Bulletin of the Iranian Mathematical Society Vol. 31 *No. 2 (2005), pp 53-56.*

THE MAXIMUM NUMBER OF EDGES IN A STRONGLY MULTIPLICATIVE GRAPH

R. KAFSHGARZAFERANI

Communicated by Cheryl Praeger

ABSTRACT. We derive a formula for the maximum number of edges in a strongly multiplicative graph as a function of its order.

Recently, L.W. Beineke and S.M. Hegde [3] introduced the notion of a strongly multiplicative graph.

Definition (Beineke, Hegde [3]). A graph with n vertices is said to be *strongly multiplicative* if its vertices can be labeled $1, 2, ..., n$, so that the values on the edges, obtained as the product of labels of the end vertices, all are distinct.

An interesting problem is to obtain a formula for the maximum number of edges $\lambda(n)$ for a strongly multiplicative graph of order n. In [3], Beineke and Hegde gave an upper bound for $\lambda(n)$. In [2], C. Adiga et al. obtained a sharper upper bound for $\lambda(n)$. Then in [1], Adiga et al. established a formula for $\lambda(n)$ in terms of the divisor function. We quote their result now.

MSC(2000): Primary 05C78
Keywords: Graph labelling, Strongly multiplicative graphs
Received: 4 May 2006 , Revised: 21 July 2006

Theorem (Adiga et al. [1]).

$$\lambda(n) = \sum_{k=1}^{n(n-1)} g(k),$$

where

$$g(k) = \min\{1, f(k)\},$$

$$f(k) = \begin{cases} [\frac{d(k)}{2}] & \text{if } \ 1 \le k \le n, \\ [\frac{d(k)}{2}] - d_n(k) & \text{if } \ n < k \le n(n-1) \end{cases}$$

and where $d(k)$ denotes the number of distinct divisors of $k, [x]$ denotes the largest integer less than or equal to x, and $d_n(k)$ denotes the number of divisors of k greater than n.

In this note we derive the following formula for $\lambda(n)$.

Theorem.

$$\lambda(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n} \sum_{k=1}^{m-1} [-\frac{\theta(m,k)}{[\sqrt{mk-1}] - k + 1}],$$

where

$$\theta(m,k) = \sum_{s=k+1}^{[\sqrt{mk-1}]} [\frac{[\frac{mk}{s}]}{\frac{mk}{s}}].$$

Proof. Let $\delta(n) = \lambda(n) - \lambda(n-1)$. Then

$$\lambda(n) = \sum_{m=2}^{n} \delta(m). \qquad (2.1)$$

Thus, in view of (2.1) it is enough to obtain a formula for $\delta(m)$. Consider the array of products

1.2	1.3	1.4	...	$1.(n-1)$	$1.n$
	2.3	2.4	...	$2.(n-1)$	$2.n$
		3.4	...	$3.(n-1)$	$3.n$
				$(n-2).(n-1)$	$(n-2).n$
					$(n-1).n.$

Let A_k denote the set of all elements of the k^{th} row. We count the number of terms in the last column which appear in other rows. If $k.n$ is divisible by s ($k \leq s-1$), then there exists an $m < n$ such that $k.n = s.m$, and hence $k.n$ repeats in the s^{th} row, i.e. $k.n \in A_s$. Observe that $k.n$ may belong to A_s, where s is the largest integer such that $k+1 < s < \sqrt{kn}$. Thus the number of repetition of $k.n$ in these rows is

$$\theta(n,k) = \sum_{s=k+1}^{[\sqrt{kn-1}]} [\frac{[\frac{nk}{s}]}{\frac{nk}{s}}].$$

By the definition of $\theta(n,k)$, it is clear that

$$0 \leq \theta(n,k) \leq [\sqrt{kn-1}] - k + 1.$$

Thus

$$[-\frac{\theta(n,k)}{[\sqrt{nk-1}]-k+1}] = \begin{cases} -1 & \text{if} \quad kn \in \bigcup_{s=k+1}^{\sqrt{kn-1}} A_s, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\delta(n) = (n-1) + \sum_{k=1}^{n-1} [-\frac{\theta(n,k)}{[\sqrt{nk-1}]-k+1}]. \qquad (2.2)$$

On using (2.2) in (2.1), we complete the proof. □

Acknowledgment

I am thankful to the referee for his valuable suggestions and comments which improved the quality of the paper. I also wish to thank Dr C. Adiga for his suggestions and help during the preparation of this paper.

References

[1] C. Adiga, H.N. Ramaswamy and D.D. Somashekara, On the strongly multiplicative graphs, *South East Asian J.Math. and Math.Sci.* **2** (2003), 45-47.

[2] C. Adiga, H.N. Ramaswamy and D.D. Somashekara, A note on strongly multiplicative graphs, *Discuss. Math. Graph Theory* **24** (2004), 81-83.

[3] L.W. Beineke and S.M. Hegde, Strongly multiplicative graphs, *Discuss. Math. Graph Theory* **21** (2001), 63-76.

R. Kafshgarzaferani
Department of Studies in Mathematics
University of Mysore
MYSORE - 570 006, INDIA
e-mail:rkzaferan@yahoo.com