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Author(s):

Y. Talebi, M. Hosseinpour and A.R. Moniri Hamzekolaee

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MODULES FOR WHICH EVERY NON-COSINGULAR SUBMODULE IS A SUMMAND

Y. TALEBI*, M. HOSSEINPOUR AND A.R. MONIRI HAMZEKOLAEE

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Abstract. A module M is lifting if and only if M is amply supplemented and every coclosed submodule of M is a direct summand. In this paper, we are interested in a generalization of lifting modules by removing the condition "amply supplemented" and just focus on modules such that every non-cosingular submodule of them is a summand. We call these modules *NS*. We investigate some general properties of *NS*-modules. Several examples are provided to separate different concepts. It is shown that every non-cosingular *NS*-module is a direct sum of indecomposable modules. We also discuss on finite direct sums of *NS*-modules.
Keywords: Non-cosingular submodule, amply supplemented module, *NS*-module.
MSC(2010): Primary: 16D10; Secondary: 16D80.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. A submodule N of a module M is denoted by $N \leq M$. The notation $N \leq_{\oplus} M$, means that N is a direct summand of M. Let M be a module and N a submodule of M. N is called a *small* submodule of M (denoted by $N \ll M$) if for any $X \leq M$, $M = N + X$ implies $X = M$. The module M is called *hollow* if every proper submodule is small in M. Let M be a module and $N, K \leq M$. We say that K is a *(weak) supplement* of N in M, provided $(N \cap K \ll M)$ $N \cap K \ll K$ and $M = N + K$. M is called *supplemented* (*weakly supplemented*) if every submodule of M has a supplement (weak supplement) in M. Following [7], M is called $\oplus$-*supplemented* if every submodule N of M has a supplement K that is a direct summand of M (in this case we call K an $\oplus$-supplement of N). As

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*Corresponding author.

a generalization of supplemented modules, a module M is called *amply supplemented* if $M = A+B$ for submodules $A, B \leq M$, then B contains a supplement of A in M. A module M is called *H-supplemented* if, given any submodule A of M, there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$. Equivalently, the module M is H-supplemented if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$ (see [6]).

A module M is called *small* if there exist modules $L \leq K$ such that $M \cong L \ll K$. For a module M let $\overline{Z}(M) = Rej(M, \mathsf{S}) = \bigcap\{Kerf \mid f : M \to U, U \in \mathsf{S}\} = \bigcap\{K \subseteq M \mid M/K \in \mathsf{S}\}$ where S denotes the class of all small modules. If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then M is called a *cosingular* (*non-cosingular*) module (see [11]). In [11], $\overline{Z}^{\alpha}(M)$ is defined by $\overline{Z}^0(M) = M$, $\overline{Z}^{\alpha+1}(M) = \overline{Z}(\overline{Z}^{\alpha}(M))$ and $\overline{Z}^{\alpha}(M) = \bigcap_{\beta<\alpha} \overline{Z}^{\beta}(M)$ if α is a limit ordinal. Hence there is a descending chain $M = \overline{Z}^0(M) \supseteq \overline{Z}(M) \supseteq \overline{Z}^2(M) \supseteq \ldots$ of submodules of M.

It is obvious that every small module is cosingular but in general the converse is not true (see [11, Remark 2.11(2)]). It is also clear that a module M is non-cosingular if and only if every nonzero factor module of M is non-small. Let M be a module and $K \leq N \leq M$. If $N/K \ll M/K$, then K is called a *coessential* submodule of N (denoted by $K \overset{ce}{\hookrightarrow} N$) in M and N is called *coessential extension* of K in M. A submodule N of M is called *coclosed* (denoted by $N \overset{cc}{\hookrightarrow} M$) if N has no proper coessential submodule. K is called a *coclosure* of N in M, if $K \overset{ce}{\hookrightarrow} N$ and $K \overset{cc}{\hookrightarrow} M$. Any module M is *lifting* if every submodule N of M contains a direct summand K of M such that $K \overset{ce}{\hookrightarrow} N$.

Lifting modules and their generalizations have been studied extensively (see for example [4–6, 8, 10]). A module M is lifting if and only if M is amply supplemented and every coclosed submodule of M is a direct summand. If we delete the assumption "M is amply supplemented" and restrict coclosed submodules to non-cosingular submodules, we can have a new generalization of lifting modules.

In this paper we define and study modules whose non-cosingular submodules are direct summand. We call these modules *NS*. In Section 2, we investigate general properties of *NS*-modules and their relation with other types of modules. We show that the class of *NS*-modules contains properly the class of lifting modules and H-supplemented modules (see Example 2.9). We show that a non-cosingular *NS*-module can be expressed as a direct sum of indecomposable modules (see Theorem 2.13).

In Section 3, we deal with (finite) direct sums of *NS*-modules. Let M has (D^*) and *-property. Let $M = M_1 \oplus \ldots \oplus M_n$ be a finite sum of relatively projective modules. Then M is *NS* if and only if each M_i is *NS* for $i = 1, \ldots, n$ (see Theorem 3.10).

2. *NS*-modules

Let R be a ring and M a right R-module. Then every non-cosingular submodule of M need not be a direct summand of M. For example, let K be a field and $R = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ for all i. Then R is a von Nuemann regular ring and by [13, 23.5(2)] and [11, Corollary 2.6], every R-module is non-cosingular. Let $L = \bigoplus_{i=1}^{\infty} K_i$. Then it is not hard to check that, L is not a direct summand of R while L is non-cosingular (In fact, for every nonzero submodule K of R, we have $L \cap K \neq 0$).

The above example leads us to study and investigate modules with every non-cosingular submodule is a summand (we call these modules NS). This new concept generalizes the definition of lifting modules. Obviously, every module with no nonzero non-cosingular submodules is NS (for example, a (small) cosingular module).

We first provide some examples of NS-modules. Before that we need the definition of a V-ring. Let R be a ring. Recall that R is a *V-ring (cosemisimple ring)*, if every simple R-module is injective. It is well-known that R is a V-ring (cosemisimple) if and only if for every R-module M, $Rad(M) = 0$ (see [13, 23.1]).

Example 2.1. (1) Let R be a commutative domain which is not a field. It is well-known from [3, Theorem 2] that R_R is a small module. So R_R is *NS.*

(2) Let R be a right V-ring. Then NS right R-modules are precisely semisimple right R-modules. It follows from the fact that over a right V-ring, every right R-module is non-cosingular (see [11, Corollary 2.6]).

(3) Since every non-cosingular simple submodule of a module M is a direct summand, then if every non-cosingular submodule of M is simple, M is *NS.*

Following [10], the module M is said to have *C^*-condition*, if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular.

Remark 2.2. Let R be a ring. Then every right R-module is NS if and only if every non-cosingular right R-module is injective. To prove the assertion, let every right R-module be NS and M a non-cosingular right R-module. Suppose that M is contained in a right R-module N. Since N is NS, then M is a direct summand of N. So, M is injective. For the converse, let M be an arbitrary right R-module and K a non-cosingular submodule of M. Then, by assumption K is injective and hence a direct summand of M.

The following introduces rings R for which every R-module is NS.

Example 2.3. (1) Let R be a right Harada ring. By [2, 28.10], every right R-module is a direct sum of an injective right R-module and a small right R-module. It follows that every non-cosingular right R-module is injective. Now by Remark 2.2, every right R-module is NS.

(2) Let R be a ring such that every right R-module has C^*. Then by [10, Theorem 2.9], every right R-module is a direct sum of an injective right R-module and a cosingular right R-module. It follows from Remark 2.2 that every right R-module is NS.

(3) Let R be a Dedekind domain which is not a field. By [8, Lemma 4.12], every non-cosingular R-module is injective. Hence every R-module is NS by Remark 2.2.

Example 2.4. (1) Let M be a module such that $\overline{Z}(M)$ is a semisimple direct summand of M. Then clearly, M is NS.

(2) Let R be a semilocal ring (i.e. $R/J(R)$ is semisimple) such that $Soc(_RR) = Soc(R_R)$. Let P be a projective right R-module. By [12, Corollary 2.7], $\overline{Z}(P) = Soc(P)$ is semisimple. If $\overline{Z}(P)$ is a direct summand of P, then P is NS by (1). For example, let K be a field and $R = K \times K[[x]]$. Then $J(R) = 0 \times (x)$. It follows that $R/J(R) \cong K \times (K[[x]]/(x))$ is semisimple. Hence R is a commutative semilocal ring with $\overline{Z}(R) = Soc(R) = K \times 0$. Clearly $\overline{Z}(R)$ is a direct summand of R. Therefore, R as a module is NS by (1).

Example 2.5. An NS-module need not be cosingular. Consider $\mathbb{Z}$-modules $M = \mathbb{Z}(p^\infty)$ and $T = \mathbb{Q}/\mathbb{Z}$. Then, M and N are NS by Example 2.3(3). In fact, they are non-cosingular.

Proposition 2.6. *Let M be an R-module. Then the following are equivalent:*

(1) *M is NS;*

(2) *For every non-cosingular submodule N of M, there is a decomposition $M = M_1 \oplus M_2$, such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$;*

(3) *For every non-cosingular submodule N of M, there is a direct summand K of M such that $K \overset{ce}{\hookrightarrow} N$;*

(4) *Every non-cosingular submodule N of M can be written as $N = A \oplus S$ where $A \leq_\oplus M$ and $S \ll M$.*

Proof. It is straightforward. □

Let M be a module, a submodule N of M is called *fully invariant* if for every $h \in End_R(M)$, $h(N) \subseteq N$. The module M is called *duo* module, if every submodule of M is fully invariant.

Some examples of duo modules are presented in [9]. We bring here examples of a non-duo module and a duo module.

Example 2.7. (1) The $\mathbb{Z}$-module $\mathbb{Q}$ is not a duo module. In fact, the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not fully invariant. Consider $\mathbb{Z}$-homomorphism $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = \frac{x}{2}$, for all $x \in \mathbb{Q}$. It is clear that $f(\mathbb{Z}) \nsubseteq \mathbb{Z}$.

(2) Let K be a field and let V be a two-dimensional vector space over K. Let the ring R be the trivial extension of V by K. Thus R is the K-vector space $K \oplus V$ and multiplication is defined in R as follows: $(a, u)(b, v) = (ab, av + bu)$

for all $a, b \in K$ and u, v in V. The R-module R is a duo module (see [9, P. 535]).

Proposition 2.8. *For a module M consider the following conditions:*
(1) *M is lifting;*
(2) *M is H-supplemented;*
(3) *M is $\oplus$-supplemented;*
(4) *M is C^*;*
(5) *M is NS.*
Then $(1) \Rightarrow (2) \Rightarrow (3)$, $(1) \Rightarrow (4) \Rightarrow (5)$, $(2) \Rightarrow (5)$ *and if M is a duo-module, then* $(3) \Rightarrow (5)$. *Moreover, if M is non-cosingular amply supplemented, then they are equivalent.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ It is easy by definitions.

$(1) \Rightarrow (4)$ It follows from [10, Proposition 2.3].

$(4) \Rightarrow (5)$ Let $N \leq M$ be a non-cosingular submodule. By assumption, N contains a direct summand K of M such that N/K is cosingular. Since N is non-cosingular, N/K is non-cosingular. Hence $N = K$ is a direct summand of M. So M is *NS*.

$(2) \Rightarrow (5)$ Let $X \leq M$ be non-cosingular. By assumption there exists a direct summand D of M such that $X \overset{ce}{\hookrightarrow} (X + D)$ and $D \overset{ce}{\hookrightarrow} (X + D)$. Since X is non-cosingular, then $(X + D)/D$ is non-cosingular. Hence $(X + D)/D$ is both non-cosingular and cosingular. Therefore, we get $X \leq D$ and consequently $X \overset{ce}{\hookrightarrow} D$. Set $M = D \oplus D'$. Then D/X is a direct summand of M/X, however it is a small submodule of M/X. Then we have $D = X$. This implies that M is *NS*.

$(3) \Rightarrow (5)$ Let $K \leq M$ be non-cosingular. There is $N \leq_{\oplus} M$ such that $M = N + K$ and $N \cap K \ll N$. Since M is $\oplus$-supplemented, it is weakly supplemented and $N \cap K \ll K$. Since M is a duo module, we get $N = (N \cap K) \oplus (N \cap K')$. Accordingly, we have $N = N \cap K'$ and $N \subseteq K'$. It follows that $M = N \oplus K$, and we conclude that $K \leq_{\oplus} M$ and M is *NS*.

$(5) \Rightarrow (1)$ Let X be a coclosed submodule of M. Then by [11, Lemma 2.3(3)], X is non-cosingular. So every coclosed submodule of M is a direct summand. Hence by [7, Proposition 4.8], M is lifting. □

The following example will show that *NS*-modules are proper generalizations of small modules, lifting modules and H-supplemented modules.

Example 2.9. (1) Let $M = \mathbb{Z}(p^{\infty}) \oplus \mathbb{Z}/\mathbb{Z}q$ as an $\mathbb{Z}$-module, where p and q are primes. Then M is *NS* by Example 2.3(3). Note that M is neither lifting nor small.

(2) Let M_1 be an H-supplemented module with a finite composition series $0 = X_0 \leq X_1 \leq \ldots \leq X_m = M$. Let $M_2 = X_m/X_{m-1} \oplus \ldots \oplus X_1/X_0$. By [6, Proposition 4.3], $M = M_1 \oplus M_2$ is H-supplemented. Then it is *NS*. But

M is not lifting in general. In particular, $M \oplus (U/V)$ is an *NS*-module but it is not lifting. (see [4, Corollary 2]).

(3) Consider the $\mathbb{Z}$-module $\mathbb{Z}$. Since $\mathbb{Z}$ is indecomposable, it is not H-supplemented by [6, Proiposition 2.9]. But $\mathbb{Z}$ is *NS* by Example 2.3(3).

Using [8, Remark 4.20], there exists a (an) non-cosingular (injective) $\mathbb{Z}$-module M such that M is not C^*. So *NS*-modules are the proper generalization of C^*-modules.

It is not hard to check that every non-cosingular H-supplemented module is C^*.

So using the above results we have the following implications:

$$\begin{array}{ccc} Lifting & \Longrightarrow & H-supplemented \\ \Downarrow & & \Downarrow \\ C^*-condition & \Longrightarrow & NS \end{array}$$

Remark 2.10. (1) Let M be an *NS*-module such that every submodule N of M with $\overline{Z}(N) \neq N$ is small in M. Then M is lifting.

(2) Let M be an *NS*-module such that every submodule of M has a coclosure. Then every non-cosingular submodule of M is lifting.

(3) Let M be an H-supplemented module such that every submodule of M has a coclosure. Then every non-cosingular submodule of M is lifting.

Proposition 2.11. *Let M be an NS-module such that $\overline{Z}(M)$ has a coclosure in M. Then $M = \overline{Z}^2(M) \oplus M'$ with $\overline{Z}^2(M)$ and M' are NS and $\overline{Z}(M') \ll M'$.*

Proof. Since $\overline{Z}(M)$ has a coclosure in M, using [11, Corollary 3.4], $\overline{Z}^2(M)$ is non-cosingular in M. Hence there exists a direct summand M' of M such that $M = \overline{Z}^2(M) \oplus M'$ with $\overline{Z}^2(M)$ and M' are *NS*. By [11, Corollary 3.4], $\overline{Z}^2(M)$ is unique coclosure of $\overline{Z}(M)$. So we get $\overline{Z}^2(M) \overset{ce}{\hookrightarrow} \overline{Z}(M)$. We also have $\overline{Z}(M) = \overline{Z}^2(M) \oplus \overline{Z}(M')$. This implies that $\overline{Z}(M') \ll M'$. □

Corollary 2.12. *Let M be an amply supplemented NS-module. Then $M = \overline{Z}^2(M) \oplus M'$ with $\overline{Z}^2(M)$ and M' are amply supplemented NS and $\overline{Z}(M') \ll M'$.*

Let $X = \sum_{\lambda \in \Lambda} X_\lambda$ be a direct sum of submodules X_λ $(\lambda \in \Lambda)$ of a module M. Then X is called a *local summand* of M if $\sum_{\lambda \in F} X_\lambda$ is a direct summand of M for each finite subset F of Λ. If $X = \sum_{\lambda \in \Lambda} X_\lambda$ is a summand of M, we say that *local summand is a direct summand* (see [7, Definition 2.15]).

Theorem 2.13. *Every non-cosingular NS module is a direct sum of indecomposable modules. If Moreover, M is supplemented, then M can be expressed as a direct sum of hollow modules.*

Proof. Let M be a non-cosingular *NS* module and $X = \sum X_i$ a local summand of M. Since each X_i is a direct summand of M, and $X_i = \overline{Z}(X_i) \leq \overline{Z}(X)$, then

$X \leq \overline{Z}(X)$. So X is non-cosingular. It follows that $X \leq_{\oplus} M$. Hence every local summand is summand. Therefore by [7, Theorem 2.17] , M is a direct sum of indecomposable modules. The last statements follows from the fact that every NS non-cosingular supplemented indecomposable module is hollow. □

Recall that an epimorphism $f : P \to M$ of R-modules is a (projective) small cover of M, if (P is projective and) $Kerf \ll P$. A ring R is perfect (semiperfect) if every R-module (finitely generated R-module) has a projective cover (see [13]).

Proposition 2.14. *If R is a right perfect (semiperfect) ring, then every (finitely generated) projective right R-module is NS.*

Proof. Let R be a right perfect ring and M a projective R-module. Let A be a non-cosingular submodule of M. Consider the canonical epimorphism $\varphi : M \to M/A$. Since M/A has a projective cover, using [1, Lemma 17.17], there exists a decomposition $M = P_1 \oplus P_2$ such that $P_2 \subseteq Ker\varphi = A$ and $(\varphi\mid_{P_1}) : P_1 \to M/A \to 0$ a projective cover. Hence, we get $A = P_2 \oplus (A \cap P_1)$ where $A \cap P_1$ is both cosingular and non-cosingular. Therefore $A = P_2$ is a direct summand of M. □

The converse of Proposition 2.14 does not hold. Consider the ring of integers $R = \mathbb{Z}$. Then every (projective) R-module is NS by Example 2.3(3). However, R is not perfect (semiperfect) (note that $R/J(R) \cong R$ is not semisimple).

A ring R is a right *max ring*, if every nonzero right R-module M has at least one maximal submodule.

Proposition 2.15. *Let R be a ring such that every right NS-module is semisimple. Then R is a right max ring.*

Proof. Since every small R-module is an *NS*-module, so by hypothesis every small R-module is semisimple. Since for a module M, $Rad(M)$ is the sum of all small submodules of M (see [1, Proposition 9.13]), so $Rad(M)$ is a semisimple submodule of M. In contrary, let M be a nonzero right R-module with no maximal submodule. Hence, $Rad(M) = M$. It follows that M is semisimple. This yields $M = Rad(M) = 0$, that contradicts $M \neq 0$. Therefore, for every nonzero module M, we have $Rad(M) \neq M$. Consequently, R is a right max ring. □

As an example of above proposition, we can focus on V-rings. Because, over a V-ring, NS-modules are precisely the semisimple ones. It is clear that a V-ring is a max ring.

Proposition 2.16. *Let M and N be two modules. Then*

(1) *The module M is NS if and only if for every $f : M \to N$ with $Kerf$ non-cosingular, Imf is NS.*

(2) *If M is NS, then for every nonzero $f : M \to N$ with $Kerf$ non-cosingular, Imf is not small in M.*

Proof. (1) ($\Longrightarrow$) Let M be *NS* and $f : M \to N$ a homomorphism with $Kerf$ non-cosingular. Then $Imf \cong M/Kerf$. Since M is NS, there exists a decomposition $M = Kerf \oplus N$. It follows that Imf is isomorphic to a submodule of M. Therefore, Imf is NS. For the converse, it suffices to choose the identity isomorphism $i : M \longrightarrow M$. Since $Keri = 0$ is non-cosingular, $M = Imf$ is NS.
(2) Since Imf is isomorphic to a direct summand of M, Imf is not a small submodule of M. □

Proposition 2.17. *Let $f : M \to M'$ be a small cover and M' an NS module such that $Rad(K) = 0$ for every non-cosingular submodule K of M. Then M is NS.*

Proof. Let $K \leq M$ be non-cosingular. Then clearly $f(K)$ is non-cosingular. Since M' is *NS*, $f(K) \oplus f(L) = M'$ for some submodule L of M. Then $M = K + L + Kerf$. Since f is a samll cover, we get $M = K + L$ and $K \cap L \subseteq Kerf \ll M$. Let $(K \cap L) + T = K$ for a submodule T of K. Therefore we have $\frac{K\cap L}{T\cap L} \cong \frac{K}{T}$. It follows that $\frac{K}{T}$ is both small and non-cosingular (since $\frac{K}{T}$ is a homomorphic image of both K and $K \cap L$). Therefore, $K = T$, yields that $K \cap L \ll K$. Now, using assumption $K \cap L = 0$. Hence $M = L \oplus K$. □

3. Direct Sums of *NS*-Modules

In this section we define the (D^*)-property. Using this concept we prove that under some assumptions a finite direct sum of *NS*-modules is *NS*. We also give a sufficient condition for an arbitrary direct sum of *NS*-modules to be *NS*.

Proposition 3.1. *Let $M = M_1 \oplus M_2$ with M_1 semisimple and M_2 NS. If every direct summand of a homomorphic image of M lifts to a direct summand of M, then M is NS.*

Proof. Let $N \leq M$ be non-cosingular. Since M_1 is semisimple, $M_1 = (N \cap M_1) \oplus M'$ for some $M' \leq M_1$; we thus get $M = [(N \cap M_1) \oplus M'] \oplus M_2$. Using modularity law, $N = (N \cap M_1) \oplus [(M' \oplus M_2) \cap N]$. Set $A = (M' \oplus M_2) \cap N$ and consider the submodule $(A + M')/M'$ of $(M_2 \oplus M')/M'$. Since $(A + M')/M'$ is a homomorphic image of N and N is non-cosingular, it follows that $(A + M')/M'$ is a direct summand of $(M_2 \oplus M')/M'$. So we get $(A + M')/M' \oplus X/M' = (M_2 \oplus M')/M'$. Hence $A + X = M_2 \oplus M'$. It follows that $N + X = A + X + N = (M_2 \oplus M') + N = M$. So $M/A = N/A + (X + A)/A$. Since $N \cap (X + A) = A + (X \cap N) \subseteq A$, therefore N/A is a direct summand of M/A. Using assumption there exists a direct summand T of M containing A such that $T/A = N/A$. Hence $N \leq_{\oplus} M$. So M is *NS*. □

Definition 3.2. We say that a module M has (D^*) *property* if for every submodule N of M there exists a non-cosingular submodule K of M such that $K \leq N$ and $K \overset{ce}{\hookrightarrow} N$. In this case we call K, a *quasi-coclosure* of N in M.

By the definition every quasi-coclosure is a coclosure. But the converse does not hold as the following example shows.

Example 3.3. Let $M = \mathbb{Z}/\mathbb{Z}p \oplus \mathbb{Z}/\mathbb{Z}p^3$. Since M is artinian, it is amply supplemented. So by [5, Proposition 1.5], $\mathbb{Z}/\mathbb{Z}p$ has a coclosure. Since $\mathbb{Z}/\mathbb{Z}p$ is simple, it is a coclosure of itself, though it is not non-cosingular. In fact $\mathbb{Z}/\mathbb{Z}p$ is a small module.

It is clear that every hollow module has (D^*) property. Also every non-cosingular amply supplemented (lifting) module has D^*.

Proposition 3.4. *Let M be a module with (D^*). Then the following statements hold:*

(1) *Every factor module of M has (D^*).*

(2) *Every non-cosingular submodule of M has (D^*).*

Proof. (1) Let $N \leq M$ and $K/N \leq M/N$. Using assumption, K has a quasi-closure L in M. It follows that $L \overset{ce}{\hookrightarrow} K$ and L is non-cosingular. So we get

$$\frac{K}{L+N} \cong \frac{K/N}{(L+N)/N} \ll \frac{M/N}{(L+N)/N} \cong \frac{M}{L+N},$$

where clearly $(L+N)/N$ is non-cosingular. Hence $(L+N) \overset{ce}{\hookrightarrow} K$ and this completes the proof.

(2) Let $N \leq M$ be non-cosingular and $K \leq N$. By assumption, there exists a non-cosingular submodule L of M such that $L \overset{ce}{\hookrightarrow} K$ in M. Since N/L is non-cosingular, $L \overset{ce}{\hookrightarrow} K$ in N by [11, Lemma 2.3(1)]. Hence N has (D^*). □

Definition 3.5. Let $M = M_1 \oplus M_2$ be a module. We say M has *-property, if the sum of a non-cosingular submodule L and a direct summand T of M with $L + T \neq M$, is a direct summand of M.

Lemma 3.6. *Let $M = M_1 \oplus M_2$ be a module with *-property. Suppose that every non-cosingular submodule N of M with the property $M = N + M_1$ or $M = N + M_2$, is a direct summand of M. Let K be a non-cosingular submodule in M such that $(K + M_i)/K$ has a quasi-coclosure in M/K for $i \in \{1, 2\}$. Then K is a direct summand of M.*

Proof. We consider the submodule $(K + M_1)/K$ of M/K. Then there exists a non-cosingular submodule N/K of M/K such that $N/K \leq (K + M_1)/K$ and $N \overset{ce}{\hookrightarrow} (K + M_1)$. It follows that $K + M_1 = N + M_1$ and $M = N + M_2$. Since N is non-cosingular in M, by hypothesis, we get $M = N \oplus N'$ for some submodule N' of M and then we have $(K + N') + M_1 = M$. If $K + N' = M$, we get

$K = N$ and so we get $K \leq_{\oplus} M$. Otherwise, by hypothesis, $K + N'$ is a direct summand of M. Let $M = (K + N') \oplus K'$ for some $K' \leq M$. It follows that $N' = (K+N') \cap (N'+K')$ and $N \cap (K+N') \cap (N'+K') = K \cap (N'+K') = 0$. Therefore we get $M = K \oplus (N' + K')$, as claimed. □

The following proposition introduces equivalent conditions for a module $M = M_1 \oplus M_2$ under some assumptions to be *NS*.

Proposition 3.7. *Let $M = M_1 \oplus M_2$ has (D^*) and *-property. Then following statements are equivalent:*

(1) *M is NS;*

(2) *Every non-cosingular submodule K of M such that $M = K + M_1$ or $M = K + M_2$ is a direct summand of M;*

(3) *Every non-cosingular submodule K of M such that $K \overset{ce}{\hookrightarrow} K + M_1$ or $K \overset{ce}{\hookrightarrow} K + M_2$ or $M = K + M_1 = K + M_2$ is a direct summand of M.*

Proof. Follows from Lemma 3.6 and [5, Theorem 2.1]. □

Let M_1 and M_2 be modules. The module M_1 is *small M_2-projective* if every homomorphism $f : M_1 \to M_2/A$ where $A \leq M_2$ and $Imf \ll M_2/A$, can be lifted to a homomorphism $g : M_1 \to M_2$. The modules M_1 and M_2 are *relatively small projective* if M_i is small M_j-projective, for every $i, j \in \{1, 2\}$, $i \neq j$. It is clear that if M_1 is M_2-projective then M_1 is small M_2-projective.

Lemma 3.8. *Let M_1 be any module, M_2 an NS-module and $M = M_1 \oplus M_2$. If M_1 is small M_2-projective, then every non-cosingular submodule N of M such that $N \overset{ce}{\hookrightarrow} (N + M_1)$ is a direct summand.*

Proof. Let N be a non-cosingular submodule of M such that $N \overset{ce}{\hookrightarrow} (N + M_1)$. By [5, Lemma 2.4], there exists a submodule N' of N such that $M = N' \oplus M_2$. Clearly, M/N' is *NS*. Since N is non-cosingular, N/N' is non-cosingular. Therefore N/N' is a direct summand of M/N'. Hence N is a direct summand of M. □

Proposition 3.9. *Let M_1 and M_2 be NS-modules such that $M = M_1 \oplus M_2$ has (D^*) and *-property. If one of the following conditions holds, then M is NS.*

(1) *M_1 is small M_2-projective and every non-cosingular submodule N of M such that $M = N + M_1$ is a direct summand.*

(2) *M_1 and M_2 are relatively small projective and every non-cosingular submodule N of M such that $M = N + M_1 = N + M_2$ is a direct summand of M.*

(3) *M_2 is M_1-projective and M_1 is small M_2-projective.*

(4) *M_1 is semisimple and small M_2-projective.*

Proof. The conclusion follows from Lemmas 3.6, 3.8 and [5, Theorem 2.8]. □

Theorem 3.10. *Let M has (D^*) and *-property. Let $M = M_1 \oplus \ldots \oplus M_n$ be a finite sum of relatively projective modules. Then M is NS if and only if each M_i is NS for $i = 1, \ldots, n$.*

Proof. The necessity is clear. Conversely, it is enough to prove that M is *NS* for $n = 2$. This follows from Proposition 3.9. □

Corollary 3.11. *Let R be a hereditary ring. Let M_1 and M_2 be R-modules such that $M = M_1 \oplus M_2$ has (D^*) and *-property. Then M is NS if and only if M_1 and M_2 is NS and every non-cosingular submodule N of M such that $M = N + M_1$ is a direct summand.*

Proof. Use [5, Lemma 2.3] and Proposition 3.9. □

Definition 3.12 ([6])**.** Let M and N be two modules. Then N is called *radical-M-projective* if, for any $K \leq M$ and any homomorphism $f : N \to M/K$ there exists a homomorphism $h : N \to M$ such that $Im(f - \pi h) \ll (M/K)$, where $\pi : M \to M/K$ is the natural epimorphism.

Proposition 3.13 ([6])**.** *Let $M = M_1 \oplus M_2$. Consider the following conditions:*

(1) *M_1 is radical-M_2-projective;*

(2) *For every $K \leq M$ with $K + M_2 = M$, there exists $M_3 \leq M$ such that $M = M_2 \oplus M_3$ and $(K + M_3)/K \ll (M/K)$.*

Then (1) ⇒ (2) and if M is amply supplemented, then (2) ⇒ (1).

Proposition 3.14. *Let $M = M_1 \oplus M_2$ such that M_1 and M_2 are NS. If M_1 is radical-M_2-projective, then every non-cosingular submodule K of M with $K + M_2 = M$, is a direct summand of M.*

Proof. Let K be a non-cosingular submodule of M such that $K + M_2 = M$. Then by Proposition 3.13, there exists $M_3 \leq M$ such that $M = M_2 \oplus M_3$ and $(K + M_3)/K \ll (M/K)$. Consider the submodule $(K + M_3)/M_3$ of M/M_3. Since $(K + M_3)/M_3$ is non-cosingular and $M/M_3 \cong M_2$ is *NS*, it follows that $(K + M_3)/M_3 \oplus L/M_3 = M/M_3$ for a submodule L of M containing M_3. We thus get $K + L = M$. On the other hand, from $M_3 \leq L$ and the modularity law we have $L = (L \cap M_2) \oplus M_3$ and hence we get $K + (L \cap M_2) + M_3 = M$. Now we have $(K + M_3)/K + ((L \cap M_2) + K)/K = M/M_3$. Since $(K + M_3)/K \ll M/K$, it implies that $(L \cap M_2) + K = M$. Further, by the above direct decomposition of M/M_3, we get $(L \cap M_2) \cap K \subseteq (M_2 \cap M_3) = 0$. We thus arrive at $M = K \oplus (L \cap M_2)$. □

We conclude the paper with a rather obvious remark that is a sufficient condition for a direct sum of *NS*-modules to be *NS*.

Remark 3.15. Let $M = \bigoplus_{i \in I} M_i$ be a duo module. Then $M = \bigoplus_{i \in I} M_i$ is *NS* if and only if each M_i is *NS*.

Proof. Let $M = \bigoplus_{i\in I} M_i$ be such that each M_i is *NS* and let $N \leq M$ be non-cosingular. Since M is a duo module, $N = \bigoplus_{i\in I}(N \cap M_i)$ and for each i, $N \cap M_i$ is non-cosingular. By assumption, for each i, we get $M_i = (N \cap M_i) \oplus N_i$ for some $N_i \leq M_i$. Then

$$M = \bigoplus_{i\in I} M_i = \bigoplus_{i\in I}[(N \cap M_i) \oplus N_i)] = [\bigoplus_{i\in I}(N \cap M_i)] \oplus [\bigoplus_{i\in I} N_i] = N \oplus N',$$

where $N' = \bigoplus_{i\in I} N_i$. Hence M is *NS*, as required. The converse is clear. □

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(Yahya Talebi) Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
E-mail address: talebi@umz.ac.ir

(Mehrab Hosseinpour) Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
E-mail address: m.hpour@umz.ac.ir

(Ali Reza Moniri Hamzekolaee) Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
E-mail address: a.monirih@umz.ac.ir