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ON QUASI P-SPACES AND THEIR APPLICATIONS IN SUBMAXIMAL AND NODEC SPACES

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Abstract. A topological space is called *submaximal* if each of its dense subsets is open and is called *nodec* if each of its nowhere dense subsets is closed. Here, we study a variety of spaces some of which have already been studied in $C(X)$. Among them are, most importantly, quasi P-spaces and pointwise quasi P-spaces. We obtain some new useful topological characterizations of quasi P-spaces and pointwise quasi P-spaces. Consequently, we obtain a close relation between these latter spaces and submaximal and nodec spaces.
Keywords: Quasi P-space, pointwise quasi P-space, submaximal space, nodec space, I-space.
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1. Preliminaries and introduction

Throughout this paper every topological space is Tychonoff. We denote by $C(X)$ (resp. $C^*(X)$) the ring of all (resp. bounded) real valued continuous functions on X. For a space X we denote by βX and υX the Stone-Cech compactification and the Hewitt realcompactification of X, respectively.

A subset A of a space X is called *C-embedded* (resp. *C^*-embedded*) if every $g \in C(A)$ (resp. $g \in C^*(A)$) is continuously extendable to X.

For each $f \in C(X)$ we use the notations $\mathrm{Z}(f)$ for $f^{-1}\{0\}$, $\mathrm{Coz}(f)$ for $X \setminus \mathrm{Z}(f)$, and $\mathrm{Z}(X)$ for $\{\mathrm{Z}(f) : f \in C(X)\}$. We call every element of $\mathrm{Z}(X)$ a *zero-set*, and its complement a *cozero-set* in X. A subset A of X is said to be *z-embedded* in X (or *z-embedded* if there is no ambiguity) if for every $F \in \mathrm{Z}(A)$, there exists a zero-set $Z \in \mathrm{Z}(X)$ such that $F = Z \cap A$. It is evident that every C^*-embedded subset is z-embedded. Also, every cozero-set and every Lindelöf subspace is z-embedded, see [24, 10.7].

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For a point $p \in \beta X$, we denote by $M^p(X)$ (or $M_p(X)$ when $p \in X$) the maximal ideal containing all $f \in C(X)$ with $p \in \mathrm{cl}_{\beta X}\mathrm{Z}(f)$, and we denote by $O^p(X)$ (or $O_p(X)$ when $p \in X$) the ideal containing all $f \in C(X)$ with $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\mathrm{Z}(f)$. We use the notations M^p and O^p instead of $M^p(X)$ and $O^p(X)$ if no ambiguity arises. For every $f \in C(X)$, the notation $\mathbf{M}_f$ (resp. $\mathbf{P}_f$) denotes the intersection of all maximal (resp. minimal prime) ideals containing f. The notions $\mathbf{P}_f$ and $\mathbf{M}_f$ and their properties were first introduced in [6, 7] and [8].

We call I a *z-ideal* (resp. *z°-ideal*) if I contains $\mathbf{M}_f$ (resp. $\mathbf{P}_f$) whenever $f \in I$.[1] The notations and terminology are those of [12].

A Tychonoff space X is called a *quasi P-space* if each prime z-ideal of $C(X)$ is either minimal or maximal. Because of the importance of of z-ideals in the study of $C(X)$, the notion quasi P-space (and other notions relevant to it) have been previously studied in the literature (see e.g., Problems 4M, 4G of [12], without being given any specific name, and [2], under the title *MZD-spaces*). In fact, the first extensive study on quasi P-spaces has appeared in [16]. In 2002, a topological characterization for quasi P-spaces was given by the third author of this paper in [18]. As a result, one can immediately conclude that: "a free union of quasi P-spaces is a quasi P-space", which is an affirmative answer to Question 9.5 of Henriksen et al. in [16]. The article [20] shows that lattices, specially algebraic frames, are strong tools for studying z-ideals. In [20, Theorem 4.5] the spaces X for which $\dim_z(X) \leq k$ are characterized, by a different method. As a related special result, in [21], Martínez obtained an answer to [16, Question 9.5].

Submaximal spaces were introduced in Bourbaki [9] as the spaces in which every subset is locally closed; i.e., every subset is open in its closure. Hewitt in [13] calls a submaximal space without isolated points an *MI-space*. In [13] it is demonstrated that in a submaximal space without isolated points every nowhere dense subset is closed and discret. After [5] got published in 1995, submaximal spaces have received plenty of attention in the general topologists' community.

The following theorem about submaximal spaces can be easily checked, see [5, Theorem 1.2]. Recall that for a subset A of X, the boundary and the set of accumulation points of A are denoted by ∂A and A', respectively.

Proposition 1.1. *For a space X the following are equivalent:*

(a) *X is a submaximal space;*
(b) *every subset A of X with empty interior is closed;*
(c) *every subset A of X with empty interior is discrete;*
(d) *∂A is discrete for every subset A of X;*
(e) *if $A \subseteq X$ and $\mathrm{int}A = \emptyset$, then $A' = \emptyset$;*

[1] Most authors use the notation *d-ideal* in place of z°-ideal.

(f) $A' = (\mathrm{int}A)'$ *for every subset* A *of* X.

A topological space X is called a *nodec space* if every nowhere dense subset of X is closed or equivalently, if every nowhere dense closed subset of X is discrete. Originally, nodec spaces appeared in van Douwen's article [10], where these spaces played a main role in the construction of countable regular maximal spaces. Because of the strong connection between nodec spaces and submaximal spaces (every submaximal space is nodec), they are addressed in many studies on submaximal spaces.

A space X is called an *I-space* if each of its points has a deleted neighborhood consisting entirely of isolated points of X. In [5], I-spaces are defined as those spaces X in which the derived set X' is closed and discrete. Obviously, the two definitions of an I-space are equivalent. Also, X is an I-space if and only if X is a scattered space of $CB(X) \leq 2$. Clearly, every I-space is submaximal.

We need the following definitions which were first introduced in [11, 16, 17, 19, 23] and [7].

Definition 1.2. (a) For a point $p \in \beta X$, we say that p is a

(i) *P-point* (with respect to X) if $M^p(X) = O^p(X)$.

(ii) *quasi P-point* (with respect to X) if prime z-ideals in $M^p(X)$ are minimal prime ideals or $M^p(X)$ itself.

(iii) *almost P-point* (with respect to X) if every $f \in M^p(X)$ is a zero divisor.

(iv) *cozero complemented point* (with respect to X) whenever prime z°-ideals contained in $M^p(X)$ are minimal prime ideals.

(v) *quasi cozero complemented point* (with respect to X) whenever every prime z°-ideal contained in $M^p(X)$ is minimal prime or maximal ideal.

(b) A Tychonoff space X is called a

(i) *P-space* (*quasi P-space*, *almost P-space*, *cozero complemented space*[2] and *quasi cozero complemented space*, respectively) if every point of βX is a P-point (quasi P-point, almost P-point, cozero complemented point and quasi cozero complemented point, respectively) with respect to X.

(ii) *pointwise quasi P-space* (*pointwise cozero complemented space*) if every point of X is a quasi P-point (cozero complemented point) with respect to X.

If there is no ambiguity, the phrase "with respect to X" may be omitted. Also, for the sake of convenience, "quasi P-", "almost P-" and "cozero complemented" are referred to as "QP-", "AP-" and "CC-", respectively.

It is well-known that if X is a QP-space, then βX need not be a QP-space (see [16]).

[2] This concept was considered in [2] and [7] under the names " $MZ^\circ D$" and " *M-space*", respectively.

The following proposition is an improved version of [4, Corollary 4.3].

Proposition 1.3. *Let A be a neighborhood of $p \in X$. Then the prime z-ideals (resp. z°-ideals) of $C(X)$ containing $O^p(X)$ are in a one-one correspondence to prime z-ideals (resp. z°-ideals) in $C(A)$ containing $O^p(A)$.*

Corollary 1.4. *Let p be a point in X and A a neighborhood of p. The point p is a P-point (AP-point, QP-point, CC-point, quasi CC-point, respectively) with respect to X if and only if p is a P-point (AP-point, QP-point, CC-point, quasi CC-point, respectively) with respect to A.*

The following is [16, Lemma 2.4].

Lemma 1.5. *Let $X = D \cup \{a\}$, where D is an infinite P-space, and let a be the unique non-P-point of X. Then,*

(a) *every prime z-ideal of $C(X)$ properly contained in $\mathbf{M}_a$ is a minimal prime ideal;*
(b) *every free maximal ideal of $C(X)$ is a minimal prime ideal.*

As a consequence, X is a QP-space.

The following lemma is well-known.

Lemma 1.6. *Let P be a prime ideal in a reduced ring R. Then, P is a minimal prime ideal if and only if $(\mathrm{Ann}(a), a) \nsubseteq P$, for every $a \in R$.*

Some of the results in this paper are close to some results in [16] and also provide generalizations. Of course, the difference between them is mainly due to the difference in the approaches; in this paper, it is mainly topological, unlike in [16] which is mainly algebraic. We try to characterize quasi P-spaces only by topological objects, such as the space βX (or X), zero-sets, cozero-sets and closure or interior operators instead of algebraic objects, such as the ring $C(X)$ and its maximal and prime ideals.

This paper is organized in the following manner. Section 1 is devoted to the introduction and preliminaries. Section 2 contains the main results of [18] and some new ones and some improvements. Some properties of QP-points and a number of necessary and sufficient conditions for a point to be a quasi P-point will be presented in Section 3. At last, Section 4 is devoted to characterizing QP-spaces and pointwise QP-spaces under natural conditions such as first countability, pseudocompactness and countable compactness.

2. Topological characterization of QP-spaces

A variety of characterizations of QP-spaces and consequently answers to some questions in [16] are derived in this section.

The following two results are counterparts of [7, Lemma 3.1] and [7, Theorem 3.2(ii)], respectively.

Lemma 2.1. *For every $f \in C(X)$,*

$$\bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2} = \sum_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2},$$

and consequently $\bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}$ is a z-ideal.

Proof. Obviously, $\bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2} \subseteq \sum_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}$. Now, Pick a $g \in \sum_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}$. It is clear that, $g = \sum_{i=1}^{n} g_i$, where $g_i \in \mathbf{M}_{f^2+h_i^2}$ and $h_i \in \mathrm{Ann}(f)$ for every $i = 1, \dots, n$. Clearly, $h = h_1^2 + h_2^2 + \cdots + h_n^2 \in \mathrm{Ann}(f)$. The inclusions $\mathbf{M}_{f^2+h_i^2} \subseteq \mathbf{M}_{f^2+h^2}$ imply that $g \in \mathbf{M}_{f^2+h^2}$. Thus,

$$\sum_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2} \subseteq \bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}.$$

Finally, since every $\mathbf{M}_{f^2+h^2}$ is a z-ideal, it follows that $\sum_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}$ and consequently $\bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}$ is a z-ideal too. □

Theorem 2.2. *The point $p \in \beta X$ is a QP-point if and only if for every $f, g \in M^p$, there exist $h \in \mathrm{Ann}(f)$ and $k \notin M^p$ such that $\mathbf{M}_{gk} \subseteq \mathbf{M}_{f^2+h^2}$.*

Proof. ($\Rightarrow$) On the contrary, suppose that there exist $f, g \in M^p$ such that $\mathbf{M}_{f^2+h^2}$ contains no $\mathbf{M}_{gk}$ for any $h \in \mathrm{Ann}(f)$ and $k \notin M^p$. Set

$$S = \{g^n k : \ k \notin M^p, \ n = 0, 1, 2, \dots\}\ , \ I = \bigcup_{h\in \mathrm{Ann}(f)} \mathbf{M}_{f^2+h^2}.$$

Clearly, S is a multiplicatively closed set and $I \cap S = \emptyset$. So, there exists a prime z-ideal $P \in \mathrm{Min}(I)$ such that $P \cap S = \emptyset$. Obviously, $P \subseteq M^p$, $g \in M^p \setminus P$ and $(f, \mathrm{Ann}(f)) \subseteq P$. Therefore, P is a prime z-ideal contained in M^p which is neither a minimal prime ideal nor a maximal ideal and this is a contradiction.

($\Leftarrow$) Let $P \subseteq M^p$ be a prime z-ideal which is not minimal prime. It is enough to show that $M^p \subseteq P$. Assume that $g \in M^p$. Lemma 1.6 implies the existence of an $f \in C(X)$ with $(f, \mathrm{Ann}(f)) \subseteq P$. By hypothesis, there exist $h \in \mathrm{Ann}(f)$ and $k \notin M^p$ such that $\mathbf{M}_{gk} \subseteq \mathbf{M}_{f^2+h^2}$. Since P is a z-ideal, clearly, it follows that $gk \in P$. Now, P is prime and $k \notin P$, so $g \in P$. □

The previous theorem gives a characterization of QP-spaces. Yet, it is far from our goal. The given characterization is not topological enough. Moreover, we are going to obtain a new definition of QP-spaces which depends only on X, not on βX. As a necessary step for our purpose, we need to know more about QP-points.

Proposition 2.3. *The following statements are equivalent for every point $p \in \beta X$:*

(a) *p is a QP-point with respect to X;*

(b) *If $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$, then there exist $A, B \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$, $p \notin \mathrm{cl}_{\beta X} B$ and $(Z_1 \setminus Z_2) \cap A \subseteq B$ (if $p \in X$, then $\mathrm{cl}_{\beta X}$ may be omitted);*

(c) *If $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$, then there exists $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$ and $p \notin \mathrm{cl}_{\beta X}((Z_1 \setminus Z_2) \cap A)$ (if $p \in X$, then $\mathrm{cl}_{\beta X}$ may be omitted).*

Proof. If we put $B = \mathrm{Z}(k)$ in Theorem 2.2, then the proof is evident. □

Theorem 2.4. *A space X is a QP-space if and only if for every $Z_1, Z_2 \in \mathrm{Z}(X)$ there exist $A, B \in Z[X]$ such that $X \setminus A \subseteq Z_1$, $Z_1 \cap Z_2 \cap B = \emptyset$ and $(Z_1 \setminus Z_2) \cap A \subseteq B$.*

Proof. ($\Rightarrow$) If $Z_1 \cap Z_2 = \emptyset$, the proof is complete by $B = Z_1$. Otherwise, for every $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$, applying Proposition 2.3, there exist $A_p, B_p \in Z[X]$ such that $X \setminus A_p \subseteq Z_1$ and $(Z_1 \setminus Z_2) \cap A_p \subseteq B_p$. Let $V_p = \beta X \setminus \mathrm{cl}_{\beta X} B_p$; then we have

$$\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) \subseteq \bigcup_{p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)} V_p.$$

By compactness, there is a finite collection $\{V_{p_i} : \ i = 1, 2, \ldots, n\}$ such that $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) \subseteq \bigcup_{i=1}^n V_{p_i} = \beta X \setminus \bigcap_{i=1}^n \mathrm{cl}_{\beta X} B_{p_i}$. Now, if we put $B = \bigcap_{i=1}^n B_{p_i}$ and $A = \bigcap_{i=1}^n A_{p_i}$, then $X \setminus A \subseteq Z_1$, $Z_1 \cap Z_2 \cap B = \emptyset$ and $(Z_1 \setminus Z_2) \cap A \subseteq B$.
($\Leftarrow$) This is an immediate conclusion of Proposition 2.3(b). □

The following corollary gives a positive answer to [16, Question 9.5], which is also answered in [21] in a differenty way.

Corollary 2.5. *Every free union of QP-spaces is a QP-space.*

Proof. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of QP-spaces and $X = \dot{\bigcup} X_\alpha$ denote the free union of X_α's. Suppose that $Z_1, Z_2 \in \mathrm{Z}(X)$. Let $Z_{1\alpha} = Z_1 \cap X_\alpha$, $Z_{2\alpha} = Z_2 \cap X_\alpha$ for all $\alpha \in I$. By Theorem 2.4, for every $\alpha \in I$, there exist $A_\alpha, B_\alpha \in \mathrm{Z}(X_\alpha)$ such that $X_\alpha \setminus A_\alpha \subseteq Z_{1\alpha}$, $Z_{1\alpha} \cap Z_{2\alpha} \cap B_\alpha = \emptyset$ and $(Z_{1\alpha} \setminus Z_{2\alpha}) \cap A_\alpha \subseteq B_\alpha$. If we put $A = \bigcup_{\alpha \in I} A_\alpha$ and $B = \bigcup_{\alpha \in I} B_\alpha$, clearly, we have $X \setminus A \subseteq Z_1$, $Z_1 \cap Z_2 \cap B = \emptyset$ and $(Z_1 \setminus Z_2) \cap A \subseteq B$. Now, by Theorem 2.4, X is a quasi P-space. □

Corollary 2.6. *For any space X, the following statements are equivalent:*

(a) *X is a QP-space.*

(b) *For every $Z_1, Z_2 \in \mathrm{Z}(X)$ there exists $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$ and $(Z_1 \setminus Z_2) \cap A \in \mathrm{Z}(X)$.*

Proof. (a)$\Rightarrow$(b) For given $Z_1, Z_2 \in \mathrm{Z}(X)$, by Theorem 2.4, there exist $A, B \in Z(X)$ such that

$$X \setminus A \subseteq Z_1 \ , \ Z_1 \cap B \subseteq X \setminus Z_2 \text{ and } (Z_1 \setminus Z_2) \cap A \subseteq B.$$

Therefore, $(Z_1 \setminus Z_2) \cap A = Z_1 \cap (X \setminus Z_2) \cap A \cap B = Z_1 \cap A \cap B \in \mathrm{Z}(X)$.
(b)$\Rightarrow$(a) This is obvious. □

Corollary 2.7. *For a z-embedded (C^*-embedded, cozero or Lindelöf) subspace Y of X,*

(a) *if $p \in Y$ is a QP-point in X, then it is a QP-point in Y;*
(b) *Y is a pointwise QP-space if X is so;*
(c) *Y is a QP-space if X is so.*

Proof. (a) Let $p \in Z_1' \cap Z_2'$ for $Z_1', Z_2' \in \mathrm{Z}(Y)$. By hypothesis, there exist $Z_1, Z_2 \in \mathrm{Z}(X)$ such that $Z_1' = Z_1 \cap Y$ and $Z_2' = Z_2 \cap Y$. By Proposition 2.3, there exist $A, B \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$, $p \notin B$ and $(Z_1 \setminus Z_2) \cap A \subseteq B$. Obviously, if we take $A' = A \cap Y$ and $B' = B \cap Y$, then we have $Y \setminus A' \subseteq Z_1'$, $p \notin B'$ and $(Z_1' \setminus Z_2') \cap A' \subseteq B'$. So, by Proposition 2.3, p is a QP-point in Y.

(b) This part is clear by (a).

(c) Let $Z_1', Z_2' \in \mathrm{Z}(Y)$. By hypothesis, there exist $Z_1, Z_2 \in \mathrm{Z}(X)$ such that $Z_1' = Z_1 \cap Y$ and $Z_2' = Z_2 \cap Y$. By Corollary 2.6, there exists an $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$, and $(Z_1 \setminus Z_2) \cap A \in \mathrm{Z}(X)$. If we take $A' = A \cap Y$, then $Y \setminus A' \subseteq Z_1'$ and $(Z_1' \setminus Z_2') \cap A' \in \mathrm{Z}(Y)$. Therefore, by Corollary 2.6, Y is a QP-space. □

3. QP-points and pointwise QP-spaces

A pointwise QP-space is the localization of a QP-space. In this section, we study some properties of QP-points and pointwise QP-spaces.

Lemma 3.1. *Let $D \subseteq X$ be a countable discrete set consisting of non P-points. Then there exists an $f \in C(X)$ such that $D \subseteq \partial \mathrm{Z}(f)$.*

Proof. Set $D = \{x_1, x_2, \ldots\}$. Since D is discrete and each x_i is not a P-point, we can find a $Z_i \in \mathrm{Z}(X)$ such that $D \subseteq Z_i$ and $x_i \notin \mathrm{int} Z_i$. Put $Z = \bigcap_{i=1}^{\infty} Z_i$, obviously, $D \subseteq Z$ and $x_i \notin \mathrm{int} Z$, for every $i \in \mathbb{N}$. Therefore, $D \subseteq \partial Z$. □

Definition 3.2. If $p \in \beta X$ has a deleted neighborhood (in βX) consisting entirely of P-points, then we say p is a *locally essential P-point* or briefly an *LEP-point* with respect to X. The space X is said to be a (*pointwise*) *locally essential P-space* or briefly (a *pointwise*) an *LEP-space* if every ($p \in X$) $p \in \beta X$ is a locally essential P-point with respect to X. If there is no ambiguity, we may omit the phrase "with respect to X".

We call $V \subseteq X$ an *X-neighborhood* of $p \in \beta X$ if there exists an open neighborhood $W \subseteq \beta X$ of p such that $W \cap X \subseteq V$. One can easily see that $p \in \beta X$ is an LEP-point if and only if p has a deleted X-neighborhood entirely of P-points. We recall that a space which contains at most one non-P-point is called an *essential P-space* (see [1]). Obviously, every essential P-space is an LEP-space and every LEP-space is a pointwise LEP-space.

Theorem 3.3. (a) *If $p \in \upsilon X$ is a limit point of a countable discrete set D of non-P-points of X, then p is not a QP-point.*

(b) *Let V be an X-neighborhood of $p \in \beta X \setminus X$ consisting entirely of P-points in X. Then p is a P-point, with respect to X (i.e., every LEP-point $p \in \beta X \setminus X$ is a P-point).*

Proof. (a) By Lemma 3.1, there exists a $Z_1 \in \mathrm{Z}(X)$ such that $D \subseteq \partial Z_1$. On the other hand, if $D = \{x_1, x_2, \ldots\}$, then for every $x_n \in D$, we can find a $Z_n \in \mathrm{Z}(X)$ such that $p \in \mathrm{cl}_{\beta X} Z_n$ and $x_n \in X \setminus Z_n$. If we put $Z_2 = \bigcap_{n\in\mathbb{N}} Z_n$, then $D \subseteq X \setminus Z_2$ and since $p \in \upsilon X$, evidently, we get

$$p \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\upsilon X} Z_n = \mathrm{cl}_{\upsilon X}(\bigcap_{n\in\mathbb{N}} Z_n) = \mathrm{cl}_{\upsilon X} Z_2.$$

Now, suppose that $A \in \mathrm{Z}(X)$ and $X \setminus A \subseteq Z_1$. Obviously $D \subseteq Z_1 \cap A$. Thus, $D \subseteq (Z_1 \setminus Z_2) \cap A$ and consequently $p \in \mathrm{cl}_{\beta X}((Z_1 \setminus Z_2) \cap A)$. Hence, by Proposition 2.3, p is not a QP-point.

(b) Suppose that $Z_1 \in \mathrm{Z}(X)$ and $p \in \mathrm{cl}_{\beta X} Z_1$. Since V is an X-neighborhood of p, there exists a $Z_2 \in \mathrm{Z}(O^p(X))$ such that $Z_2 \subseteq V$. Thus, $Z_1 \cap Z_2$ is a zero-set in the P-space V and hence $Z_1 \cap Z_2$ is open in V and consequently is open in X. Therefore, $Z_1 \cap Z_2$ is clopen in X and hence $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$ is clopen in βX. Since $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2) \subseteq \mathrm{cl}_{\beta X} Z_1$, it follows that $Z_1 \in \mathrm{Z}(O^p(X))$. Therefore, $M^p(X) = O^p(X)$. In conclusion, p is a P-point with respect to X. □

It is easy to see that part (b) is not true for $p \in X$, even if p is the only non-isolated point of X. For instance, take $X = \mathbb{N}^* = \mathbb{N} \cup \{p\}$ to be the one point compactification of $\mathbb{N}$.

By the above theorem and Proposition 1.3, we immediately have the following corollary.

Corollary 3.4. *Every LEP-point is a QP-point and so every (pointwise) LEP-space is a (pointwise) QP-space.*

Example 3.5. Theorem 3.3(a), is not valid when $p \notin \upsilon X$. To see this, let $\mathbb{N}^* = \mathbb{N} \cup \{p\}$ be the one point compactification of $\mathbb{N}$ and $\{\mathbb{N}^*_m\}_{m\in\mathbb{N}}$ be a family of mutually disjoint copies of $\mathbb{N}^*$. If we take $X = \bigcup_{m\in\mathbb{N}} \mathbb{N}^*_m$ as the free union of $\mathbb{N}^*_m$'s, then $A = \{p_m\}$ is a countable discrete set of non-P-points and has a limit point $p \in \beta X \setminus \upsilon X$ (X is countable, so $\upsilon X = X$), whereas p is a QP-point by Corollary 2.5.

Example 3.6. The countable condition in part (a) of Theorem 3.3 is necessary. For example, let $X = W^* = \{\alpha : \ \alpha \text{ is an ordinal } \leq \omega_1\}$ where ω_1 is the first uncountable ordinal. Let A be the set of all limit ordinals less than ω_1, then ω_1 is the limit point of $A \setminus A'$ that is an uncountable discrete set of non-P-points, but we know that ω_1 is a P-point. This example also shows that the set of non-P-points of a space need not be closed, even if it is discrete.

Example 3.7. The converse of Corollary 3.4 does not hold. Let X be an infinite countable maximal completely regular space which is nodec as well (as it is shown in [10, Example 3.3], this space exists). Such a space is a perfectly normal submaximal space satisfying Theorem 3.15. Therefore, X is a QP-space. But, since every point of that is a G_δ point, no point of X is a P-point. Hence, X is not an LEP-space. This is also an example of a QP-space with no P-points.

Example 3.8. Theorem 3.3 motivates to ask the following: Assuming that the set of P-points is a dense subset of X, is X a QP-space? The answer is negative, because $\beta\mathbb{N}\setminus\mathbb{N}$ is a space with a dense set of P-points which is not a QP-space (see [16, Proposition 2.7]). Also, the space $W = \{\alpha :\ \alpha \text{ is an ordinal} < \omega_1\}$ is not a pointwise QP-space (since the point ω_0^2 is the limit of a countable discrete set of non-P-points), but every point in W is a G_δ-point and the set of isolated points is a dense subset of W.

Part (c) of the following proposition generalizes [16, Lemma 2.4].

Proposition 3.9. (a) *In a pointwise QP-space, every countable discrete set of non-P-points is closed.*

(b) *The set of non-P-points of X is discrete if and only if X is a pointwise LEP-space and in this case, X is a pointwise QP-space.*

(c) *If the set of non-P-points of X is finite, then X is an LEP-space and so is a QP-space; in fact, X is a finite free union of essentially P-spaces.*

Proof. This is easily checked by Theorem 3.3 and Corollary 3.4. □

Corollary 3.10. *If $x \in X$ is a QP-point and X has a countable basis at x, then x is an LEP-point. Thus, a first countable space is pointwise QP-space if and only if it is a pointwise LEP-space.*

Proof. This immediately follows from parts (a) of Theorem 3.3 and Proposition 3.9. □

Recall that a point x in a space X is said to be of *countable tightness* if for every $A \subseteq X$ and $x \in \mathrm{cl}A$ there exists a countable set $B \subseteq A$ for which $x \in \mathrm{cl}B$. A space is *countably tight* if it is of countable tightness at each of its points. Example 3.7 shows that we cannot replace the first axiom of countability in Corollary 3.10 by countable tightness.

Question 3.11. Does the converse of part (a) of Proposition 3.9 hold? In other words, does the assumption that "every countable discrete set of non-P-points of X is closed" imply that X is a pointwise QP-space?

At the end of this section, we consider some conditions which makes a pointwise QP-space into a QP-space. First, we need a definition and some assertions.

Definition 3.12. A space X is called an *SZ-space*, whenever for every $Z \in \mathrm{Z}(X)$ we have $\mathrm{cl}(X \setminus Z) \in \mathrm{Z}(X)$.

Obviously, every perfectly normal space is an SZ-space. But, the converse is not true. For example, W is an SZ-space and not a perfectly normal space.

Lemma 3.13. *A space X is an SZ-space if and only if for every $Z \in \mathrm{Z}(X)$ there exists an $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z$ and $Z \cap A = \partial Z$.*

Proof. ($\Rightarrow$) For $Z \in \mathrm{Z}(X)$, take $A = \mathrm{cl}(X \setminus Z)$. Now we get $X \setminus A \subseteq Z$ and $Z \cap A = Z \cap \mathrm{cl}(X \setminus Z) = \partial Z$.
($\Leftarrow$) Suppose that $Z \in \mathrm{Z}(X)$, then there exists an $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z$ and $Z \cap A = \partial Z$. Thus, we have $A \cap \mathrm{int} Z = \emptyset$ and so $A \subseteq \mathrm{cl}(X \setminus Z) \subseteq A$. Therefore, $\mathrm{cl}(X \setminus Z) = A \in \mathrm{Z}(X)$. □

Lemma 3.14. *If X is a pointwise QP-space and also is an SZ-space, then for every $Z_1, Z_2 \in \mathrm{Z}(X)$ there exist $A \in \mathrm{Z}(X)$ and an open set $V \subseteq X$ containing $Z_1 \cap Z_2$ such that $X \setminus A \subseteq Z_1$ and $V \cap Z_1 \cap A \subseteq Z_2$.*

Proof. Since X is a pointwise QP-space, by Proposition 2.3, for every $x \in Z_1 \cap Z_2$, there exist $A_x, B_x \in \mathrm{Z}(X)$ such that $X \setminus A_x \subseteq Z_1$, $x \notin B_x$ and $(Z_1 \setminus Z_2) \cap A_x \subseteq B_x$. This implies that for every $x \in Z_1 \cap Z_2$, there exist a cozero-set neighborhood V_x of x and an $A_x \in \mathrm{Z}(X)$ such that $X \setminus A_x \subseteq Z_1$ and

$$Z_1 \cap A_x \cap V_x \subseteq Z_2. \tag{3.1}$$

Since X is an SZ-space, by Lemma 3.13, there exists an $A \in \mathrm{Z}(X)$ such that $X \setminus A \subseteq Z_1$ and $Z_1 \cap A = \partial Z_1$. Also, for every A_x we have $\partial Z_1 \subseteq Z_1 \cap A_x$. Consequently, using (3.1), for every $x \in Z_1 \cap Z_2$, we can write $Z_1 \cap A \cap V_x = \partial Z_1 \cap V_x \subseteq Z_1 \cap A_x \cap V_x \subseteq Z_2$. The proof is complete, by taking $V = \bigcup_{x \in Z_1 \cap Z_2} V_x$. □

Theorem 3.15. *The following assertions hold.*

(a) *If X is a normal pointwise QP-space and is also an SZ-space, then X is a QP-space.*
(b) *If X is a Lindelöf pointwise QP-space, then X is a QP-space.*

Proof. (a) Since X is normal, for every $Z_1, Z_2 \in \mathrm{Z}(X)$ in Lemma 3.14, there exists a cozero-set W such that $Z_1 \cap Z_2 \subseteq W \subseteq V$. Therefore, $Z_1 \cap A \cap W \subseteq Z_2$ and consequently, by Theorem 2.4, X is a QP-space.

(b) Assume $Z_1, Z_2 \in \mathrm{Z}(X)$, analogous to part (a), for every $x \in Z_1 \cap Z_2$, there exist a cozero-set neighborhood V_x of x and an $A_x \in \mathrm{Z}(X)$ for which $X \setminus A_x \subseteq Z_1$ and (1) holds. Clearly, $Z_1 \cap Z_2 \subseteq \bigcup_{x \in Z_1 \cap Z_2} V_x$. Since X is Lindelöf and $Z_1 \cap Z_2$ is a closed subset of X, there exists a countable subset D of $Z_1 \cap Z_2$ such that $Z_1 \cap Z_2 \subseteq \bigcup_{x \in D} V_x$. It is clear that if we take $V = \bigcup_{x \in D} V_x$ and $A = \bigcap_{x \in D} A_x$, then clearly, V is a cozero-set containing $Z_1 \cap Z_2$ and $A \in \mathrm{Z}(X)$. It is obvious that $X \setminus A \subseteq Z_1$ and by (1), $Z_1 \cap Z_2 \cap V \subseteq Z_2$. Therefore, by Theorem 2.4, X is a QP-space. □

Example 3.8 shows that in the above theorem, the condition that X is a pointwise QP-space cannot be omitted, even if X has a dense subset of P-points.

Question 3.16. Is there an example of a space which is a non-normal SZ-space and a pointwise QP-space but not a QP-space?

4. Further characterizations for nodec and submaximal spaces

In this section, we will characterize QP-spaces and pointwise QP-spaces under some further conditions, such as being perfectly normal, first countable, pseudocompact, countably compact and locally compact. Also, the strong relation between QP, pointwise QP, submaximal and nodec spaces and their similarities under some conditions will be shown.

Lemma 4.1. *The following assertions hold.*

(a) *For a pseudocompact QP-space X, the set of non-isolated points is finite and every P-point is isolated.*

(b) *For a countably compact pointwise QP-space X, the set of non-isolated points is finite and every P-point is isolated.*

Proof. (a) If the set of non-P-points of X is infinite, then it contains an infinite countable discrete subset D. By Theorem 3.3, no limit point of D in $\beta X = \upsilon X$ is a QP-point, which is a contradiction. Hence, the set of non-P-points of X is finite. To complete the proof, it is enough to show that every P-point is isolated. To see this, assume that $x \in X$ is a P-point in X. Thus, x has a zero-set neighborhood Z consisting only of P-points. Since Z is clopen, so it is C-embedded in X, which implies that it is a pseudocompact P-space. Hence, Z must be finite; i.e., x is an isolated point.

(b) Let D be as in (a). Since X is a pointwise QP-space, it follows that D is closed in X, which contradicts the countable compactness. Thus, the set of non-P-points of X is finite. Hence, every P-point $x \in X$ has a zero-set neighborhood which is a countably compact P-space and consequently it is finite. Therefore, x is an isolated point. □

Example 4.2. In part (a) of the preceding lemma we cannot replace QP-space with pointwise QP-space. For instance, consider the space Ψ in [12, Problem 5.I] which is a pseudocompact pointwise QP-space (indeed, it is an I-space) but the set of its non-isolated points is infinite. This also shows that the space Ψ, as mentioned in [16], is a pointwise QP-space but not a QP-space.

The following theorem improves [16, Theorem 4.1].

Theorem 4.3. *For a space X, the following statements are equivalent:*

(a) *X is a compact QP-space;*

(b) *X is a pseudocompact QP-space;*

(c) *X is a countably compact QP-space;*
(d) *X is a countably compact pointwise QP-space;*
(e) *X is a finite topological union of a number of the one point compactification of discrete spaces.*

Proof. (e) $\Rightarrow$ (a) $\Rightarrow$ (b) and (e) $\Rightarrow$ (c) $\Rightarrow$ (d) are clearly true.

(b) $\Rightarrow$ (e) By Lemma 4.1, there exists a finite family of mutually disjoint open sets $\{V_i\}_{i=1}^n$ such that $X = \bigcup_{i=1}^n V_i$ and V_i has only one non-isolated point σ_i. It is enough to show that for every σ_i, $1 \leq i \leq n$, there is a local base of neighborhoods consisting of cofinite subsets of V_i. To do so, let G_i be an open neighborhood of σ_i. If $V_i \setminus G_i$ is infinite, then V_i has a C-embedded copy of $\mathbb{N}$ which contradicts the pseudocompactness of V_i.

(d) $\Rightarrow$ (e) This is similar to (b) $\Rightarrow$ (e). □

In the theory of $C(X)$, most properties may be expressed in terms of zero-sets. Thus, we present, inspired by Proposition 1.1, the following definition which helps us find good connections between submaximal spaces and quasi P-spaces.

Definition 4.4. (a) A point $a \in X$ is a ∂Z-D (resp. Z-lol) point if for every $Z \in \mathrm{Z}(X)$, whenever $a \in \partial Z$ (resp. $a \in Z'$), a is an isolated point of ∂Z (resp. $a \in (\mathrm{int}Z)'$);

(b) We say that X is a ∂Z-D-space (resp. Z-lol-space) if every $x \in X$ is a ∂Z-D-point (resp. Z-lol-point);

Clearly, X is a ∂Z-D-space (resp. Z-lol-space) if and only if ∂Z is discrete (resp. $(\mathrm{int}Z)' = Z'$) for every $Z \in Z(X)$.

It is easy to see that for every open subset U of X and every point $x \in U$, x is a ∂Z-D-point (resp. Z-lol-point) in X if and only if it is a ∂Z-D-point (resp. Z-lol-point) in U.

Lemma 4.5. *Consider a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of mutually disjoint non-void spaces. For every $\lambda \in \Lambda$, pick a point $x_\lambda \in X_\lambda$. Suppose that X is the quotient space of the disjoint union of X_λ's by identifying x_λ's as a point σ.*

(a) *The point σ is a ∂Z-D-point in X if and only if for every $\lambda \in \Lambda$, σ is a ∂Z-D-point in X_λ.*

(b) *If σ is a Z-lol-point in X, then it is a Z-lol-point in X_{λ_0} for some $\lambda_0 \in \Lambda$. Conversely, assume that σ is a Z-lol-point and not a G_δ-point in X_{λ_0} for some $\lambda_0 \in \Lambda$, then σ is a Z-lol-point in X.*

Proof. The proof is easy and follows from the definitions. □

Note that, in part (b) of the above lemma, if σ is a Z-lol-point in X_{λ_0} for some $\lambda_0 \in \Lambda$, then σ need not be a Z-lol-point in X. For instance, let $\mathbb{N}^* = \mathbb{N} \cup \{x_1\}$ be the one point compactification of $\mathbb{N}$ and $\Sigma = \mathbb{N} \cup \{x_2\}$ be the space which is introduced in [12, 4M]. Now, put $\Lambda = \{1, 2\}$, $X_1 = \mathbb{N}^*$ and

$X_2 = \Sigma$. Suppose that X is the space induced by X_1 and X_2 as in Lemma 4.5. One can easily see that σ is a Z-*lol*-point in X_2 while it is not a Z-*lol*-point in X.

Before proceeding further, it is worth making clear the relation between the above notions and submaximal (nodec) spaces. Obviously, every submaximal (nodec) space is a ∂Z-D-space and every ∂Z-D-space is a Z-*lol*-space. As the following examples show, the converse of these implications is not valid.

Example 4.6. Let λ be an ordinal and

$$Y_\lambda = \{\alpha < \lambda : \ \alpha \text{ is a non-limit ordinal}\} \cup \{\lambda\}.$$

Paste to every $x \in Y_\lambda \setminus \{\lambda\}$ a copy of Y_λ at the end point λ. Denote the resulted quotient space by X_λ. For more information about such a topological structure see [3]. Clearly, if we take $\lambda = \omega_1$ (i.e., the first uncountable ordinal), then X_λ is a ∂Z-D-space which is not submaximal.

Example 4.7. Let X_λ be the space introduced in Example 4.6. Suppose that X is the quotient space of the disjoint union of X_{ω_1} and X_ω by identifying $\{\omega, \omega_1\}$ to a point. By Lemma 4.5, X is a Z-*lol*-space which is not a ∂Z-D-space.

Below, we give some relations between some foregoing concepts.

Proposition 4.8. *In every basically disconnected space X, the two concepts of ∂Z-D-space and Z-lol-space are equivalent.*

Proof. Let X be a basically disconnected Z-*lol*-space and $x \in \partial Z$ where $Z \in Z[X]$. Since $\mathrm{int}Z$ is closed, we have $x \notin (\mathrm{int}Z)'$. Therefore, $x \notin Z'$ and it is an isolated point of ∂Z. The converse is true, in general. $\square$

Proposition 4.9. *Every point of a Z-lol-space (consequently, a submaximal space) X is either a G_δ-point or an AP-point.*

Proof. Suppose that $x \in X$ is not a G_δ-point and $x \in Z \in \mathrm{Z}(X)$. Hence, $x \in Z'$ and since X is a Z-*lol* space, it follows that $Z^\circ \neq \emptyset$. Therefore, x is an AP-point. $\square$

It is obvious that every QP-point is a quasi CC-point. Now, if $p \in \beta X$ is not an AP-point (i.e., $M^p(X)$ is not a z°-ideal), then the notions quasi CC-point and CC-point are equivalent. Hence, we obtain the following two results.

Lemma 4.10. *Let X be a pointwise QP-space, then every point of X is either an AP-point or a CC-point.*

Obviously, Proposition 4.9 and Lemma 4.10 are also true for submaximal spaces, nodec spaces and ∂Z-D-spaces.

Proposition 4.11. *Assume that X is a pointwise QP-space and every $x \in X$ is a G_δ-point. Then X is a pointwise CC-space.*

Proof. Since every point is a zero-set, every AP-point is isolated and so is a CC-point. Thus, by Lemma 4.10, every element of the space is a CC-point. $\square$

Theorem 4.12. *Every ∂Z-D-point $p \in X$ is a QP-point. The converse is true when p is a G_δ-point of X.*

Proof. Let p be a ∂Z-D-point of X. If $p \in \operatorname{int}\mathrm{Z}(f)$, then there exists an $h \in \operatorname{Ann}(f)$ such that $p \in \operatorname{Coz}(h) \subseteq \operatorname{int}\mathrm{Z}(f)$. Thus, $A = B = \mathrm{Z}(h)$ satisfies the part (b) of Proposition 2.3. Otherwise, $p \in \partial\mathrm{Z}(f)$ and by assumption, there exists a $k \in C(X)$ such that $\operatorname{Coz}(k) \cap \partial\mathrm{Z}(f) \subseteq \{p\}$. Define $h \in \operatorname{Ann}(f)$ by $h(x) = g(x)k(x)$ where $x \in \operatorname{int}\mathrm{Z}(f)$, and $h(x) = 0$ where $x \notin \operatorname{int}\mathrm{Z}(f)$. We can see that h is a continuous function and

$$[\mathrm{Z}(h) \cap \mathrm{Z}(f) \cap \operatorname{Coz}(g)] \cap \operatorname{Coz}(k) = [\mathrm{Z}(h) \cap \partial\mathrm{Z}(f) \cap \operatorname{Coz}(g) \cap \operatorname{Coz}(k)] \cup [\mathrm{Z}(h) \cap \operatorname{int}\mathrm{Z}(f) \cap \operatorname{Coz}(g) \cap \operatorname{Coz}(k)] = \emptyset \cup [\mathrm{Z}(gk) \cap \operatorname{Coz}(gk)] = \emptyset.$$

Therefore, $A = \mathrm{Z}(h)$ and $B = \mathrm{Z}(k)$ satisfy Proposition 2.3.

Conversely, let p be a G_δ-point and also a QP-point of X and $p \in \partial\mathrm{Z}(f)$ for some $f \in C(X)$. Then $\{p\}$ is a zero-set in X, namely $\mathrm{Z}(g)$ and hence, by Proposition 2.3, there ex ist $h \in \operatorname{Ann}(f)$ and $k \in C(X)$ such that $p \in \operatorname{Coz}(k)$ and $\mathrm{Z}(h) \cap \mathrm{Z}(f) \cap \operatorname{Coz}(g) \cap \operatorname{Coz}(k) = \emptyset$. Therefore, $\partial\mathrm{Z}(f) \cap \operatorname{Coz}(k) = \{p\}$ and p is an isolated point in $\partial\mathrm{Z}(f)$. $\square$

Corollary 4.13. *Every ∂Z-D-space X is a pointwise QP-space. The converse is true when every point of X is a G_δ-point. Therefore, if every point of X is a G_δ-point, then X is a ∂Z-D-space if and only if X is a pointwise QP-space.*

Corollary 4.14. *The following statements hold.*

(a) *Any nodec (consequently, submaximal) space is a pointwise QP-space.*
(b) *In a nodec (consequently, submaximal) space, any countable discrete subset of non-P-points is a closed set.*

So, the following question may arise:

Question 4.15. Suppose that every point of X is a G_δ-point. Are the concepts of pointwise QP-space and nodec space equivalent?

Suppose that $x \in \mathbb{R}^n$ and $A = \{y \in \mathbb{R}^n : \ x_1 \leq y_1\}$. Clearly, x is a G_δ-point, $A \in \mathrm{Z}(X)$ and $x \in \partial A$. Since x is not an isolated point in ∂A, by Theorem 4.12, it is not a QP-point. Thus, $\mathbb{R}^n$ has no QP-points, whereas $\mathbb{R}^n$ is a perfectly normal CC-space. This fact shows that although every QP-space is a quasi CC-space, the quasi CC-space is so far from even pointwise QP-space. This also shows that the condition of pointwise QP-space in Theorem 3.15 is necessary. In addition, it is good to mention that Theorem 4.12 is not satisfied by a Z-*lol*-point or even by a Z-*lol*-space. For example, consider the Z-*lol* space in Example 4.7 and let σ be the juncture point. Then σ is a Z-*lol*-point and a limit point of a countable discrete set of non-P-points, so it cannot be a QP-point.

However, the following result can be asserted:

Lemma 4.16. *Let X be a Z-lol-space (∂Z-D-space, nodec space). If I is any non-maximal z-ideal in $C(X)$ which contains O_x for some $x \in X$, then every element of I is a zero divisor.*

Proof. Suppose that for some $f \in I$, $\mathrm{int}\mathrm{Z}(f) = \emptyset$. Since X is a Z-*lol*-space, $\mathrm{Z}(f)$ is discrete. Therefore, there exists $g \in O_x$ such that $\mathrm{Z}(g) \cap \mathrm{Z}(f) = \{x\}$. Since I contains f and g, it follows that $\{x\} \in \mathrm{Z}(I)$ and hence $I = M_x$, which is a contradiction. □

Proposition 4.17. *If X is a Z-lol- and a CC-space, then X is a pointwise QP-space.*

Proof. Let P be a prime non-maximal z-ideal in M_x. By Lemma 4.16, every element of P is a zero divisor. Since X is a CC-space, by [17, Theorem 1.3] P is a minimal prime ideal. □

If X is scattered, then the concepts of I-space, submaximal and nodec space are equivalent. Also, by the foregoing discussions, every nodec space is a ∂Z-D-space and every ∂Z-D-space is a pointwise QP-space. The spaces X and Y, in the following example, respectively, show that the converse is not true, even if X is scattered.

Example 4.18.
(a) Let Y_{ω_1} be the space defined in Example 4.6 (for $\lambda = \omega_1$). Paste a copy of Y_{ω_1} at the point ω_1 to each isolated point of $\mathbb{N}^*$ and denote the resulted quotient space by X. The space X has the following properties:
   (i) X is scattered and it is not an I-space. Hence, it is neither submaximal nor nodec.
   (ii) X is a ∂Z-D-space.
(b) Paste a copy of $\mathbb{N}^*$ at the point ω_0 to each isolated point of Y_{ω_1} to obtain the space Y. The space Y has the following properties:
   (i) ω_1 is a QP-point in Y, but it is neither a G_δ-point nor an LEP-point.
   (ii) Y is scattered.
   (iii) Y is a Lindelöf pointwise QP-space and hence it is a QP-space.
   (iv) Y is not a ∂Z-D-space (consider the zero-set Y_{ω_1}).

Proposition 4.19. *Let X be a perfectly normal space, then the following statements are equivalent:*

(a) *X is a QP-space;*
(b) *X is a pointwise QP-space;*

(c) *X is a ∂Z-D-space;*
(d) *X is a Z-lol-space;*
(e) *X is a nodec space.*

Proof. The implications (a)⇒(b), (c)⇒(d) and (e)⇒(b) are clear.

(b)⇒(c) Every $\{x\}$ is a zero-set in X and so x is a G_δ-point. This then follows from Corollary 4.13.

(d)⇒(e) Let $A \subseteq X$ and $\overline{A}^{\circ} = \emptyset$. Since $\overline{A}$ is a zero-set and X is a Z-*lol*-space, we have $\emptyset = (\overline{A}^{\circ})' = (\overline{A})' = A'$. Therefore, A is a closed set.

(b) ⇒(a) Since every perfectly normal space is an SZ-space, by Theorem 3.15, the proof is clear. □

Since every countable space is perfectly normal, the following corollary is an immediate consequence of the above theorem.

Corollary 4.20. *Every countable nodec (consequently, submaximal) space X is a QP-space.*

Theorem 4.21. (a) *If X is countably compact, then the following statements are equivalent:*
(i) *X is a finite disjoint union of one point compactification of some discrete spaces;*
(ii) *X is an I-space (equivalently, X is scattered of $CB(X) \leq 2$);*
(iii) *X is submaximal;*
(iv) *X is a nodec space;*
(v) *X is a ∂Z-D-space;*
(vi) *X is a pointwise QP-space;*
(vii) *X is a QP-space.*
(b) *If X is also a CC-space, then the above statements are equivalent to:*
(viii) *X is a Z-lol-space.*
(c) *If X is a perfectly normal pseudocompact space, then all the above statements are equivalent.*

Proof. (a) The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi) and (i) ⇒ (vii) ⇒ (vi) are obtained easily by Proposition 1.1, Theorem 4.3 and Corollary 4.13. Also, the implication (vi) ⇒(i) is deduced from Theorem 4.3.

(b) The implication (v) ⇒ (viii) is obvious. Since X is a CC-space, then the converse follows from the part (a) and Proposition 4.17.

(c) We know every pseudocompact normal space is a countably compact space. Therefore, by the part (a) and Proposition 4.19 we are done. □

Corollary 4.22. *Every normal pseudocompact pointwise QP-space is a QP-space.*

Theorem 4.23. *If X is either a first countable, or a locally compact space, then the following statements are equivalent:*

(a) X *is an* I*-space;*
(b) X *is submaximal;*
(c) X *is a pointwise* QP*-space.*

Proof. The implications (a)$\Rightarrow$ (b) $\Rightarrow$ (c) are evident.

(c) $\Rightarrow$ (a) At first, suppose that X is first countable. By Corollary 3.10, X is a pointwise LEP-space. Note that every G_δ-P-point is an isolated point, so every $x \in X$ has a deleted neighborhood entirely of isolated points. In the case of locally compact, the proof is easy using the fact that every compact subspace of X is C^*-embedded and applying Corollary 2.7 and Theorem 4.21. □

Corollary 4.24. *If* X *is a normal locally compact pointwise* QP*-space, then it is a* QP*-space.*

Proof. This is a conclusion of the above theorem and [16, Theorem 4.3]. □

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