ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 875–883

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 875–883
Online ISSN: 1735-8515

ON REDUCIBILITY OF WEIGHTED COMPOSITION OPERATORS

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In this paper, we study two types of the reducing subspaces for the weighted composition operator $W : f \to u \cdot f \circ \varphi$ on $L^2(\Sigma)$. A necessary and sufficient condition is given for W to possess the reducing subspaces of the form $L^2(\Sigma_B)$ where $B \in \Sigma_{\sigma(u)}$. Moreover, we pose some necessary and some sufficient conditions under which the subspaces of the form $L^2(\mathcal{A})$ reduce W. All of these are basically discussed using the conditional expectation properties. To explain the results, some examples are then presented.
Keywords: Reducing subspace, weighted composition operators, conditional expectation.
MSC(2010): Primary 47B37; Secondary: 47B38.

1. **Introduction and preliminaries**

Interesting results concerning the reducibility of composition operator C_φ are found in [1]. In this paper, we attempt to give some necessary and sufficient conditions for a weighted composition operator $W \in B(L^2(\Sigma))$, to possess two types of reducing subspaces of the forms $L^2(\Sigma_A)$ and $L^2(\mathcal{A})$.
Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. Given a $B \in \Sigma$, by $\mathcal{A}_B$ we mean $\{A \cap B : A \in \mathcal{A}\}$ and B^c stands for the complement of B. Also we shall abbreviate the subspace $L^2(B, \Sigma_B, \mu_{|\Sigma_B})$ to $L^2(\Sigma_B)$ which is isometrically isomorphic to $\{f \in L^2(\Sigma) : \chi_{B^c} f = 0\}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^\circ(\Sigma)$. The subspace $L^\infty(\Sigma)$ consists of those Σ-measurable functions on X which are essentially bounded. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. The characteristic function of a set A will be denoted by χ_A and χ_X means the

Article electronically published on 30 June, 2017.
Received: 26 October 2015, Accepted: 10 April 2016.
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constant function 1. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each non-negative function $f \in L^\circ(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu,$$

where A is an $\mathcal{A}$-measurable set for which $\int_F f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ uniquely defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. The mapping $E^{\mathcal{A}}$ is a linear orthogonal projection onto $L^2(\mathcal{A})$. If $\mathcal{B} \subseteq \mathcal{A} \subseteq \Sigma$, then $E^{\mathcal{B}}_{\mathcal{A}}$ denotes the appropriate conditional expectation from $L^2(\mathcal{A})$ onto $L^2(\mathcal{B})$. We shall abbreviate the notation $E^{\mathcal{A}}_{\Sigma}$ to $E^{\mathcal{A}}$. Then $E^{\mathcal{B}}_{\mathcal{A}} E^{\mathcal{A}} = E^{\mathcal{B}}$. For more details on conditional expectation see [11].

Let $\varphi : X \to X$ be a Σ- measurable transformation of X. Denote by $\mu \circ \varphi^{-1}$ the measure on Σ given by $\mu \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A))$ for $A \in \Sigma$. We say that φ is non-singular if $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. Put $h = d\mu \circ \varphi^{-1}/d\mu$. By $\varphi^{-1}(\Sigma)$ we mean the relative completion of the σ-algebra generated by $\{\varphi^{-1}(A) : A \in \Sigma\}$. In this case, the conditional expectation $E^{\varphi^{-1}(\Sigma)}$ is understood. For a non-singular measurable transformation φ of X and a Σ-measurable weight function $u : X \to [0, \infty)$, the weighted composition operator on $L^2(\Sigma)$ is defined by $W(f) = u \cdot f \circ \varphi$. It is shown in [8] that W is bounded if and only if $J := hE^{\varphi^{-1}(\Sigma)}(u^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Even though φ is not invertible, the function $E^{\varphi^{-1}(\Sigma)}(\cdot) \circ \varphi^{-1}$ is well defined since $E^{\varphi^{-1}(\Sigma)}(\cdot) \circ \varphi^{-1} = g \circ \varphi$ for some $g \in L^\circ(\Sigma)$ which is uniquely determined on $\sigma(h)$ ([4]). For a bounded weighted composition operator W we can write $W = M_u \circ C_\varphi$, where M_u is the multiplication and C_φ is the composition operator. For more details the interested reader is referred to [1,2,11,12]. Throughout this paper, we assume that φ is non-singular, $u \geq 0$ and $J \in L^\infty(\Sigma)$.

The role of conditional expectation operator is important in this note. We shall frequently use the following general properties of $E^{\mathcal{A}}$ and W acting on $L^2(\Sigma)$. The proofs of these facts and some related discussions may be found in [1, 6–8, 11].

L(1) If f is an $\mathcal{A}$-measurable function, then $E^{\mathcal{A}}(fg) = fE^{\mathcal{A}}(g)$;
L(2) If $f \geq 0$ then $E^{\mathcal{A}}(f) \geq 0$; if $f > 0$ then $E^{\mathcal{A}}(f) > 0$;
L(3) $\sigma(f) \subseteq \sigma(E^{\mathcal{A}}(f))$, for each nonnegative $f \in L^2(\Sigma)$;
L(4) $E^{\mathcal{A}}(|f|^2) = |E^{\mathcal{A}}(f)|^2$ if and only if $f \in L^\circ(\mathcal{A})$;
L(5) $\varphi^{-1}(\sigma(h)) = X$, i.e., $h \circ \varphi > 0$;

L(6) (Change of variable) $\int_{\varphi^{-1}(A)} gf\circ\varphi d\mu = \int_A hE^{\varphi^{-1}(\Sigma)}(g)\circ\varphi^{-1} f d\mu$, for all $g \in L^2(\Sigma)$ and $A \in \Sigma$;

L(7) $W^*f = hE^{\varphi^{-1}(\Sigma)}(uf)\circ\varphi^{-1}$;

L(8) $W^*Wf = hE^{\varphi^{-1}(\Sigma)}(u^2)\circ\varphi^{-1}f$;

L(9) $WW^*f = u(h\circ\varphi)E^{\varphi^{-1}(\Sigma)}(uf)$;

L(10) $E^{\varphi^{-1}(\mathcal{A})}(L^2(\mathcal{A})) = \overline{C_\varphi(L^2(\mathcal{A}))} = \{f \in L^2(\mathcal{A}) : f \text{ is } \varphi^{-1}(\mathcal{A})\text{-measurable}\}$.

Let $\mathcal{H}$ be a real or complex Hilbert space. The set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ is denoted by $B(\mathcal{H})$. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and the range of an operator $T \in B(\mathcal{H})$. Recall that a closed subspace $M \subseteq \mathcal{H}$ is said to be invariant for an operator $T \in B(\mathcal{H})$ whenever $T(M) \subseteq M$. If M and its orthogonal complement $M^\perp$ are both invariant for T, then we say that M reduces T. The problem of classifying the reducing subspaces of T is equivalent to finding the orthogonal projections in $\{T\}'$, the commutant algebra of T. In this case, an operator T can be written with respect to the decomposition $\mathcal{H} = M \oplus M^\perp$ as a 2×2 matrix with linear transformation entries,

$$[T] = \begin{bmatrix} PTP & 0 \\ 0 & (I-P)T(I-P) \end{bmatrix},$$

where P is an orthogonal projection onto M, $PTP \in B(M)$ and $(I-P)T(I-P) \in B(M^\perp)$. So M is a reducing subspace of T if and only if $PT(I-P) = 0$ and $(I-P)TP = 0$. One may consult [10] for further information.

2. Reducibility of weighted composition operators

In order to characterize the reducibility of weighted composition operators we first need to know the behavior of the orthogonal projections onto a reducing subspace. For this we shall need the following known facts.

Lemma 2.1 ([5]). *For a closed subspace M of $\mathcal{H}$ and $T \in B(\mathcal{H})$, let P be the orthogonal projection onto M. Then the following are equivalent:*

(a) *M is a reducing subspace of T;*
(b) *$TP = PT$;*
(c) *$T^*P = PT^*$.*

In this case, P commutes with TT^* and T^*T.

Lemma 2.2 ([1, Corollary 3]). *Let $\mathcal{A}$ and $\mathcal{B}$ be two complete σ-finite subalgebras in Σ. Then the following are equivalent:*

(a) *$E^{\mathcal{A}}E^{\mathcal{B}}$ is an orthogonal projection;*
(b) *$E^{\mathcal{A}}E^{\mathcal{B}} = E^{\mathcal{B}}E^{\mathcal{A}}$;*
(c) *$E^{\mathcal{A}}E^{\mathcal{B}} = E^{\mathcal{A}\cap\mathcal{B}}$.*

Let P be the orthogonal projection onto a reducing subspace of $L^2(\Sigma)$ for W. By Lemma 2.1, L(7), L(8) and L(9) we obtain the following proposition.

Proposition 2.3. *Let W be a weighted composition operator induced by the pair (u, φ), and let P be the orthogonal projection onto a reducing subspace of $L^2(\Sigma)$ for W. Then for each $f \in L^2(\Sigma)$,*

(a) $P(uf \circ \varphi) = u(Pf) \circ \varphi$;
(b) $P(hE^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1}) = hE^{\varphi^{-1}(\Sigma)}(uPf) \circ \varphi^{-1}$;
(c) $P(Jf) = JPf$;
(d) $P(uh \circ \varphi E^{\varphi^{-1}(\Sigma)}(uf)) = uh \circ \varphi E^{\varphi^{-1}(\Sigma)}(uPf)$.

It should be mentioned that the part (c) of the following proposition was originally proved by C. Burnap and A. Lambert in [1, Theorem 5(a)].

Proposition 2.4. *Let $B \in \Sigma$ with $\mu(B) > 0$ and let $C_\varphi \in B(L^2(\Sigma))$. Then the following assertions hold:*

(a) $\varphi^{-1}(B) \subseteq B$ *if and only if* $L^2(\Sigma_B)$ *is an invariant subspace of* C_φ;
(b) $\varphi^{-1}(B) \supseteq B$ *if and only if* $L^2(\Sigma_B)$ *is an invariant subspace of* C_φ^*;
(c) $L^2(\Sigma_B)$ *reduces* C_φ *if and only if* $\varphi^{-1}(B) = B$.

Proof. Let $B \in \Sigma$ with $\mu(B) > 0$ be arbitrary. Then $L^2(\Sigma) = L^2(\Sigma_B) \oplus L^2(\Sigma_{B^c})$, where $L^2(\Sigma_B)$ is isometrically isomorphic to $\{f \in L^2(\Sigma) : f = 0 \text{ on } B^c\}$. If $\varphi^{-1}(B) \subseteq B$, then we get $\varphi^{-1}(\Sigma_B) = \varphi^{-1}(\Sigma) \cap \varphi^{-1}(B) \subseteq \Sigma \cap B = \Sigma_B$. Since $(B, \Sigma_B, \mu_{|\Sigma_B})$ is a relatively complete σ-finite measure space, using L(10), we get $C_\varphi(L^2(\Sigma_B)) \subseteq L^2(\varphi^{-1}(\Sigma_B))$, and $C_\varphi(L^2(\Sigma_B)) \subseteq L^2(\Sigma_B)$. Hence $L^2(\Sigma_B)$ is an invariant subspace of C_φ. Assuming $\varphi^{-1}(B) \supseteq B$ implies $\varphi^{-1}(\Sigma_{B^c}) \subseteq \varphi^{-1}(\Sigma) \cap B^c \subseteq \Sigma_{B^c}$ and $C_\varphi(L^2(\Sigma_{B^c})) \subseteq L^2(\varphi^{-1}(\Sigma_{B^c})) \subseteq L^2(\Sigma_{B^c})$. Consequently, if $\varphi^{-1}(B) = B$, then $L^2(\Sigma_B)$ reduces C_φ. On the other hand, if $L^2(\Sigma_B)$ and $L^2(\Sigma_{B^c})$ are both invariant under C_φ, then by the same argument we get that $\varphi^{-1}(\Sigma_B) \subseteq \Sigma_B$ and $\varphi^{-1}(\Sigma_{B^c}) \subseteq \Sigma_{B^c}$. Thus, $\varphi^{-1}(B) = B$. By these observations the desired results are established. □

In the following theorem we try to restate a similar fact for the combination of a multiplication and a composition operator.

Theorem 2.5. *Let $W \in B(L^2(\Sigma))$ and $B \in \Sigma_{\sigma(u)}$. Then $L^2(\Sigma_B)$ reduces W if and only if $\varphi^{-1}(B) = B$. In particular, if $\sigma(u) \subseteq \varphi^{-1}(\sigma(u))$, then $L^2(\Sigma_{\sigma(u)})$ is reducing for W.*

Proof. Let $\varphi^{-1}(B) = B$ and put $P = M_{\chi_B}$. Then for each $f \in L^2(\Sigma)$, we have $\chi_B W f = \chi_{\varphi^{-1}(B)}(uf \circ \varphi) = u\chi_B \circ \varphi f \circ \varphi = u(\chi_B f) \circ \varphi$. Hence $PW = WP$ and so $L^2(\Sigma_B)$ reduces W. Conversely, let $L^2(\Sigma_B)$ reduces W. Then by Proposition 2.3(a), one gets

$$u\chi_B f \circ \varphi = u\chi_{\varphi^{-1}(B)} f \circ \varphi. \tag{2.1}$$

Since Σ and $\varphi^{-1}(\Sigma)$ are σ-finite, then X can be written as $X = \cup_{i=1}^{\infty} X_i = \cup_{j=1}^{\infty} Y_j$, for mutually disjoint sets $X_i \in \Sigma$ and $Y_j \in \varphi^{-1}(\Sigma)$ with finite measures.

It is easy to see that $\{\varphi^{-1}(X_i)\cap Y_j\}_{i,j}$ is also a partition of X. Put $f=\chi_{X_i}$ in (2.1). Then $u\chi_{B\cap\varphi^{-1}(X_i)}=u\chi_{\varphi^{-1}(B)\cap\varphi^{-1}(X_i)}$, and so

$$\begin{aligned} u\chi_B &= u\chi_{\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}(\varphi^{-1}(X_i)\cap Y_j\cap B)} \\ &= u\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\chi_{\varphi^{-1}(X_i)\cap Y_j\cap B} \\ &= u\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\chi_{Y_j\cap\varphi^{-1}(X_i)\cap\varphi^{-1}(B)} \\ &= u\chi_{\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}(Y_j\cap\varphi^{-1}(X_i))\cap\varphi^{-1}(B)} \\ &= u\chi_{\varphi^{-1}(B)}. \end{aligned}$$

From $B\subseteq\sigma(u)$ we deduce that $B=\varphi^{-1}(B)$. When $\sigma(u)\subseteq\varphi^{-1}(\sigma(u))$, φ maps $\sigma(u)$ into $\sigma(u)$ and so $L^2(\Sigma)$ can be decomposed as $L^2(\Sigma)=L^2(\Sigma_{\sigma(u)})\oplus L^2(\Sigma_{\sigma(u)^c})$. Now, the desired conclusion follows from [3, Lemma 2.3]. □

Let $\mathcal{A}\subseteq\Sigma$ be a relatively complete σ-finite algebra. In the following we pose some necessary and sufficient conditions, of course not simultaneously, on which the subspace $L^2(\mathcal{A})$ reduces W.

Theorem 2.6. *If $L^2(\mathcal{A})$ reduces W, then $(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}\subseteq\mathcal{A}_{\sigma(u)}$ and u, $J\in L^\circ(\mathcal{A})$.*

Proof. The reducibility of W implies that $u\chi_{\varphi^{-1}(A)}=W(\chi_A)\in L^2(\mathcal{A})$, for all $A\in\mathcal{A}$ with finite measure. Therefore $\sigma(u\chi_{\varphi^{-1}(A)})=\sigma(u)\cap\varphi^{-1}(A)\in\mathcal{A}$, and so $\varphi^{-1}(A)\cap\sigma(u)\in\mathcal{A}_{\sigma(u)}$ for each $A\in\mathcal{A}$. Thus, $(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}\subseteq\mathcal{A}_{\sigma(u)}$. Let $\{C_n\}\subseteq\mathcal{A}$, $\mu(C_n)<\infty$ and $X=\bigcup_{n=1}^{\infty}C_n$. Thus, $X=\bigcup_{n=1}^{\infty}\varphi^{-1}(C_n)$. Hence we get that $u\chi_{\varphi^{-1}(C_n)\cap\sigma(u)}=u\chi_{\varphi^{-1}(C_n)}=W(\chi_{c_n})\in L^2(\mathcal{A})$, for each $n\in\mathbb{N}$. This implies that $u\in L^\circ(\mathcal{A})$. Finally, it just remains to show that J is $\mathcal{A}$-measurable. Since $\mathcal{R}(E^{\mathcal{A}})=L^2(\mathcal{A})$ reduces W, then by Lemma 2.1, $E^{\mathcal{A}}W^*W=W^*WE^{\mathcal{A}}$. By L(8), $W^*W=M_J$. It follows that $E^{\mathcal{A}}(Jf)=JE^{\mathcal{A}}(f)$, for each $f\in L^2(\Sigma)$. Let $\{B_n\}$ be a sequence of finite measure elements in Σ increasing to X. Then $E^{\mathcal{A}}(\chi_{B_n})\uparrow E^{\mathcal{A}}(1)=1$ and hence $E^{\mathcal{A}}(J\chi_{B_n})\uparrow J$. Since $E^{\mathcal{A}}(J\chi_{B_n})$ is $\mathcal{A}$-measurable for each $n\in\mathbb{N}$, we conclude that $J\in L^\circ(\mathcal{A})$. □

Corollary 2.7. *If $L^2(\mathcal{A})$ reduces W and $h\circ\varphi\in L^\circ(\mathcal{A})$. Then $E^{(\varphi^{-1}(\Sigma))_{\sigma(u)}}E^{\mathcal{A}_{\sigma(u)}}=E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}$ and $E^{\mathcal{A}_{\sigma(u)}}E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}=E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}$.*

Proof. By Theorem 2.6 we know that $u\in L^\circ(\mathcal{A})$. By applying these assumptions to Proposition 2.3, part (d) we obtain that

$$E^{(\varphi^{-1}(\Sigma))_{\sigma(u)}}E^{\mathcal{A}_{\sigma(u)}}=E^{\mathcal{A}_{\sigma(u)}}E^{(\varphi^{-1}(\Sigma))_{\sigma(u)}}.$$

Now by Lemma 2.2,

$$E^{(\varphi^{-1}(\Sigma))_{\sigma(u)}}E^{\mathcal{A}_{\sigma(u)}} = E^{\mathcal{A}_{\sigma(u)}\cap(\varphi^{-1}(\Sigma))_{\sigma(u)}}.$$

So we only have to show that $E^{\mathcal{A}_{\sigma(u)}\cap(\varphi^{-1}(\Sigma))_{\sigma(u)}} = E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}$. Again by Theorem 2.6 we have

$$\mathcal{A}_{\sigma(u)} \cap (\varphi^{-1}(\Sigma))_{\sigma(u)} \supseteq (\varphi^{-1}(\mathcal{A}))_{\sigma(u)} \cap (\varphi^{-1}(\Sigma))_{\sigma(u)} = (\varphi^{-1}(\mathcal{A}))_{\sigma(u)}.$$

Consequently

$$E^{\mathcal{A}_{\sigma(u)}\cap(\varphi^{-1}(\Sigma))_{\sigma(u)}} \geq E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}.$$

On other hand, let $f \in L^2((\varphi^{-1}(\Sigma))_{\sigma(u)} \cap \mathcal{A}_{\sigma(u)})$ be an arbitrary. Then $f = \chi_{\sigma(u)} g \circ \varphi$ for some $g \in L^\circ(\Sigma)$ with $g = 0$ on $\sigma(h)^c$. At this moment, we may assume that f is non-negative. Hence g is non-negative as well, thus $E^{\mathcal{A}}$ can be applied to g, since all non-negative functions are conditionable. Because $u \in L^\circ(\mathcal{A})$, $uf = ug\circ\varphi \in L^\circ(\mathcal{A})$, we have $E^{\mathcal{A}}(ug\circ\varphi) = ug\circ\varphi$. On the other hand, in light of Proposition 2.3 (a), the fact that $L^2(\mathcal{A})$ reduces W should imply that $E^{\mathcal{A}}(ug\circ\varphi) = u(E^{\mathcal{A}}(g))\circ\varphi$ (even though g might not belong to $L^2(\Sigma)$). Combining these, one gets that $uf = ug\circ\varphi = E^{\mathcal{A}}(ug\circ\varphi) = u(E^{\mathcal{A}}(g))\circ\varphi$. This yields $f = \chi_{\sigma(u)}(E^{\mathcal{A}}(g))\circ\varphi$, which means that $f \in L^\circ((\varphi^{-1}(\mathcal{A}))_{\sigma(u)})$. In this stage, one should easily pass from non-negative f's to arbitrary ones. Indeed, for a real case we have $E^{\mathcal{A}}(f) = E^{\mathcal{A}}(f^+) - E^{\mathcal{A}}(f^-)$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{0, -f\}$. If f is complex-valued, then $E^{\mathcal{A}}(f) = E^{\mathcal{A}}(Ref) + iE^{\mathcal{A}}(Imf)$, where Ref and Imf are the real and imaginary parts of f, respectively. Eventually, we conclude that

$$E^{\mathcal{A}_{\sigma(u)}\cap(\varphi^{-1}(\Sigma))_{\sigma(u)}} \leq E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}.$$

Hence

$$E^{(\varphi^{-1}(\Sigma))_{\sigma(u)}}E^{\mathcal{A}_{\sigma(u)}} = E^{\mathcal{A}_{\sigma(u)}\cap(\varphi^{-1}(\Sigma))_{\sigma(u)}} = E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}.$$

The equation $E^{\mathcal{A}_{\sigma(u)}}E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}} = E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}$ is precisely followed by the inclusion $(\varphi^{-1}(\mathcal{A}))_{\sigma(u)} \subseteq \mathcal{A}_{\sigma(u)}$ and the fact that $E^{(\varphi^{-1}(\mathcal{A}))_{\sigma(u)}}$ is the projection onto $L^2((\varphi^{-1}(\mathcal{A}))_{\sigma(u)})$. □

Theorem 2.8. *Assume that $u, J \in L^\circ(\mathcal{A})$ and $E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}}M_u = E^{\varphi^{-1}(\mathcal{A})}M_u$ on $L^\circ(\Sigma)$. Then $L^2(\mathcal{A})$ reduces W.*

Proof. Since $W(L^2(\Sigma)) \subseteq L^2(\Sigma)$ and also the set of all $\mathcal{A}$-measurable simple functions are dense in $L^2(\mathcal{A})$, it is sufficient to show that $W(\chi_A)$ and $W^*(\chi_A)$ are $\mathcal{A}$-measurable for each $A \in \mathcal{A}$ with finite measure. After taking adjoint on our hypothesis, we obtain $M_u E^{\mathcal{A}} E^{\varphi^{-1}(\Sigma)}(f) = M_u E^{\varphi^{-1}(\mathcal{A})}(f)$ for each $f \in L^\circ(\Sigma)$. Set $f = \chi_{\varphi^{-1}(A)}$. Since $E^{\varphi^{-1}(\Sigma)}(\chi_A\circ\varphi) = \chi_A\circ\varphi = \chi_{\varphi^{-1}(A)} = f$ and u is $\mathcal{A}$-measurable, then $M_u E^{\mathcal{A}} E^{\varphi^{-1}(\Sigma)}(f) = M_u E^{\mathcal{A}}(f) = E^{\mathcal{A}}(uf)$ and $M_u E^{\varphi^{-1}(\mathcal{A})}(f) = uf$. It follows that $E^{\mathcal{A}}(uf) = uf$ and so $W(\chi_A) =$

$u\chi_{\varphi^{-1}(A)} = uf \in L^\circ(\mathcal{A})$.

Now, let $E^{\varphi^{-1}(\mathcal{A})}(u\chi_A) = g\circ\varphi$ for some $g \in L^\circ(\mathcal{A})$. Since $u\chi_A = E^{\mathcal{A}}(u\chi_A)$, we obtain

$$\begin{aligned} W^*(\chi_A) &= hE^{\varphi^{-1}(\Sigma)}(u\chi_A)\circ\varphi^{-1} = hE^{\varphi^{-1}(\Sigma)}M_u(\chi_A)\circ\varphi^{-1} \\ &= hE^{\varphi^{-1}(\mathcal{A})}(u\chi_A)\circ\varphi^{-1} = h(g\circ\varphi)\circ\varphi^{-1} = hg \in L^\circ(\mathcal{A}). \end{aligned}$$

This completes the proof. □

Corollary 2.9. *Let C_φ be a bounded composition operator on $L^2(\Sigma)$. If $L^2(\mathcal{A})$ reduces C_φ, then*

(a) $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ *and* $h \in L^\infty(\mathcal{A})$;
(b) $E^{\mathcal{A}}E^{\varphi^{-1}(\mathcal{A})} = E^{\varphi^{-1}(\mathcal{A})}$;
(c) $E^{\mathcal{A}}E^{\varphi^{-1}(\Sigma)} = E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}$;
(d) $E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)} = E^{\varphi^{-1}(\mathcal{A})}$;
(e) $C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}}C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}}C_\varphi$.

Proof. (a) It suffices to put $u = 1$ in Theorem 2.6.

(b) It follows immediately from (a) and the fact that $E^{\varphi^{-1}(\mathcal{A})}$ is the projection onto space of $\varphi^{-1}(\mathcal{A})$-measurable functions.

(c) Put $u = 1$ and $P = E^{\mathcal{A}}$ in Proposition 2.3(d). Then by L(5) and Lemma 2.2 we obtain $E^{\mathcal{A}}E^{\varphi^{-1}(\Sigma)} = E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}} = E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}$.

(d) Let $f \in L^2(\Sigma)$ be an arbitrary function. Then $E^{\varphi^{-1}(\Sigma)}(f) = g\circ\varphi$, for some $g \in L^2(\Sigma)$. Since h is $\mathcal{A}$-measurable and $L^2(\Sigma)\cap L^\infty(\Sigma)$ is dense in $L^2(\Sigma)$, using [9, Proposition 3] we have

$$E^{\varphi^{-1}(\mathcal{A})}_{\varphi^{-1}(\Sigma)}(g\circ\varphi) = E^{\mathcal{A}}(g)\circ\varphi.$$

It follows that

$$\begin{aligned} E^{\varphi^{-1}(\mathcal{A})}(f) &= E^{\varphi^{-1}(\mathcal{A})}_{\varphi^{-1}(\Sigma)}E^{\varphi^{-1}(\Sigma)}(f) = E^{\varphi^{-1}(\mathcal{A})}_{\varphi^{-1}(\Sigma)}(g\circ\varphi) \\ &= E^{\mathcal{A}}(g)\circ\varphi = E^{\mathcal{A}}(g\circ\varphi) \quad \text{(by Proposition 2.3}(a)) \\ &= E^{\mathcal{A}}E^{\varphi^{-1}(\Sigma)}(f) = E^{\mathcal{A}\cap\varphi^{-1}(\Sigma)}(f). \end{aligned}$$

Note that the last equation holds by an application of (c).

(e) From the general theory of reducing subspaces (see [10]) and the fact that $E^{\mathcal{A}}$ is the orthogonal projection onto $L^2(\mathcal{A})$ which reduces C_φ, the statement is trivially deduced.

□

Corollary 2.10. *The following assertions hold.*

(a) *Let $\varphi^{-2}(\Sigma) \subseteq \Sigma$ be a complete σ-finite subalgebra, and let u, $h \in L^\circ(\varphi^{-1}(\Sigma))$. If $M_uE^{\varphi^{-1}(\Sigma)} = E^{\varphi^{-2}(\Sigma)}M_u$, then $L^2(\varphi^{-1}(\Sigma))$ reduces W.*

(b) *If* $u \in L^\circ(\mathcal{A})$ *and* $L^2(\mathcal{A})$ *reduces* C_φ*, then* $L^2(\mathcal{A})$ *reduces* W*.*
(c) *If* $\sigma(u) = X$ *and* $L^2(\mathcal{A})$ *reduces* W*, then* $L^2(\mathcal{A})$ *reduces* C_φ*.*
(d) $L^2(\mathcal{A})$ *reduces* C_φ *if and only if* $h \in L^\circ(\mathcal{A})$ *and* $E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}} = E^{\varphi^{-1}(\mathcal{A})}$*.*
(e) $L^2(\mathcal{A})$ *reduces* M_u *if and only if* $u \in L^\circ(\mathcal{A})$*.*

Proof. (a) Put $\mathcal{A} = \varphi^{-1}(\Sigma)$. Because u is $\varphi^{-1}(\Sigma)$-measurable, $E^{\varphi^{-1}(\Sigma)}M_u = M_uE^{\varphi^{-1}(\Sigma)}$. Now, the desired conclusion follows by Theorem 2.8.
(b) Let $f \in L^2(\Sigma)$. By Proposition 2.3(a), $E^{\mathcal{A}}(f)\circ\varphi = E^{\mathcal{A}}(f\circ\varphi)$. Hence $WE^{\mathcal{A}}(f) = uE^{\mathcal{A}}(f)\circ\varphi = uE^{\mathcal{A}}(f\circ\varphi) = E^{\mathcal{A}}(u\cdot f\circ\varphi) = E^{\mathcal{A}}W(f)$.
(c) $L^2(\mathcal{A})$ reduces W, then $WE^{\mathcal{A}} = E^{\mathcal{A}}W$ and $u \in L^\circ(\mathcal{A})$. Thus, $uE^{\mathcal{A}}(f)\circ\varphi = E^{\mathcal{A}}(u\cdot f\circ\varphi) = uE^{\mathcal{A}}(f\circ\varphi)$ for each $f \in L^2(\Sigma)$. Because $u > 0$, we have $E^{\mathcal{A}}(f)\circ\varphi = E^{\mathcal{A}}(f\circ\varphi)$, and so $C_\varphi E^{\mathcal{A}} = E^{\mathcal{A}}C_\varphi$.
(d) Put $u = 1$ in Theorem 2.8. Then $h \in L^\circ(\mathcal{A})$ and $E^{\varphi^{-1}(\Sigma)}E^{\mathcal{A}} = E^{\varphi^{-1}(\mathcal{A})}$. The converse follows from Theorem 2.6 and Corollary 2.9(d). This result is originally due to Burnap and Lambert [1, Theorem 5(b)].
(e) It follows from Theorem 2.6 and Theorem 2.8.

□

Example 2.11. Let $X = [-1, 1]$. Suppose that the σ-algebra Σ consists of all Lebesgue measurable subsets of X. Let μ be the Lebesgue measure on X. The transformation $\varphi : X \to X$ is given by

$$\varphi(x) = \begin{cases} x, & x \in [-1, 0) \\ 1 - x, & x \in [0, 1]. \end{cases}$$

The weight function u is defined on X by $u(x) = x$. Then by Theorem 2.5 all subspaces of $L^2(\Sigma)$ of the form $L^2(\Sigma_A)$ reduce $W : L^2(\Sigma) \to L^2(\Sigma)$, where A is an arbitrary Lebesgue measurable subset of $[-1, 0]$. Put $A = [-1, 0]$. Note that the matrix blocks form of the weighted composition operator with respect to the closed subspaces $L^2(\Sigma_A)$ and $\mathcal{N}(E^{\Sigma_A})$ is represented as follows

$$[W]\begin{bmatrix} E^{\Sigma_A}(f) \\ f - E^{\Sigma_A}(f) \end{bmatrix} = \begin{bmatrix} E^{\Sigma_A}M_uC_\varphi & E^{\Sigma_A}M_uC_\varphi \\ (1 - E^{\Sigma_A})M_uC_\varphi & (1 - E^{\Sigma_A})M_uC_\varphi \end{bmatrix}\begin{bmatrix} E^{\Sigma_A}(f) \\ f - E^{\Sigma_A}(f) \end{bmatrix}.$$

By Corollary 2.9(e), $E^{\Sigma_A}M_uC_\varphi = 0 = (1 - E^{\Sigma_A})M_uC_\varphi$. In this circumstance the matrix of weighted composition operator with respect to the decomposition $L^2(\Sigma) = L^2([-1, 0]) \oplus L^2([0, 1])$ becomes

$$[W] = \begin{bmatrix} M_{E^{\Sigma_A}(u)} & 0 \\ 0 & T \end{bmatrix},$$

where $T : L^2([0, 1]) \to L^2([0, 1])$ is defined by $Tf(x) = xf(1 - x)$.

Example 2.12. Let $X = [-\frac{1}{2}, \frac{1}{2}]$, $d\mu = dx$, Σ be the Lebesgue sets, and let $\mathcal{A} \subseteq \Sigma$ be the σ-subalgebra generated by the symmetric subsets about the origin. Let $0 < a \leq \frac{1}{2}$ and $f \in L^2(\Sigma)$. Then

$$\begin{aligned}\int_{-a}^{a} E^{\mathcal{A}}(f)(x)dx &= \int_{-a}^{a} f(x)dx \\ &= \int_{-a}^{a} \{\frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}\}dx \\ &= \int_{-a}^{a} \frac{f(x)+f(-x)}{2}dx.\end{aligned}$$

Thus, $E^{\mathcal{A}}(f)(x) = \frac{f(x)+f(-x)}{2}$. Therefore, by Corollary 2.10(e), $L^2(\mathcal{A})$ reduces M_u if and only if u is an even function.

Acknowledgement.

The authors are very grateful to the referee(s) for careful reading of the paper and for a number of helpful comments and corrections which improved the presentation of this paper.

References

[1] C. Burnap and A. Lambert, Reducibility of composition operators on L^2, *J. Math. Anal. Appl.* **178** (1993), no. 1, 87–101.

[2] J.T. Campbell, M. Embry-Wardrop, R.J. Fleming and S.K. Narayan, Normal and quasinormal weighted composition operators, *Glasg. Math. J.* **33** (1991), no. 3, 275–279.

[3] J.T. Campbell and W.E. Hornor, Localising and seminormal composition operators on L^2, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), no. 2, 301–316.

[4] J.T. Campbell and J. Jamison, On some classes of weighted composition operators, *Glasg. Math. J.* **32** (1990), no. 2, 87–94.

[5] J.B. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1990.

[6] P. Dodds, C. Huijsmans and B. de Pagter, Characterizations of conditional expectation-type operators, *Pacific J. Math.* **141** (1990), no. 1, 55–77.

[7] J. Herron, Weighted conditional expectation operators, *Oper. Matrices* **5** (2011), no. 1, 107–118.

[8] T. Hoover, A. Lambert and J. Quinn, The Markov process determined by a weighted composition operator, *Studia Math.* **72** (1982), no. 3, 225–235.

[9] A. Lambert and B.M. Weinstock, Descriptions of conditional expectations induced by non-measure preserving transformations, *Proc. Amer. Math. Soc.* **123** (1995), no. 3, 897–903.

[10] H. Radjavi and P. Rosenthal, Invariant Subspaces, Dover Publications, 2nd edition, Mineola, NY, 2003.

[11] M.M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.

[12] R.K. Singh and J.S. Manhas, Composition Operators on Function Spaces, North-Holland Mathematics Studies, North-Holland Publishing Co. Amsterdam, 1993.

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