y^nA \subseteq x^nA$ which leads to the fact that A is an AVR. □

The following result corresponds to [10, Theorem 8] and [9, Theorem 2.15].

Theorem 3.2. *Let R be a quasi-local ring with maximal ideal M. If M is a regular ideal, then R is a PAVR if and only if $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$, where*

$$Rad_V(M) = \{ x \in V \mid x^n \in M \text{ for some } n \geq 1 \}.$$

Proof. Let R be a pseudo-almost valuation ring and $x, y \in V$. Then $x = a/b$ and $y = c/d$ for some $a, c \in R$ and regular elements $b, d \in R$. Suppose that $x^nV \nsubseteq y^nV$ for every $n \geq 1$. Since M is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $(ad)^nR \subseteq (bc)^nR$ or $(bc)^nM \subseteq (ad)^nM$. If $(ad)^nR \subseteq (bc)^nR$, then $(a/b)^nR \subseteq (c/d)^nR$, and so $x^nV \subseteq y^nV$, which is a contradiction. Hence $(bc)^nM \subseteq (ad)^nM$ for some $n \geq 1$. Let $e \in M$ be a regular element of R. Then there is an $f \in M$ such that $ec^nb^n = fa^nd^n$. Thus, we have $(c/d)^n = (f/e)(a/b)^n$. Since R is a pseudo-almost valuation ring, there is an $m \geq 1$ such that $e^mR \subseteq f^mR$ or $f^mM \subseteq e^mM$. It is clear that if $f^mM \subseteq e^mM$, then $(f/e)^m \in V$. Thus, we get $y^{mn} = (c/d)^{mn} = (f/e)^m(a/b)^{mn} \in x^{mn}V$, and so $y^{mn}V \subseteq x^{mn}V$. We now assume that $e^mR \subseteq f^mR$ for some $m \geq 1$. Then f is a regular element and $(e/f)^m \in R \subseteq V$ which leads to $x^{mn} = (a/b)^{mn} = (e/f)^m(c/d)^{mn} \in y^{mn}V$. Hence $x^{mn}V \subseteq y^{mn}V$. Therefore V is an almost valuation ring.

Now, suppose that $x \in V$ is a non-unit element of V. Then $x = a/b$ for some $a, b \in R$. Since R is a pseudo-almost valuation ring with maximal ideal M, there is an $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nM \subseteq a^nM$. If $b^nM \subseteq a^nM$ for some $n \geq 1$ and $d \in M$ is a regular element, then there exists a $c \in M$ such that $db^n = ca^n$. Since d and b are regular, a is regular as well. Hence x is a unit of V, which is a contradiction. Therefore, one easily obtains that $a^nR \subseteq b^nR$ for some $n \geq 1$. Thus $x^n = a^n/b^n \in R$ is a non-unit of R because

x is a non-unit of V. It follows that $x^n \in M$, and so $x \in Rad_V(M)$. Hence $Rad_V(M)$ is the maximal ideal of V.

Conversely, suppose that $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$ and $a, b \in R$. Thus there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Now assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is a $(c/d) \in V$ such that $a^n = (c/d)b^n$. If $(c/d) \in Rad_V(M)$, then there is an $m \geq 1$ such that $(c/d)^m \in M$. Thus, we get $a^{mn} = (c/d)^m b^{mn} \in b^{mn}M$, and so $a^{mn}R \subseteq b^{mn}R$. If $(c/d) \notin Rad_V(M)$, then c/d is a unit of V, and so $dM = cM$ which leads to $da^nM = cb^nM = db^nM$. It follows that $a^nM = b^nM$ because d is a regular element. Therefore R is a pseudo-almost valuation ring. □

It was shown in [9, Proposition 2.16] that if an integral domain R admits a nonzero principal pseudo-strongly prime ideal P, then R is an almost valuation domain with maximal ideal P. We have the following result:

Proposition 3.3. *Let R be a ring and P a regular principal ideal of R. If P is a pseudo-strongly prime ideal of R, then R is an almost valuation ring with maximal ideal P.*

Proof. Suppose that $P = (p)$ for some regular prime element $p \in R$. If P is a non-maximal ideal of R, then there is a non-unit $r \in R \backslash P$. Based on [14, Proposition 2.4], let n be the least positive integer such that $p^n = r^n d$ for some $d \in P$. It follows that $d = ps$ for some $s \in R$ because $d \in P$. If $n = 1$, then we get $p = rd$ which leads to $p = psr$. Since p is regular, r is a unit, which is a contradiction. Hence $n > 1$, and we have $p^{n-1} = r^{n-1}(rs)$, which is a contradiction to our choice of n. Thus P is a maximal ideal of R. Hence R is a PAVR, and so $(P : P)$ is an almost valuation ring by Theorem 3.2. Since $P = (p)$, we have $(P : P) = R$. □

We recall that an overring V of R is said to be a *root extension* of R if for every $x \in V$, there is an integer $n \geq 1$ such that $x^n \in R$. The following theorem corresponds to [9, Theorem 2.17].

Theorem 3.4. *Let R be a quasi-local ring with maximal ideal M. Suppose that V is an almost valuation overring of R such that M is an ideal of V and $Rad_V(M)$ is the maximal ideal of V. Then R is an almost valuation ring if and only if V is a root extension of R.*

Proof. If $V = R$, then there is nothing to prove. Hence assume that $V \neq R$ and R is an almost valuation ring. Let $x \in V \backslash R$. Then $x = a/b$ for some $a, b \in R$, where b is a regular element. If $x \in Rad_V(M)$, then $x^n \in M \subset R$ for some $n \geq 1$. Now, assume that $x \notin Rad_V(M)$. Since $Rad_V(M)$ is the maximal ideal of V, x is a unit of V, and so a is a regular element of R. Since R is an almost valuation ring, there is an $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$. If $a^nR \subseteq b^nR$, then $x^n \in R$, and consequently if $b^nR \subseteq a^nR$, then $x^{-n} \in R$.

If $x^{-n} \in M$, then $x^{-1} \in Rad_V(M)$, which is a contradiction. Hence x^{-n} is a unit of R, and so $x^n \in R$. Therefore V is a root extension of R.

Conversely, suppose that V is a root extension of R and $a, b \in R$. Since V is an almost valuation ring, there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is a $y \in V$ such that $a^n = b^ny$. Since V is a root extension of R, there is an $m \geq 1$ such that $y^m \in R$, and so $a^{mn} = b^{mn}y^m \in b^{mn}R$. Hence, we get $a^{mn}R \subseteq b^{mn}R$, and consequently R is an almost valuation ring. □

It is known ([4, Proposition 2.6]) that a PVD is a pullback of a valuation domain. Badawi shows that a pseudo-almost valuation domain is a pullback of an almost valuation domain, see [9, Theorem 2.19]. Our next result shows how to construct a PAVR as a pullback of an AVR.

Theorem 3.5. *Let V be an almost valuation ring with nonzero maximal ideal N and $F = V/M$, where M is an ideal of V such that $\sqrt{M} = N$. Let $\alpha : V \longrightarrow F$ be the canonical epimorphism and H be a subring of F. If H is a field and $R = \alpha^{-1}(H)$, then the pullback $R = V \times_F H$ is a PAVR with maximal ideal M. In particular, if H is properly contained in F and V is not a root extension of R, then R is a PAVR which is not an almost valuation ring.*

Proof. In view of the construction stated in the hypothesis, it is well known that M is an ideal of ring R such that $R/M \simeq H$, and so M is a maximal ideal of R. Let $a, b \in R$. Since V is an almost valuation ring, there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is an $x \in V$ such that $a^n = b^nx$. If $x \in N$, then $x^m \in M \subset R$ for some m, and so $a^{mn}R \subseteq b^{mn}R$. If $x \notin N$, then x is a unit of V which leads to $b^n = a^nx^{-1}$. Since M is an ideal of V, we obtain that $b^nM = a^nx^{-1}M \subseteq a^nM$. Hence M is a pseudo-strongly prime ideal of R, and it follows that R is a PAVR. The remaining is clear from Theorem 3.4. □

In the light of Theorems 3.2 and 3.5, we have the following corollary which is an analog of Anderson and Dobbs [4, Proposition 2.6], and Badawi [9, Corollary 2.21].

Corollary 3.6. *Pseudo-almost valuation rings are precisely the pullbacks in the category of commutative rings (with 1) of diagrams of the form*

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ H & \rightarrow & F, \end{array}$$

where V is an almost valuation ring having maximal ideal $Rad_V(M)$ for some ideal M of V, $F = V/M$, the vertical map is the canonical surjection, H is a field contained in F, and the horizontal map is inclusion.

The following result is an analog of Badawi [9, Theorem 4.1].

Theorem 3.7. *Let R be a PAVR with maximal ideal M. Suppose that V is an overring of R such that $1/e \in V$ for some regular element $e \in M$. Then V is an almost valuation ring.*

Proof. Let $x, y \in V$ such that $x^nV \nsubseteq y^nV$ for every $n \geq 1$. Then $x = a/b$ and $y = c/d$ for some $a, c \in R$ and regular elements $b, d \in R$. If $(ad)^nR \subseteq (bc)^nR$ for some $n \geq 1$, then it is clear that $x^nV \subseteq y^nV$, which is a contradiction. Since M is a pseudo-strongly prime ideal of R, there is an $n \geq 1$ such that $(bc)^nM \subseteq (ad)^nM$. We now assume that $e \in M$ is a regular element of R such that $1/e \in V$. Thus $e(bc)^n \in (ad)^nM \subseteq (ad)^nV$, and consequently $(bc)^n \in (1/e)(ad)^nV \subseteq (ad)^nV$. Therefore, $y^nV \subseteq x^nV$. □

The following result is an analog of Anderson ([1, Proposition 4.2 and 4.3]) and Badawi ([9, Theorem 4.3]).

Theorem 3.8. *Let P be a regular pseudo-strongly prime ideal of R. Then $(P : P)$ is an almost valuation ring.*

Proof. Set $V = (P : P)$. Let $x, y \in V$ such that $x^nV \nsubseteq y^nV$ for every $n \geq 1$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and regular elements $s, t \in R$. If $(at)^nR \subseteq (bs)^nR$ for some $n \geq 1$, then $x^nV \subseteq y^nV$, which is a contradiction. Since P is pseudo-strongly prime, there is an $n \geq 1$ such that $(bs)^nP \subseteq (at)^nP$. Since P is regular, there exists a regular element $e \in P$. Therefore, there exists an element $p \in P$ such that $(bs)^ne = (at)^np$. Thus $(b/t)^n = (a/s)^n(p/e)$. Also, there is an $m \geq 1$ such that $e^mR \subseteq p^mR$ or $p^mP \subseteq e^mP$ because P is a pseudo-strongly prime ideal of R. If $e^mR \subseteq p^mR$, then we have $e^m = p^mr$ for some $r \in R$. Thus p is regular and $(e/p)^m = r \in R \subseteq V$. Hence, $x^{mn} = (a/s)^{mn} = (b/t)^{mn}(e/p)^m \in y^{mn}V$, and we conclude that $x^{mn}V \subseteq y^{mn}V$, which is a contradiction. Therefore, $p^mP \subseteq e^mP$ for some $n \geq 1$ which leads to $(p/e)^mP \subseteq P$. It follows that $(p/e)^m \in V$, and so $y^{mn} = (b/t)^{mn} = (a/s)^{mn}(p/e)^m \in x^{mn}V$. Thus $y^{mn}V \subseteq x^{mn}V$ which shows that V is an almost valuation ring. □

Acknowledgments

We would like to thank the referee for many helpful suggestions and comments.

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