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CLIFFORD-FISCHER THEORY APPLIED TO A GROUP OF THE FORM $2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$

A. B. M. BASHEER* AND J. MOORI

(Communicated by Ali Reza Ashrafi)

Abstract. In our paper [A. B. M. Basheer and J. Moori, On a group of the form $2^{10}{:}(U_5(2){:}2)$] we calculated the inertia factors, Fischer matrices and the ordinary character table of the split extension $2^{10}{:}(U_5(2){:}2)$ by means of Clifford-Fischer Theory. The second inertia factor group of $2^{10}{:}(U_5(2){:}2)$ is a group of the form $2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$. The purpose of this paper is the determination of the conjugacy classes of $\overline{G}$ using the coset analysis method, the determination of the inertia factors, the computations of the Fischer matrices and the ordinary character table of the split extension $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$ by means of Clifford-Fischer Theory. Through various theoretical and computational aspects we were able to determine the structures of the inertia factor groups. These are the groups $H_1 = H_2 = (3^{1+2}{:}8){:}2$, $H_3 = QD_{16}$ and $H_4 = D_{12}$. The Fischer matrices $\mathcal{F}_i$ of $\overline{G}$, which are complex valued matrices, are all listed in this paper and their sizes range between 2 and 5. The full character table of $\overline{G}$, which is 41×41 complex valued matrix, is available in the PhD thesis of the first author, which could be accessed online.
Keywords: Group extensions, extra-special $p-$group, Clifford theory, inertia groups, Fischer matrices, character table.
MSC(2010): Primary: 20C15; Secondary: 20C40.

1. Introduction

Let $U = U_5(2)$ be the special unitary group consisting of 5×5 matrices over $\mathbb{F}_4$ that preserves a non-singular Hermitian form. The outer automorphism of U is 2 (see the $\mathbb{ATLAS}$ [6]) and thus the full automorphism group of U is a group of the form $U_5(2){:}2$. This group has a $10-$dimensional absolutely irreducible module over $\mathbb{F}_2$. Hence a split extension of the form $2^{10}{:}(U_5(2){:}2)$ does exist. In [5] we studied the Fischer matrices and the ordinary character table of $2^{10}{:}(U_5(2){:}2)$ using the theory of Clifford-Fischer Matrices. One of the

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inertia factor groups of $2^{10}{:}(U_5(2){:}2)$ was the group $2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$, which is denoted by $\overline{G}$ in the present paper. By $\mathbb{ATLAS}$ the second largest maximal subgroup of $U_5(2){:}2$ is a group of the form $2_{-}^{1+6}{:}3_{-}^{1+2}{:}2S_4$. Our group $\overline{G}$ is not maximal in $U_5(2){:}2$ but sits maximally, of index 3, in $2_{-}^{1+6}{:}3_{-}^{1+2}{:}2S_4$. Clearly $\overline{G}$ is of extension type (more precisely a split extension of the extra-special $2-$group of order 128, of "$-$" type, by a group of the form $(3^{1+2}{:}8){:}2)$. In this paper we are interested in determining the conjugacy classes, inertia factor groups and calculating the Fischer matrices and hence the ordinary character table of $\overline{G}$ using the coset analysis technique together with the theory of Clifford-Fischer Matrices. This character table is important as it is required for the computations of the ordinary character table of $2^{10}{:}(U_5(2){:}2)$. It will turn out that the character table of $\overline{G}$ is a 41×41 complex valued matrix and it is partitioned into four blocks corresponding to the four inertia factor groups $H_1 = H_2 = (3^{1+2}{:}8){:}2,\ \ H_3 = QD_{16}$ and $H_4 = D_{12}$ (see Section 3). If one only interested in the calculation of the character table, then it could be computed by using GAP or Magma and the generators $\overline{g}_1$ and $\overline{g}_2$ of $\overline{G}$. But Clifford-Fischer Theory provides many other interesting information on the group and on the character table, in particular the character table produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma. Also providing various examples for the applications of Clifford-Fischer Theory to both split and non-split extensions is making sense, since each group requires individual approach. The readers (particulary young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1] and [2,3] and [4].

In [5] we were able to generate $\overline{G}$ inside $2^{10}{:}(U_5(2){:}2)$ in terms of 11×11 matrices over $\mathbb{F}_2$. In fact $2^{10}{:}(U_5(2){:}2) \leq PSL(11,2)$ and thus $\overline{G} \leq PSL(11,2)$. The following two elements $\overline{g}_1$ and $\overline{g}_2$ generate $\overline{G}$.

$$\overline{g}_1=\begin{pmatrix}
0&1&0&1&0&1&0&1&1&1&0\\
0&1&0&1&0&1&1&0&1&1&0\\
1&1&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&1&0\\
1&1&0&1&1&1&0&0&0&0&0\\
1&1&1&0&0&0&1&1&1&1&0\\
0&1&0&0&0&1&0&0&1&0&0\\
0&1&0&1&1&0&1&1&0&1&0\\
0&1&0&1&0&0&0&0&1&1&0\\
0&1&1&1&1&1&1&1&0&1&0\\
0&0&0&1&0&0&0&0&1&1&1
\end{pmatrix},\overline{g}_2=\begin{pmatrix}
0&1&1&0&1&0&0&1&1&0&0\\
1&1&1&0&0&1&1&1&0&0&0\\
1&1&0&0&0&0&0&1&1&0&0\\
0&1&1&0&0&1&1&1&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0\\
0&1&1&0&0&0&0&1&1&0&0\\
0&0&0&1&0&0&1&0&1&0&0\\
0&0&1&0&0&1&0&0&0&0&0\\
1&0&1&1&0&1&1&1&0&0&0\\
0&0&1&1&1&1&0&1&1&0&0\\
1&0&1&1&0&0&0&0&0&0&1
\end{pmatrix},$$

with $o(\overline{g}_1) = 24,\ o(\overline{g}_2) = 16$ and $o(\overline{g}_1\overline{g}_2) = 12$.

Let $N := 2_{-}^{1+6}$ and $G := (3^{1+2}{:}8){:}2 \cong \overline{G}/2_{-}^{1+6}$. Then $\overline{G} = N{:}G$. We have found that $\overline{G}$ has 11 normal subgroups of orders 1, 2, 128, 384, 3456, 6912, 13824, 27648, 27648, 27648 and 55296. The normal subgroup of order 128 is an extra-special 2−group isomorphic to N. Also in $\overline{G}$ we were able to locate 4 complements of N, namely $(3^{1+2}{:}Q_8){:}2$ (3 times) and $(3^{1+2}{:}8){:}2$, where Q_8 is the quaternion group of order 8. Clearly we are interested in the fourth complement $(3^{1+2}{:}8){:}2$, since it can be identified with G. The following two elements σ_1 and σ_2 are 11−dimensional matrices over $\mathbb{F}_2$ that generate the complement $G = (3^{1+2}{:}8){:}2$.

$$\sigma_1=\begin{pmatrix} 1&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0\\ 1&1&0&1&0&1&0&1&0&1&0\\ 0&0&1&0&1&0&1&1&1&1&0\\ 1&0&1&1&1&1&0&0&0&1&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0&1&1&0\\ 0&0&0&1&1&1&0&1&1&1&0\\ 0&0&0&1&0&0&1&0&0&0&0\\ 1&0&0&1&0&0&1&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \sigma_2=\begin{pmatrix} 1&0&1&0&1&0&0&0&1&0&0\\ 1&0&1&0&0&1&1&0&0&1&0\\ 0&0&1&1&0&0&0&0&1&0&0\\ 1&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&1&0&1&0\\ 1&1&1&0&0&0&1&0&0&1&0\\ 0&1&1&1&0&1&1&1&0&1&0\\ 1&1&0&0&0&0&0&0&0&1&0\\ 1&0&0&1&0&0&1&1&1&0&0\\ 1&1&1&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

The character table of the group $G = (3^{1+2}{:}8){:}2$ is given in Table 1.

Table 1. The character table of $G = (3^{1+2}{:}8){:}2$

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}
	1_1	2_1	2_2	3_1	3_2	4_1	4_2	6_1	6_2	8_1	8_2	12_1	12_2	12_3
$\lvert C_G(g_i)\rvert$	432	48	12	216	18	24	12	24	6	8	8	12	12	12
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	1	1	1	1	1	−1	−1	−1	1	1	1
χ_3	1	1	−1	1	1	1	−1	1	−1	1	1	−1	−1	1
χ_4	1	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	1
χ_5	2	2	0	2	2	−2	0	2	0	0	0	0	0	−2
χ_6	2	−2	0	2	2	0	0	−2	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	0	0
χ_7	2	−2	0	2	2	0	0	−2	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	0	0
χ_8	6	−2	0	−3	0	2	−2	1	0	0	0	1	1	−1
χ_9	6	−2	0	−3	0	2	2	1	0	0	0	−1	−1	−1
χ_{10}	6	−2	0	−3	0	−2	0	1	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$	1
χ_{11}	6	−2	0	−3	0	−2	0	1	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	1
χ_{12}	8	0	−2	8	−1	0	0	0	1	0	0	0	0	0
χ_{13}	8	0	2	8	−1	0	0	0	−1	0	0	0	0	0
χ_{14}	12	4	0	−6	0	0	0	−2	0	0	0	0	0	0

2. The Conjugacy Classes of $\overline{G} = 2_-^{1+6}{:}((3^{1+2}{:}8){:}2)$

In this section we calculate the conjugacy classes of $\overline{G}$ using the coset analysis technique (see [1] or [9] and [10] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of G. Recall that N is a group of order $2^{1+6} = 128$ and has $2^6 + 1 = 65$ conjugacy classes. The action of $N = \langle n_1, n_2, \cdots, n_7\rangle$ on the identity coset $N1_G = N$ produces the 65 conjugacy classes of N, where we know that N has

- singleton conjugacy class consisting of 1_N,
- singleton conjugacy class consisting of the central involution σ of N,
- 27 conjugacy classes, each class consists of two non-central involutions,
- 36 conjugacy classes, each class consists of two elements of order 4.

Using GAP, the action of $\overline{G} = \langle \overline{g}_1, \overline{g}_2 \rangle$ (or just $C_G(1_G) = G = \langle \sigma_1, \sigma_2 \rangle$) on these 65 orbits leaves invariant $\{1_N\} := \Omega_1$ and $\{\sigma\} := \Omega_2$, while fuse the 27 orbits of non-central involutions into a single orbit Ω_3 and also fuse the 36 orbits of elements of order 4 altogether into a single orbit Ω_4. Thus in $\overline{G}$, we get four conjugacy classes of sizes 1, 1, 54 and 72. Similarly one can apply this to the other 13 cosets Ng_i, where g_i is a representative of a conjugacy class of G (for the classes of G, see Table 1). Corresponding to the 14 conjugacy classes of $G = (3^{1+2}{:}8){:}2$, we obtain 41 conjugacy classes for $\overline{G}$. We list these classes in Table 2, where the notations used in this table are as in [1].

3. **The Inertia Factor Groups of** $\overline{G} = 2_-^{1+6}{:}((3^{1+2}{:}8){:}2)$

We have seen in Section 2 that the action of $\overline{G}$ on N produced four orbits of lengths 1, 1, 54 and 72. By a theorem of Brauer (for example see Theorem 5.1.5 of [12]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce four orbits $\Delta_1,\ \Delta_2,\ \Delta_3$ and Δ_4 and let $\theta_1,\ \theta_2,\ \theta_3$ and θ_4 be respective representatives of these orbits. It is well known that any extra-special $p-$group of order p^{1+2m} has $p^{2m}+1$ irreducible characters (p^{2m} linear characters of the vector space p^{2m} are inflated to the full extension p^{1+2m} and $p-1$ faithful irreducible characters each of degree p^m). Thus the group $N = 2_-^{1+6}$ has 65 irreducible characters in which 64 characters are linear and one unique faithful character θ_2 of degree 8 (the values of θ_2 are as follows: $\theta_2(1_N) = 8,\ \theta_2(\sigma) = -8$ and $\theta_2(t) = 0$ for any $t \in N \backslash \{1_N, \sigma\}$). It is necessary that the action of $\overline{G}$ (or just G) on $\mathrm{Irr}(N)$ leaves invariant the identity character $\theta_1 := \mathbf{1}_N$ and the character θ_2. Therefore the corresponding inertia factors of θ_1 and θ_2 are $H_1 = H_2 = G = (3^{1+2}{:}8){:}2$. Let $\Delta_1 = \{\mathbf{1}_N\}$ and $\Delta_2 = \{\theta_2\}$. Thus $|\Delta_3| + |\Delta_4| = 63$. Since $|G| = 432 = 2^4 \times 3^3$, the set $\Omega = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 108, 144, 216, 432\}$ consists of all the divisors of $|G|$. Since $|\Delta_3|, |\Delta_4| \in \Omega$ and $|\Delta_3| + |\Delta_4| = 63$, we can see that the only possibility for $(|\Delta_3|, |\Delta_4|)$ is that $(|\Delta_3|, |\Delta_4|) = (9, 54)$ or $(27, 36)$ (the cases $(54, 9)$ and $(36, 27)$ are excluded with out loss of generality). Let H_3 and H_4 be the respective inertia factor groups of characters from the orbits Δ_3 and Δ_4. The rest of this section is devoted to determine $(|\Delta_3|, |\Delta_4|)$ and (H_3, H_4). It is well-known that the identity character $\mathbf{1}_N$ is extendable to its inertia group (since $\mathbf{1}_{\overline{G}}\downarrow_N^{\overline{G}} = \mathbf{1}_N$). Using Magma or GAP one can easily obtain that the Schur multiplier $M(G) = M((3^{1+2}{:}8){:}2)$ is trivial. This shows that the character θ_2 is also extendable to an ordinary character of $\overline{G}$. Now since the characters of Δ_3 and Δ_4 are linear characters of N, it follows by application of Theorem 5.1.18 of [12] that these characters are extendable to ordinary characters of their respective inertia groups. Thus for the construction of the character table of $\overline{G}$, all the character tables of the inertia factors that we will use are the ordinary ones. Using this and the fact that $\overline{G}$ has 41 conjugacy classes (see Table 2) we deduce that $|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| =$

TABLE 2. The conjugacy classes of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$

$[g_i]_G$	k_i	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1_1$	$k_1 = 65$	$m_{11} = 1$	g_{11}	1	1	55296
		$m_{12} = 1$	g_{12}	2	1	55296
		$m_{13} = 54$	g_{13}	2	54	1024
		$m_{14} = 72$	g_{14}	4	72	768
$g_2 = 2_1$	$k_2 = 17$	$m_{21} = 4$	g_{21}	2	36	1536
		$m_{22} = 4$	g_{22}	2	36	1536
		$m_{23} = 24$	g_{23}	4	216	256
		$m_{24} = 96$	g_{24}	4	864	64
$g_3 = 2_2$	$k_3 = 9$	$m_{31} = 8$	g_{31}	8	288	192
		$m_{32} = 8$	g_{32}	8	288	192
		$m_{33} = 24$	g_{33}	2	576	96
		$m_{34} = 48$	g_{34}	8	1728	32
		$m_{35} = 48$	g_{35}	4	1728	32
$g_4 = 3_1$	$k_4 = 2$	$m_{41} = 64$	g_{41}	3	128	432
		$m_{42} = 64$	g_{42}	6	128	432
$g_5 = 3_2$	$k_5 = 5$	$m_{51} = 16$	g_{51}	3	384	192
		$m_{52} = 16$	g_{52}	6	384	192
		$m_{53} = 96$	g_{53}	12	2304	24
$g_6 = 4_1$	$k_6 = 5$	$m_{61} = 16$	g_{61}	4	288	192
		$m_{62} = 16$	g_{62}	4	288	192
		$m_{63} = 96$	g_{63}	8	1728	32
$g_7 = 4_2$	$k_7 = 5$	$m_{71} = 16$	g_{71}	4	576	96
		$m_{72} = 16$	g_{72}	4	576	96
		$m_{73} = 96$	g_{73}	8	3456	16
$g_8 = 6_1$	$k_8 = 2$	$m_{81} = 64$	g_{81}	6	1152	48
		$m_{82} = 64$	g_{82}	6	1152	48
$g_9 = 6_2$	$k_9 = 3$	$m_{91} = 32$	g_{91}	24	2304	24
		$m_{92} = 32$	g_{92}	24	2304	24
		$m_{93} = 64$	g_{93}	6	4608	12
$g_{10} = 8_1$	$k_{10} = 3$	$m_{10,1} = 32$	$g_{10,1}$	8	1728	32
		$m_{10,2} = 32$	$g_{10,2}$	8	1728	32
		$m_{10,3} = 64$	$g_{10,3}$	16	3456	16
$g_{11} = 8_2$	$k_{11} = 3$	$m_{11,1} = 32$	$g_{11,1}$	8	1728	32
		$m_{11,2} = 32$	$g_{11,2}$	8	1728	32
		$m_{11,3} = 64$	$g_{11,3}$	16	3456	16
$g_{12} = 12_1$	$k_{12} = 2$	$m_{12,1} = 64$	$g_{12,1}$	12	2304	24
		$m_{12,2} = 64$	$g_{12,2}$	12	2304	24
$g_{13} = 12_2$	$k_{13} = 2$	$m_{13,1} = 64$	$g_{13,1}$	12	2304	24
		$m_{13,2} = 64$	$g_{13,2}$	12	2304	24
$g_{14} = 12_3$	$k_{14} = 2$	$m_{14,1} = 64$	$g_{14,1}$	12	2304	24
		$m_{14,2} = 64$	$g_{14,2}$	12	2304	24

41. Since $|\mathrm{Irr}(H_1)| = |\mathrm{Irr}(H_2)| = 14$, we get that $|\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| = 13$. This is a key result in determining the inertia factors H_3 and H_4.

Proposition 3.1. *The case* $(|\Delta_3|, |\Delta_4|) = (9, 54)$ *is not possible*

Proof. Assume that $(|\Delta_3|, |\Delta_4|) = (9, 54)$. Then $|H_3| = 48$ and $|H_4| = 8$. Using GAP, it is easy to see that the group G has four conjugacy classes of maximal subgroups, where each class is represented by $M[1] := (3^{1+2}{:}4){:}2$, $M[2] := 3^{1+2}{:}8$, $M[3] := 3^{1+2}{:}Q_8$ and $M[4] := (3 \times Q_8){:}2$. Note that the respective indices of $M[1]$, $M[2]$, $M[3]$ and $M[4]$ in G are 2, 2, 2 and 9. This shows that $H_3 = M[4] = (3 \times Q_8){:}2$. Using GAP, we can see that $|\mathrm{Irr}(H_3)| = |\mathrm{Irr}((3 \times Q_8){:}2)| = 12$. Using this and the fact that $|\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| = 13$, we deduce that $|\mathrm{Irr}(H_4)| = 1$, but this clearly a contradiction as we know that $|H_4| = 8$. Thus $(|\Delta_3|, |\Delta_4|) \neq (9, 54)$. □

From Proposition 3.1, it follows that $(|\Delta_3|, |\Delta_4|) = (27, 36)$, that is $|H_3| = 16$ and $|H_4| = 12$. From the elementary group theory, we know that there are, up to isomorphism, 14 groups of order 16 and 5 groups of order 12 (see GAP). Since we are interested in two groups H_3 and H_4 of orders 16 and 12 such that $|\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| = 13$, it follows that there are 3 groups of order 16, each having 7 characters and 2 groups of order 12 each having 6 characters. The three groups of order 16 each having 7 characters are D_{16}, QD_{16} and Q_{16}, while the two groups of order 12 each having 6 characters are 3:4 and D_{12}, where QD_{16} is the *quasidihedral* group of order 16 and Q_{16} is the *quaternion* group of order 16. For conjugacy classes, character tables and structure of these groups, readers are advised to consult GAP. According to the above, possible pairs representing (H_3, H_4) are

$$(H_3, H_4) \in \{(D_{16}, 3{:}4), (D_{16}, D_{12}), (QD_{16}, 3{:}4), (QD_{16}, D_{12}), (Q_{16}, 3{:}4), (Q_{16}, D_{12})\}.$$

Proposition 3.2. *The pair $(H_3, H_4) = (QD_{16}, D_{12})$ is feasible.*

Proof. Recall that the Fischer matrix $\mathcal{F}_i$ of $\overline{G}$, corresponding to g_i, is of size $c(g_i)$ for each $i \in \{1, 2, \cdots, 14\}$. Also for each i, there are exactly two rows in $\mathcal{F}_i$ correspond to H_1 and H_2 and these two rows are labeled by $(1, 1)$ and $(2, 1)$. Therefore the two groups H_3 and H_4 contribute together $c(g_i) - 2$ rows to $\mathcal{F}_i$, $\forall i \in \{1, 2, \cdots, 14\}$. From Table 2 we can deduce the values of all $c(g_i)$. In particular we have $c(g_2) = 4$, $c(g_3) = 5$ and $c(g_6) = c(g_7) = 3$. We will use this to show that (H_3, H_4) can be (QD_{16}, D_{12}). The group D_{16} has three classes of involutions while the two groups 3:4 and D_{12} have one and three classes of involutions respectively. Thus if $(H_3, H_4) = (D_{16}, 3{:}4)$ or (D_{16}, D_{12}), then the sizes (recall that we have two rows from H_1 and H_2) of the Fischer matrices $\mathcal{F}_2$ and $\mathcal{F}_3$ will be 6×4 and 8×5 respectively, but this clearly contradicts the fact that the Fischer matrices of any extension are squares (see Proposition 3.6(i) of [2]). Hence $(H_3, H_4) \notin \{(D_{16}, 3{:}4), (D_{16}, D_{12})\}$. The group Q_{16} has three classes of elements of order 4. For similar reasons if $(H_3, H_4) = (Q_{16}, 3{:}4)$ or (Q_{16}, D_{12}), then the Fischer matrices $\mathcal{F}_6$ and $\mathcal{F}_7$ will be of sizes 7×3 and 5×3, respectively, which is a contradiction again. Thus $(H_3, H_4) \notin \{(Q_{16}, 3{:}4), (Q_{16}, D_{12})\}$. Finally if $(H_3, H_4) = (QD_{16}, 3{:}4)$, then the Fischer matrices $\mathcal{F}_6$ and $\mathcal{F}_7$ will be of sizes 6×3 and 6×3, respectively, which is a contradiction again. Hence the case $(H_3, H_4) = (QD_{16}, D_{12})$ is feasible, that is $H_3 = QD_{16}$ and $H_4 = D_{12}$. □

Proposition 3.2 asserts that (QD_{16}, D_{12}) could be a possible candidate for the inertia factor groups H_3 and H_4 respectively, but we are not yet sure whether the group $G = (3^{1+2}{:}8){:}2 = \langle \sigma_1, \sigma_2 \rangle$ does contain subgroups of orders 16 and 12 that are isomorphic to QD_{16} and D_{12} respectively. By further computations (done through GAP) on the structures of the maximal subgroups $M[1]$, $M[2]$, $M[3]$ and $M[4]$ of G, we have found that the group G contains two subgroups of orders 16 and 12 such that these subgroups are isomorphic to QD_{16} and D_{12} respectively. Hence $H_3 = QD_{16}$ and $H_4 = D_{12}$. As subgroups of $G = \langle \sigma_1, \sigma_2 \rangle$, the group $H_3 = QD_{16}$ is generated as follows:

$$c_1=\begin{pmatrix}0&0&1&0&1&0&1&0&1&1&0\\0&0&1&1&0&1&1&0&1&0&0\\0&1&0&0&1&0&0&1&1&1&0\\1&0&1&1&1&0&0&1&0&0&0\\0&1&1&1&1&1&0&1&0&0&0\\0&0&1&0&1&1&0&0&1&1&0\\1&0&1&0&1&0&0&0&1&1&0\\1&1&0&0&0&0&1&1&1&0&0\\1&1&0&1&1&1&0&0&1&1&0\\0&0&1&0&1&0&0&0&1&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}, c_2=\begin{pmatrix}1&0&1&0&1&0&0&0&1&0&0\\0&1&1&0&1&0&0&0&1&0&0\\1&1&0&1&0&1&1&0&1&1&0\\0&0&1&0&0&1&1&0&0&1&0\\1&1&1&1&1&1&1&0&1&1&0\\0&0&1&0&1&1&0&0&1&0&0\\0&0&1&0&1&0&1&0&1&0&0\\0&1&0&1&1&1&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\1&1&1&0&1&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix},$$

while $H_4 = D_{12}$ is generated by:

$$d_1=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&1&0&0\\0&0&1&1&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\1&1&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}, d_2=\begin{pmatrix}1&1&0&0&0&0&1&0&1&1&0\\1&1&0&1&1&1&1&0&1&0&0\\0&0&1&1&1&1&0&1&0&1&0\\0&0&0&0&1&1&0&1&0&0&0\\0&0&0&0&1&0&0&1&1&0&0\\1&1&0&0&0&1&0&0&1&1&0\\0&1&0&0&0&0&0&0&1&1&0\\1&1&0&1&1&1&0&1&0&1&0\\1&1&0&1&1&1&0&0&1&1&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix},$$

$$d_3 = \begin{pmatrix}0&0&0&1&1&1&0&0&1&0&0\\0&0&0&1&1&1&0&0&0&0&0\\1&0&0&0&1&0&1&1&0&0&0\\1&1&1&1&1&0&0&1&1&1&0\\0&1&0&0&1&0&1&1&0&0&0\\0&1&0&1&1&0&0&0&0&0&0\\0&1&1&1&0&1&1&0&1&0&0\\1&1&0&0&1&1&1&0&0&0&0\\0&0&1&0&1&0&0&0&0&0&0\\1&1&0&1&1&1&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}.$$

The fusions of classes of $H_3 = QD_{16}$ and $H_4 = D_{12}$ into classes of $G = (3^{1+2}{:}8){:}2$ are given in Table 3.

The character tables of H_3 and H_4 are well-known and they are available in GAP. For the convenience we supply the character tables of H_3 and H_4 in Tables 4 and 5 respectively.

4. Fischer Matrices of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$

In this section we calculate the Fischer matrices of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$. From Section 3 of Basheer and Moori [2] we recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the

TABLE 3. The fusions of conjugacy classes of H_3 and H_4 into classes of G

Inertia Factor Groups H_3 & H_4	Class of H_i	$\hookrightarrow$	Class of G	Class of H_i	$\hookrightarrow$	Class of G
$H_3 = QD_{16}$	g_{131}		1_1	g_{731}		4_2
	g_{231}		2_1	$g_{10,31}$		8_1
	g_{331}		2_2	$g_{11,31}$		8_2
	g_{631}		4_1			
$H_4 = D_{12}$	g_{141}		1_1	g_{241}		2_1
	g_{341}		2_2	g_{541}		3_2
	g_{342}		2_2	g_{941}		6_2

TABLE 4. The character table of $H_3 = QD_{16}$

	g_{131}	g_{231}	g_{331}	g_{631}	g_{731}	$g_{10,31}$	$g_{11,31}$
$o(g_{i3m})$	1	2	2	4	4	8	8
$\lvert C_{H_3}(g_{ikm})\rvert$	16	16	4	8	4	8	8
χ_1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	1	1
χ_3	1	1	-1	1	1	-1	-1
χ_4	1	1	1	1	-1	-1	-1
χ_5	2	2	0	-2	0	0	0
χ_6	2	-2	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$
χ_7	2	-2	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$

TABLE 5. The character table of $H_4 = D_{12}$

	g_{141}	g_{341}	g_{342}	g_{241}	g_{541}	g_{941}
$o(g_{i4m})$	1	2	2	2	3	6
$\lvert C_{H_4}(g_{ikm})\rvert$	12	12	4	4	6	6
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	-1	-1	1	1	-1
χ_4	1	1	-1	-1	1	1
χ_5	2	-2	0	0	-1	1
χ_6	2	2	0	0	-1	-1

sizes of the centralizers of $g_{ij},\ 1 \leq j \leq c(g_i)$, in $\overline{G}$ and m_{ij} respectively. Also the rows of $\mathcal{F}_i$ are partitioned into parts $\mathcal{F}_{ik},\ 1 \leq k \leq t$, corresponding to the inertia factors $H_1, H_2, \cdots, H_t$, where each $\mathcal{F}_{ik}$ consists of $c(g_{ik})$ rows correspond to the $\alpha_k^{-1}-$regular classes (which are the H_k-classes that fuse to class $[g_i]_G$). Thus every row of $\mathcal{F}_i$ is labeled by the pair (k, m), where $1 \leq k \leq t$ and $1 \leq m \leq c(g_{ik})$. In Table 2 we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij},\ 1 \leq i \leq 14,\ 1 \leq j \leq c(g_i)$. Also the fusions of classes of H_3 and H_4 are given in Table 3. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 2 that the sizes of the Fischer matrices of $\overline{G} = 2^{1+6}_{-}$:$((3^{1+2}$:8):2) range between 2 and 5 for every $i \in \{1, 2, \cdots, 14\}$.

We have used the arithmetical properties of the Fischer matrices, given in Proposition 3.6 of [2], to calculate some of the entries of these matrices. In addition to these properties, we have the following important lemmas, which help us more in determining some entries of the Fischer matrices of $\overline{G}$.

Lemma 4.1. *For every Fischer matrix $\mathcal{F}_i$, of size $c(g_i)$, the sum of the $1^{st}, 3^{rd}$, $4^{th}, \cdots$, $c(g_i)^{th}$ rows equal (componentwise) the square of the modulus of the 2^{nd} row.*

Proof. The proof is similar to the proof of Lemma 6 of Pahlings [13]. □

Lemma 4.2. *For every Fischer matrix $\mathcal{F}_i$, we can order the g_{ij}, for $1 \leq j \leq c(g_i)$, so that the second row of $\mathcal{F}_i$ is of the form $[z_i \quad -z_i \quad 0 \quad \cdots \quad 0]$ and we may choose the $g_{i2} = \sigma g_{i1}$, where σ is the central involution in $\overline{G}$. Furthermore*

$$a_{i1}^{(k,m)} = a_{i2}^{(k,m)} = \frac{|C_{H_k}(g_{i11})|}{|C_{H_k}(g_{ikm})|} \quad \text{for } k \in \{1,3,4\},\ 1 \leq m \leq c(g_{ik}). \tag{4.1}$$

Proof. The proof is similar to the proof of Lemma 7 of [13]. □

Note 4.3. The proof of Lemma 7 of [13] contained a very important piece of information that is the last row of every Fischer matrix of $2_+^{1+22}{\cdot}Co_2$ is $[\eta(g_{i1}) \quad \eta(g_{i2}) \quad \cdots \quad \eta(g_{is_i})]$, where s_i in his notation has the same meaning of $c(g_i)$ in our notation. In our group $\overline{G}$, the second row of every Fischer matrix $\mathcal{F}_i$ is given by $[\theta_2(g_{i1}) \quad \theta_2(g_{i2}) \quad 0 \quad \cdots \quad 0]$.

Note 4.4. Observe that with Lemma 4.1, Equation (4.1) and Note 4.3 we know the first two columns and the second row of every Fischer matrix $\mathcal{F}_i$. Also from Proposition 3.6(iii) of [2], we know the first row of every Fischer matrix $\mathcal{F}_i$. This reduces the number of unknowns in every Fischer matrix of size $c(g_i)$ to $c(g_i)^2 - 4c(g_i) + 4$.

Using the row and column orthogonality relations that the Fischer matrices satisfy (Proposition 3.6 of [2]) we have built an algebraic system of equations. With the help of the symbolic mathematical package Maxima [8], we were able to solve these systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}
$o(g_{1j})$		2	2	2	4
$\lvert C_{\overline{G}}(g_{1j})\rvert$		55296	55296	1024	768
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$				
$(1,1)$	432	1	1	1	1
$(2,1)$	432	8	-8	0	0
$(3,1)$	16	27	27	-5	3
$(4,1)$	12	36	36	4	-4
m_{1j}		1	1	54	72

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}
$o(g_{2j})$		2	2	4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		1536	1536	256	64
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$				
$(1,1)$	48	1	1	1	1
$(2,1)$	48	4	-4	0	0
$(3,1)$	16	3	3	3	-1
$(4,1)$	4	12	12	-4	0
m_{2j}		4	4	24	96

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}	g_{35}
$o(g_{3j})$		8	8	2	4	8
$\lvert C_{\overline{G}}(g_{3j})\rvert$		192	192	96	32	32
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$					
$(1,1)$	12	1	1	1	1	1
$(2,1)$	12	$2i\sqrt{2}$	$-2i\sqrt{2}$	0	0	0
$(3,1)$	4	3	3	3	-1	-1
$(4,1)$	12	1	1	-1	-1	1
$(4,2)$	4	3	3	-3	1	-1
m_{3j}		8	8	16	48	48

$\mathcal{F}_4$

g_4		g_{41}	g_{42}
$o(g_{4j})$		3	6
$\lvert C_{\overline{G}}(g_{4j})\rvert$		432	432
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$		
$(1,1)$	216	1	1
$(2,1)$	216	-1	1
m_{4j}		64	64

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}
$o(g_{5j})$		3	6	12
$\lvert C_{\overline{G}}(g_{5j})\rvert$		144	144	24
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$			
$(1,1)$	18	1	1	1
$(2,1)$	18	2	-2	0
$(4,1)$	6	3	3	-1
m_{5j}		16	16	96

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}
$o(g_{6j})$		4	4	8
$\lvert C_{\overline{G}}(g_{6j})\rvert$		192	192	32
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$			
$(1,1)$	24	1	1	1
$(2,1)$	24	2	-2	0
$(3,1)$	8	3	3	-1
m_{6j}		16	16	96

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}
$o(g_{7j})$		4	4	8
$\lvert C_{\overline{G}}(g_{7j})\rvert$		96	96	16
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$			
$(1,1)$	12	1	1	1
$(2,1)$	12	2	-2	0
$(3,1)$	4	3	3	-1
m_{7j}		16	16	96

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		6	6
$\lvert C_{\overline{G}}(g_{8j})\rvert$		48	48
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{8j}		64	64

$\mathcal{F}_9$

g_9		g_{91}	g_{92}	g_{93}
$o(g_{9j})$		6	24	24
$\lvert C_{\overline{G}}(g_{9j})\rvert$		24	24	12
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$			
$(1,1)$	6	1	1	1
$(2,1)$	6	$i\sqrt{2}$	$-i\sqrt{2}$	0
$(4,1)$	6	1	1	-1
m_{9j}		32	32	64

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$	$g_{10,3}$
$o(g_{10j})$		8	8	16
$\lvert C_{\overline{G}}(g_{10j})\rvert$		32	32	16
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$			
$(1,1)$	8	1	1	1
$(2,1)$	8	$i\sqrt{2}$	$-i\sqrt{2}$	0
$(3,1)$	8	1	1	-1
m_{10j}		32	32	64

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$	$g_{11,3}$
$o(g_{11j})$		8	8	16
$\lvert C_{\overline{G}}(g_{11j})\rvert$		32	32	16
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$			
$(1,1)$	8	1	1	1
$(2,1)$	8	$i\sqrt{2}$	$-i\sqrt{2}$	0
$(3,1)$	8	1	1	-1
m_{11j}		32	32	64

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$
$o(g_{12j})$		12	12
$\lvert C_{\overline{G}}(g_{12j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$		
$(1,1)$	12	1	1
$(2,1)$	12	1	-1
m_{12j}		64	64

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		12	12
$\lvert C_{\overline{G}}(g_{13j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$		
$(1,1)$	12	1	1
$(2,1)$	12	1	-1
m_{13j}		64	64

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$
$o(g_{14j})$		12	12
$\lvert C_{\overline{G}}(g_{14j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$		
$(1,1)$	12	1	1
$(2,1)$	12	1	-1
m_{14j}		64	64

Remark 4.5. In the above matrices, if we omit the first column and the second row of every Fischer matrix $\mathcal{F}_i$, we obtain the corresponding Fischer matrix $\widetilde{\mathcal{F}_i}$ of the split extension $H = 2^6{:}((3^{1+2}{:}8){:}2)$. This shows that the group $H = 2^6{:}((3^{1+2}{:}8){:}2)$ has $\sum_{i=1}^{14} c(g_i) - 14 = 41 - 14 = 27$ conjugacy classes, which is also equal to $14 + 7 + 6 = |\mathrm{Irr}((3^{1+2}{:}8){:}2)| + |\mathrm{Irr}(QD_{16})| + |\mathrm{Irr}(D_{12})| =$ the number of irreducible characters of H.

5. The Character Table of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$

Through Sections 2, 3 and 4 we have found the following items:

- the conjugacy classes of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$ (Table 2),
- the fusions of classes of the inertia factors H_3 and H_4 into classes of G (Table 3),
- the character tables of the inertia factors $H_1 = H_2$, H_3 and H_4 (Tables 1, 4 and 5 respectively),
- the Fischer matrices of $\overline{G}$ (see Section 4).

The character table of $\overline{G}$ can be constructed easily by following the description of Subsection 3.1 of [2]. The character table of $\overline{G}$ is a $41 \times 41\mathbb{C}-$valued matrix. The full character table of $\overline{G}$ is available in the PhD thesis [1]. This character table is not yet incorporated into the GAP library but our aim is to do so. We conclude this section by remarking that a table showing the fusion of classes of $\overline{G}$ into classes of $U_5(2){:}2$ has been supplied in [1].

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