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PI-EXTENDING MODULES VIA NONTRIVIAL COMPLEX BUNDLES AND ABELIAN ENDOMORPHISM RINGS

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Abstract. A module is said to be PI-extending provided that every projection invariant submodule is essential in a direct summand of the module. In this paper, we focus on direct summands and indecomposable decompositions of PI-extending modules. To this end, we provide several counter examples including the tangent bundles of complex spheres of dimensions bigger than or equal to 5 and certain hyper surfaces in projective spaces over complex numbers and obtain results when the PI-extending property is inherited by direct summands. Moreover, we show that under some module theoretical conditions PI-extending modules with Abelian endomorphism rings have indecomposable decompositions. Finally, under suitable hypotheses, we apply our former results to obtain that the finite exchange property implies the full exchange property.

Keywords: Extending module, projective invariant, tangent bundle, exchange property.
MSC(2010): Primary: 16D50; Secondary: 16D70, 16D80.

1. Introduction

All rings are associative with unity and modules are unital right modules. We use R to denote such a ring and M to denote a right R-module. Recall that a module M is called *extending* or CS, or said to satisfy the C_1 condition if every submodule is essential in a direct summand of M (equivalently, every complement submodule is a direct summand). This condition has been proven to be an important common generalization of the injective, semisimple and uniform module notions. There have been numerous generalizations of the extending property including the C_{11} and PI-extending conditions. Recall that a module is called C_{11}-*module* [15] if each of its submodule has a complement which is a direct summand of the module. A module M is called PI-extending [3] if each projection invariant submodule (i.e., a submodule which is invariant

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under all idempotent endomorphisms of M) is essential in a direct summand of M.

From [15], every extending module is a C_{11}-module but the converse is not true in general. Moreover, the class of C_{11}-modules is properly contained in the class of PI-extending modules [3]. Also the class of PI-extending modules is closed under direct sums [3, Corollary 4.11] but not direct summands [3, Example 5.5].

In this paper, we obtain results which provide more counter examples on direct summands of the aforementioned generalized extending conditions. Incidentally, our result in this trend include the tangent bundles of complex spheres of dimensions bigger than or equal to 5 and certain hyper surfaces in projective spaces over complex numbers. In contrast to the former counter examples, we obtain results when the PI-extending property is inherited by direct summands. Moreover we prove that a PI-extending module M with C_3 which has Abelian endomorphism ring over a ring with acc on right annihilators of the form $r(m)$ where $m \in M$ has an indecomposable decomposition. In particular, our former result gives that the finite exchange property implies full exchange property. We also apply our decomposition result to the nonsingular modules.

Recall from [10], the following conditions for a right R-module M:

- C_2 : for each direct summand N of M and each monomorphism $\psi : N \to M$, the submodule $\psi(N)$ is a direct summand of M.
- C_3 : for all direct summands K and L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.

It is clear that C_2 implies C_3 but not conversely [10]. Following [10], a module M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \bigoplus_{i \in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$ for modules $B_i \leq A_i$. A family $\{N_i\}_{i \in I}$ of independent submodules of a module M is said to be a *local (direct) summand* if, for any finite subset $A \subseteq I$, $\bigoplus_{i \in A} N_i$ is a direct summand of M [4]. Further a ring is called *Abelian* if every idempotent is central. For any unexplained terminology, definition and notation see [1,2,4,10].

2. Direct summands of PI-extending modules

In this section we deal with whether direct summands of PI-extending modules inherit the PI-extending notion. To this end, our results yield an abundance to provide counter examples on using algebraic topology. In contrast, we prove when the direct summands of a PI-extending module enjoy with the property.

Lemma 2.1. [3, Corollary 3.2] *A module M is PI-extending if and only if for every projection invariant submodule N of M, there exists a direct summand K of M such that $N \cap K = 0$ and $N \oplus K$ is essential in M.*

Lemma 2.2. [3, Corollary 4.11] *Any direct sum of PI-extending modules is also a PI-extending module. In particular, any direct sum of uniform modules is a PI-extending module.*

Proposition 2.3. *Let R be a ring such that the right R-module R is a PI-extending module and every direct summand of a PI-extending module is a PI-extending module. Then every indecomposable projective right R-module is uniform.*

Proof. Let P be an indecomposable projective R-module. Then there exists a free R-module F such that $F = P \oplus P^{'}$ for some submodule $P^{'}$ of F. By Lemma 2.2, F is a PI-extending module and, by hypothesis, so does P. Thus, P is uniform. □

In view of Proposition 2.3, we are interested in when indecomposable projective modules are uniform. If R is a commutative Noetherian ring of Krull dimension 1, then every finitely generated indecomposable projective R-module is uniform, and hence, by Lemma 2.2, every finitely generated projective R-module is PI-extending [2, Theorem IV 2.5]. In contrast, there are many examples of right Noetherian domains R of Krull dimension 1 and finitely generated indecomposable projective right R-modules P which are not uniform and hence are not PI-extending despite the fact that they are direct summands of free right R-modules which are PI-extending (see [5, 6, 9]).

On the other hand, [3, Example 5.5] (see, also [14, Example 4]) provides a commutative Noetherian domain of Krull dimension 2 and a finitely generated indecomposable projective right R-module P which is not uniform. Actually the module P corresponds to the tangent bundle of a real sphere of odd dimension over its coordinate ring which is nontrivial. One might expect to have more counter-examples via nontrivial complex bundles we have the following two results in this trend.

Theorem 2.4. *Let $\mathbb{C}$ be the complex field and S be the polynomial ring $\mathbb{C}[z_0, z_1, \ldots, z_n, w_0, w_1, \ldots, w_n]$ where $n \in \mathbb{Z}$, $n \geq 2$. Let R be the ring S/Ss where $s = (\sum_j z_j w_j) - 1$. Then the right R-module $M = \bigoplus_{i=1}^{2n+2} R$ is a PI-extending module but Mcontains a direct summand Kwhich is not PI-extending.*

Proof. By Lemma 2.2, M_R is PI-extending. By [7, p.125, 5.4.9], the tangent bundle of sphere $S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_j z_j \bar{z_j} = 1\}$ is a nontrivial vector bundle (i.e. indecomposable R-module) K, say and $M_R \cong K \oplus K^{'}$

where $K^{'} \cong R$. Since K_R is not uniform, it is easy to see that K_R is not PI-extending. □

Surprisingly, we may provide more examples which based on certain hyper surfaces in projective spaces, $\mathbb{P}^{n+1}_{\mathbb{C}}$ over complex numbers.

Theorem 2.5. *Let X be the hyper surface in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined by the equation $x_0^m + x_1^m + \cdots + x_{n+1}^m = 0$ for $n \geq 2$. Let $R = \mathbb{C}[x_1, \ldots, x_{n+1}]/(\sum_{i=1}^{n+1} x_i^m + 1)$ be the coordinate ring. There exist PI-extending R-modules but contain direct summands which are not PI-extending for $m \geq n+2$.*

Proof. By [11], there are indecomposable projective R-modules of rank n over R. It follows that $F_R = K \oplus W$ where F_R is a free module, K is indecomposable and projective R-module of rank n. From Lemma 2.2, F_R is PI-extending. Now K_R is not uniform. Thus K_R is not PI-extending. □

In contrast to above facts, our next results in the rest of this section concern when a direct summand of a PI-extending module is a PI-extending module.

Proposition 2.6. *Let $M = M_1 \oplus M_2$. Then M_1 is PI-extending if and only if for every projection invariant submodule N of M_1 there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$ and $K \oplus N$ is an essential submodule of M.*

Proof. Suppose M_1 is PI-extending. Let N be any projection invariant submodule of M_1. By Lemma 2.1, there exists a direct summand L of M_1 such that $N \cap L = 0$ and $N \oplus L$ is essential in M_1. Then $M_1 = L \oplus L^{'}$ for some $L^{'}$ submodule of M_1. So $L \oplus M_2$ is a direct summand of M. It is clear that, $M_2 \subseteq K \oplus M_2$, $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2) \oplus N$ is essential in M. Conversely, suppose that M_1 has the stated property. Let H be a projection invariant submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$ and $K \oplus H$ is an essential submodule of M. Now $K = K \cap (M_1 \oplus M_2) = M_2 \oplus (K \cap M_1)$ so that $K \cap M_1$ is a direct summand of M, and hence also of M_1. Moreover, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$ which is an essential submodule of M_1. By Lemma 2.1, M_1 is PI-extending. □

Theorem 2.7. *Let $M = M_1 \oplus M_2$ be a PI-extending module such that M_2 is a projection invariant submodule of M and for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is a PI-extending module.*

Proof. Let N be any projection invariant submodule of M_1. By [3, Lemma 4.13], $N \oplus M_2$ is a projection invariant submodule of M. By Lemma 2.1, there exists a direct summand K of M such that $(N \oplus M_2) \cap K = 0$ and $N \oplus M_2 \oplus K$

is an essential submodule of M. Since $K \cap M_2 \subseteq K \cap (N \oplus M_2) = 0$, $K \oplus M_2$ is a direct summand of M by hypothesis. Now the result follows by Proposition 2.6. □

Corollary 2.8. *Let $M = M_1 \oplus M_2$ be a PI-extending module with C_3 condition. If M_2 is a projection invariant submodule of M, then M_1 is a PI-extending module.*

Proof. Immediate from Theorem 2.7. □

Corollary 2.9. *Let $M = M_1 \oplus M_2$ be a PI-extending module with C_2 condition. If M_2 is a projection invariant submodule of M, then M_1 is a PI-extending module.*

Proof. Since C_2 implies C_3, Corollary 2.8 yields the result. □

3. Decompositions into indecomposable submodules

Throughout this section we assume that modules have Abelian endomorphism rings. For any module M, S will stand for the $End(M_R)$. Since our main concern is based on projection invariant direct summands, we should point out that if S is Abelian then every direct summand is projection invariant. However, the reverse of this fact does not hold [3, Example 2.2(3)]. We focus our attention on decomposing a PI-extending module with C_3 (C_2) and with an Abelian endomorphism ring into indecomposable (and hence uniform) submodules. Specifically, we obtain that the finite exchange property implies the full exchange property. Recall that C_3 condition does not imply S is Abelian, see for example [8, Example 2.10]. Now our following easy example illustrates that S is being Abelian and C_3 condition are different from each other.

Example 3.1. Let M be the $\mathbb{Z}$-module $\bigoplus_{i=1}^{\infty} \mathbb{Z}$. Then $End(M_{\mathbb{Z}}) \cong \prod_{i=1}^{\infty} \mathbb{Z}$ which is Abelian. However, $M_{\mathbb{Z}}$ does not satisfy C_3, by [13, Example 9]

Next we mention a result without proof which is recently obtained in [8, Theorem 2.12]. First note that Δ will denote the ideal $\{f \in S \,|\, ker f$ is essential in $M\}$.

Theorem 3.2. *Let M_R be a PI-extending module with C_2 condition and let S be Abelian. Then S/Δ is a (von Neumann) regular ring.*

We continue investigation on PI-extending modules with $C_3(C_2)$. Let us begin with the following example which makes it clear that the converse of Theorem 3.2 does not hold.

Example 3.3. Let V be an infinite dimensional vector space over a division ring D and let $S = End_D(V)$. Let $\{x_1, x_2, \dots\}$ be a basis of V. It is clear that V_D is a PI-extending module with C_2. Since $\Delta = 0$, S is a von Neumann regular ring. However, S is not an Abelian ring. To see this, define $\sigma : V \to V$ by $\sigma(x_i) = x_{i+1}$ for $i = 1, 2, 3, \dots$, and $\pi : V \to x_i D$ by $\pi(x_i) = x_i$ and $\pi(x_j) = 0$ for $i \neq j$. Now $\sigma\pi(x_i) = \sigma(x_i) = x_{i+1}$ but $\pi\sigma(x_i) = \pi(x_{i+1}) = 0$.

Proposition 3.4. *Let M be a PI-extending module with C_3. If S is Abelian then every direct summand of M is PI-extending.*

Proof. Let N be a direct summand of M. Then $M = N \oplus N^{'}$ for some submodule $N^{'}$ of M. Let $\pi : M \to N$ be the canonical projection. Then $N^{'} = ker\pi$. Let $f^2 = f \in S$. Thus $f(N^{'}) = f(ker\pi)$. Since S is Abelian $f(N^{'}) = f(ker\pi) \subseteq ker\pi = N^{'}$, so $N^{'}$ is a projection invariant submodule of M. Let K be any projection invariant submodule of N. From [3, Lemma 4.13], $K \oplus N^{'}$ is a projection invariant submodule of M. By Lemma 1.1, there exists a direct summand L of N such that $(K \oplus N^{'}) \cap L = 0$ and $K \oplus N^{'} \oplus L$ is essential in M. Since M has C_3, $N^{'} \oplus L$ is a direct summand of M. Note that $N^{'} \oplus L = N^{'} \oplus \pi(L)$ and hence $\pi(L)$ is a direct summand of N. Moreover, $K \oplus N^{'} \oplus L = K \oplus \pi(L) \oplus N^{'}$ is essential in M which gives that $K \oplus \pi(L)$ is essential in N. Thus N is PI-extending. □

Our next result and its corollary are based on relative injectivity. We obtain that a direct summand of a PI-extending module is also PI-extending.

Proposition 3.5. *Let M be a PI-extending module with S is Abelian. Let K be any direct summand of M such that M/K is K-injective. Then K is PI-extending.*

Proof. There exists a submodule $K^{'}$ of M such that $M = K \oplus K^{'}$. By hypothesis, $K^{'}$ is K-injective. Let L be a direct summand of M such that $L \cap K^{'} = 0$. By [4, Lemma 7.5] there exists a submodule H of M such that $H \cap K^{'} = 0$, $M = H \oplus K^{'}$ and $L \subseteq H$. Since S is Abelian L is a projection invariant submodule of M. Now, $M = L \oplus L^{'}$ for some $L^{'} \leq M$ so $H = H \cap (L \oplus L^{'}) = L \oplus (H \cap L^{'})$ which gives that L is a direct summand of H. Hence $L \oplus K^{'}$ is a direct summand of M. So by Theorem 2.7, K is PI-extending. □

Our next example shows that the converse of Proposition 3.5 is not true.

Example 3.6. [8, Example 2.10] Let $R = \mathbb{Z}_{(p)}$ be the localization of integers $\mathbb{Z}$ at a prime p. Put $M_R = \mathbb{Z}_{(p)} \oplus \mathbb{Z}(p^\infty)$. Then M_R is PI-extending module and $\mathbb{Z}(p^\infty)$ is an injective direct summand of M which is also PI-extending. Now, take $f = \begin{bmatrix} 0 & 0 \\ h & 1 \end{bmatrix}$ and $g = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where $h : \mathbb{Z}_{(p)} \to \mathbb{Z}(p^\infty)$ is defined by $h(1) = \frac{\overline{1}}{p}$. Then $f, g \in S$ and $f^2 = f$. It is clear that $fg \neq gf$, i.e., S is not Abelian.

Corollary 3.7. *Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. Let S be Abelian. Then M is PI-extending if and only if M_1 is PI-extending.*

Proof. Assume M is PI-extending. By Proposition 3.5, M_1 is PI-extending. Conversely, if M_1 is PI-extending and M_2 is an injective submodule, then M is PI-extending by Lemma 2.2. □

Now we obtain an indecomposable decomposition of a PI-extending module. Furthermore we apply our result to get exchange property.

Theorem 3.8. *Let R be a ring and M be an R-module such that R satisfies acc on right annihilators of the form $r(m)$, where $m \in M$. If M is PI-extending module with C_3 and S is Abelian, then M has an indecomposable decomposition.*

Proof. Let $\{X_\lambda \mid \lambda \in I\}$ be an independent family of submodules of M and let $X = \bigoplus_{\lambda \in I} X_\lambda$ be a local summand of M. Define $\pi_k : X \to \bigoplus_{k \in I, k \neq \lambda} X_k$. Thus $f(X) = f(\bigoplus_{\lambda \in I} X_\lambda) = \bigoplus_{\lambda \in I} f(X_\lambda) = \bigoplus_{\lambda \in I} f(ker\pi_\lambda)$ where $f^2 = f \in S$. Since S is Abelian, $f(ker\pi_\lambda) \subseteq ker\pi_\lambda$. Then $f(X) = \bigoplus_{\lambda \in I} f(ker\pi_\lambda) \subseteq \bigoplus_{\lambda \in I} ker\pi_\lambda = X$. Thus X is a projection invariant submodule of M. By Lemma 2.1, there exists a direct summand K of M such that $X \cap K = 0$ and $X \oplus K$ is essential in M. Now, consider $X \oplus K$. For any finite subset $A \subseteq I$, $Y = \bigoplus_{\lambda \in A} X_\lambda$ is a direct summand of M. Since M has C_3, $Y \oplus K$ is a direct summand of M. Thus $X \oplus K$ is a local summand of M. By [15, Lemma 4.5], $X \oplus K$ is a complement in M. But $X \oplus K$ is essential submodule of M, then $M = X \oplus K$. So, [10, Theorem 2.17] yields that M has an indecomposable decomposition. □

Corollary 3.9. *Let R be a ring and M be an R-module such that R satisfies acc on right annihilators of the form $r(m)$, $m \in M$. If M is a PI-extending module with C_3 and S is Abelian, then M is a direct sum of uniform submodules.*

Proof. Since an indecomposable PI-extending module is uniform, the result follows by Theorem 3.8 and Proposition 3.4. □

Corollary 3.10. *Let R be a right Noetherian ring and M be a right R-module with C_3 and S is Abelian. Then M is PI-extending if and only if M is a direct sum of uniform submodules.*

Proof. Immediate by Corollary 3.9 and Lemma 2.2. □

Proposition 3.11. *Let R be a ring and M be an R-module such that R satisfies acc on $r(m)$ where $m \in M$. If M_R is PI-extending with C_3 and S is Abelian then the finite exchange property implies full exchange property.*

Proof. The proof follows from Theorem 3.8 and [16, Corollary 6]. □

Example 3.12. Let P be the set of prime integers. Let $R = \prod_{p \in P} \mathbb{Z}/\mathbb{Z}p$ be the product of fields. Then R_R has finite exchange property. Moreover, R_R is injective but it is not a direct sum of indecomposable modules [12, Example 13]

Theorem 3.13. *The following statements are equivalent for a nonsingular PI-extending module M with C_3 and S is Abelian,*

(i) *M has an indecomposable decomposition,*
(ii) *Every finitely generated submodule of M has finite uniform dimension,*
(iii) *Every cyclic submodule of M has finite uniform dimension,*
(iv) *R satisfies acc on $r(m)$ where $m \in M$.*

Proof. $(i) \Rightarrow (ii)$. There exists an index set I and indecomposable submodules M_i $(i \in I)$ of M such that $M = \bigoplus_I M_i$. By Proposition 3.4, M_i is also PI-extending. It follows that M_i is uniform. If L is a finitely generated submodule of M then $L \subseteq \bigoplus_{i \in J} M_i$ for some finite subset J of I and hence L has finite uniform dimension.

$(ii) \Rightarrow (iii)$. It is obvious.

$(iii) \Rightarrow (iv)$. Let $m \in M$. Suppose that $r(m)$ is essential in a right ideal A of R. Let $a \in A$. Then there exists an essential right ideal E of R_R such that $aE \subseteq r(m)$. It follows that $maE = 0$ and hence $ma = 0$ so $a \in r(m)$. Thus $r(m) = A$. Therefore $r(m)$ is a complement in R-module R for each $m \in M$. Moreover $R/r(m) \cong mR$, gives that the R-module $R/r(m)$ has finite uniform dimension. Now (iv) follows by [4, Theorem 5.10].

$(iv) \Rightarrow (i)$. It follows from Theorem 3.8. □

Corollary 3.14. *Let M be a nonsingular module such that mR has finite uniform dimension for each $m \in M$. If M is PI-extending with C_3 and S is Abelian then the finite exchange property implies full exchange property.*

Proof. By Theorem 3.13 and [16, Corollary 6]. □

Corollary 3.15. *Let R be a ring with finite uniform dimension and M be a nonsingular PI-extending module with C_3 and S is Abelian. Then the finite exchange property implies full exchange property.*

Proof. Let R and M be as stated. Let $m \in M$. Then $r(m)$ is a complement in the right R-module R. Hence the R-module $R/r(m)$ has finite uniform dimension. Since $mR \cong R/r(m)$, Corollary 3.14 completes the proof. □

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