Bulletin of the Iranian Mathematical Society Vol. 31 *No.* 2 (*2005*), *pp 71-81.*

INHOMOGENEOUS TWO-PARAMETER ABSTRACT CAUCHY PROBLEM

M. KHANEHGIR*, M. JANFADA AND A. NIKNAM

ABSTRACT. We use the semigroup theory to study the inhomogeneous two-parameter abstract Cauchy problem 2-IACP

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1,t_2) = H_i u(t_1,t_2) + f(t_1,t_2) \\ i=1,2 \quad t_i \in [0\,,\,a_i) \\ u(0,0)=u_0,\ u_0 \in X, \end{cases}$$

where X is a Banach space, $H_i : D(H_i) \subseteq X \to X$, $i=1,2$, is a densely-defined closed linear operator and $f:[0,a_1)\times[0,a_2)\to X$ is a continuous function $(a_1,a_2>0)$. We discuss the existence and uniqueness of solution of 2-IACP. In fact, we claim that if $(H_1\,,H_2)$ is the generator of a C_0-two-parameter semigroup $\{W(t_1\,,t_2)\}_{t_1,t_2\geq 0}$, then 2-IACP with some conditions has a unique solution.

1. Introduction

Let X be a Banach space, $B(X)$ is the Banach space of all bounded linear operators on X and $\mathbb{R}^n_+ = \{(t_1,t_2,...,t_n) : t_i \geq 0,\ i=1,2,...,n\}$. By an n-parameter semigroup of operators we mean a homomorphism $W\colon (\mathbb{R}^n_+,+) \to B(X)$ for which $W(0)=I$ and denote it by $(X,\mathbb{R}^n_+,W)$. Let now $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. Trivially for $s\in\mathbb{R}^+$, the component $u_i(s)=W(se_i)$ of W defines a one-parameter semigroup of

MSC(2000): Primary 34A12, 34A30; Secondary 47D03, 47D99, 35D05

Keywords: N-parameter semigroup, Homogeneous n-parameter abstract Cauchy problem, Inhomogeneous two-parameter abstract Cauchy problem

Received: 02 June 2006 , Revised: 22 June 2006

*Corresponding author

operators, $i = 1, 2, ..., n$. Also for each integers $0 \leq i, j \leq n$, the n-parameter semigroup property implies that, $u_i(s)u_j(s') = u_j(s')u_i(s)$. The n-parameter semigroup $(X, \mathbb{R}_+^n, W)$ is called strongly (respectively, uniformly) continuous if for each $i = 1, 2, ..., n$, the one-parameter components $u_i(s) = W(se_i)$ are strongly (respectively, uniformly) continuous. One can prove that the n-parameter semigroup $(X, \mathbb{R}_+^n, W)$ is strongly continuous if and only if $\lim_{t\in\mathbb{R}_+^n, t\to 0} W(t)x = x$, for all $x\in X$, and it is uniformly continuous if and only if $\lim_{t\in\mathbb{R}_+^n, t\to 0} W(t) = I$. Consider an n-parameter semigroup of operators $(X, \mathbb{R}_+^n, W)$ and let $H_i, i = 1, 2, ..., n$, be the infinitesimal generator of the component semigroup $\{u_i(t)\}_{t\geq 0}$ of W, $i = 1, 2, ..., n$. We shall think of $(H_1, H_2, ..., H_n)$ as the infinitesimal generator of $(X, \mathbb{R}_+^n, W)$.
N-parameter semigroups of operators introduced by Hille in 1944 and one can see some of their properties in [3]. For some new results in the theory of n-parameter semigroups and their applications one can see [4] and [5].
If W is a C_0-n-parameter semigroup of operators then by the Hille-Yosida theorem, H_i, $i = 1, 2, ..., n$, is a closed and densely defined operator.
Let $D(H_i) \subseteq X$ be the domain of H_i, $i = 1, 2, ..., n$. For $x\in X_1 = \bigcap_{i=1}^n D(H_i)$ we define $\| x \|_1 = \| x \| + \sum_{i=1}^n \| H_i x \|$.

In [1] one can see that if $(X, \mathbb{R}_+^n, W)$ is a C_0-n-parameter semigroup of operators with the infinitesimal generator $(H_1, H_2, ..., H_n)$ then
(a) There is $M \geq 1$ and $\omega_i\in\mathbb{R}$, $i = 1, 2, ..., n$, such that $\|W(t_1, t_2, ..., t_n)\| \leq Me^{\sum_{i=1}^n t_i\omega_i}$. So $\|W(t_1, ..., t_n)\|$ is bounded in any compact subset of $\mathbb{R}_+^n$;
(b) If $x\in D(H_i)$, so does $W(t)x$, for each $t\in\mathbb{R}_+^n$, and $H_iW(t)x = W(t)H_ix$, $i = 1, 2, ..., n$;
(c) $X_1 = \bigcap_{i=1}^n D(H_i)$ is a dense subspace of X and moreover $(X_1, \|.\|_1)$ is a Banach space;
(d) For each integers $1 \leq i, j \leq n, D(H_iH_j)\bigcap D(H_i) \subseteq D(H_jH_i)$ and for every $x \in D(H_iH_j)\bigcap D(H_i)$, $H_iH_jx = H_jH_ix$.

Also we have the Hille-Yosida theorem for n-parameter semigroups as follows ([4]):
$(H_1, ..., H_n)$ is the infinitesimal generator of a C_0-n-parameter semigroup $\{W(t)\}_{t\in\mathbb{R}_+^n}$ satisfying $\| W(t_1, ..., t_n) \| \leq M_0e^{\sum_{i=1}^n t_i\omega_i}$ for some $M_0 \geq 1$ and $\omega_i > 0$,$i = 1, ..., n$, if and only if
(a) H_i is a closed densely defined operator, $i = 1, ..., n$, and

$R(\lambda', H_j)R(\lambda, H_i) = R(\lambda, H_i)R(\lambda', H_j)$, for all $\lambda > \omega_i$, $\lambda' > \omega_j$ and integers $1 \leq i, j \leq n$,
(b) For each $i = 1, ..., n$, $[\omega_i, \infty) \subseteq \rho(H_i)$ and there is $M \geq 1$ such that $\| R(\lambda, H_i)^k \| \leq \frac{M}{(Re\lambda - \omega_i)^k}$, $i = 1, ..., n$, and $Re\lambda > \omega_i$.

Now we introduce n-ACP and its solution ([4]).
Suppose that X is a Banach space, H_i are closed linear operators from $D(H_i) \subseteq X$ into X and $a_i > 0$, $i = 1, 2, ..., n$. Then a continuous X-valued function u:$[0, a_1] \times ... \times [0, a_n] \to X$with continuous partial derivatives which satisfies the following n-parameter abstract Cauchy problem n-ACP

$$(1.1) \qquad \begin{cases} \frac{\partial}{\partial t_i} u(t_1, t_2, ..., t_i, ..., t_n) = H_i u(t_1, ..., t_n) \\ i = 1, 2, ..., n \qquad t_i \in [0 \, , \, a_i] \\ u(0) = u_0 \qquad u_0 \in \bigcap_{i=1}^{n} D(H_i) \end{cases}$$

is called a solution of the initial value problem (1.1).
It is proved ([4, Theorem. 2.1]) that if $(H_1, H_2, ..., H_n)$ is the infinitesimal generator of a C_0-n-parameter semigroup $(X, I\!R^n_+, W)$, then (1.1) has the unique solution $u(t_1, t_2, ..., t_n) = W(t_1, t_2, ..., t_n)u_0$ for each $u_0 \in \bigcap_{i=1}^{n} D(H_i)$, where $(t_1, t_2, ..., t_n) \in [0, a_1] \times ... \times [0, a_n]$,
For convenience we denote by I_a the positive two-cell $[0, a_1] \times [0, a_2]$ where $a = (a_1, a_2) \in I\!R^2_+$. So one can see that for a closed linear operator $A : D(A) \subseteq X \to X$, the two-parameter initial value problem

$$(1.2) \qquad \begin{cases} \frac{\partial}{\partial t_1} u(t_1, t_2) - \frac{\partial}{\partial t_2} u(t_1, t_2) = Au(t_1, t_2) \qquad (t_1, t_2) \in I_a \\ u(0) = x \qquad\qquad x \in D(A), \end{cases}$$

doesn't have a unique solution for each $x \in D(A)$ in both I_a and $I_{a'}$ (this can be proved in a similar way as in the proof of Theorem 2.5 of [4]). The initial value problem (1.2) can have a solution, for example if (H_1, H_2) is the generator of a C_0-two-parameter semigroup $\{W(t_1, t_2)\}_{t_1, t_2 \geq 0}$ and $A = H_1 - H_2$ then obviously $u(t_1, t_2) = W(t_1, t_2)x$ is a solution of (1.2) in any positive two-cell I_a and for the initial value $x \in \bigcap_{i=1}^{2} D(H_i) = D(A)$.

In this paper we intend to study the inhomogeneous two-parameter abstract Cauchy problem 2-IACP

$$(1.3)\quad \begin{cases} \frac{\partial}{\partial t_i}u(t_1,t_2) = H_i u(t_1,t_2) + f(t_1,t_2) \\ \\ u(0,0) = u_0, \quad u_0 \in \bigcap_{i=1}^{2} D(H_i), \end{cases} \qquad t_i \in [0\,,\,a_i),\ \ i = 1,2,$$

where $H_i : D(H_i) \subseteq X \to X$, $i = 1,2,$ is a densely-defined closed linear operator and $f : [0,a_1) \times [0,a_2) \to X$ is a continuous function and $a_1, a_2 > 0$.
By a (classical) solution of 2-IACP we mean a continuous X-valued function $u : [0,a_1) \times [0,a_2) \to X$ having continuous partial derivatives such that $u(t_1,t_2) \in \bigcap_{i=1}^{2} D(H_i)$ for all $(t_1,t_2) \in (0,a_1) \times (0,a_2)$ and u satisfies 2-IACP. In the next section we study conditions under which 2-IACP has a unique solution.

We end this section with the definition of the Bochner line integral that we use in the next section. Suppose that $M(x,y)$ and $N(x,y)$ are continuous functions of two variables from the open disk B in $I\!R^2$ to a Banach space X and suppose also that C is a curve in B with parametric equations

$$\begin{cases} x = f(t) \\ \\ y = g(t) \end{cases} \qquad a \le t \le b$$

Such that f, g have continuous first derivative on $[a,b]$. In this case Bochner line integral $M(x,y)dx + N(x,y)dy$ on C which is defined by

$$\oint_C M(x,y)dx + N(x,y)dy$$

is

$$\int_a^b [M(f(t),g(t))f'(t) + N(f(t),g(t))g'(t)]dt.$$

Also it is easily proved that if $\frac{\partial M}{\partial y}(x,y)$ and $\frac{\partial N}{\partial x}(x,y)$ are continuous in B and there exists a function ϕ such that $\nabla\phi(x,y) = M(x,y)\underline{i} + N(x,y)\underline{j}$, where $\underline{i}$ and $\underline{j}$ are the unit vectors of axes X and Y, respectively, and C is a sectionally smooth curve (it means $f(t)$ and $g(t)$ are differentiable functions except probably in the finite points) in B from the point (x_1,y_1) to point (x_2,y_2), then the Bochner line integral

$$\oint_C M(x,y)dx + N(x,y)dy$$

is independent of the path C and

$$\oint_C M(x,y)dx + N(x,y)dy = \phi(x_2,y_2) - \phi(x_1,y_1).$$

2. Inhomogeneous two-parameter abstract Cauchy problem

Consider 2-IACP as we mentioned before.
Let $\{W(t_1,t_2)\}_{t_1,t_2\geq 0}$ be the C_0-two-parameter semigroup generated by (H_1,H_2) and let $u(t_1,t_2)$ be a solution of 2-IACP. Then the X-valued function of two-variables $g(s_1,s_2) = W(t_1-s_1,t_2-s_2)u(s_1,s_2)$ has partial derivatives for $0 < s_1 < t_1$, $0 < s_2 < t_2$ and for i=1,2, we have

$$\begin{aligned} \frac{\partial g}{\partial s_i} &= -H_iW(t_1-s_1,t_2-s_2)u(s_1,s_2) \\ &\quad +W(t_1-s_1,t_2-s_2)H_iu(s_1,s_2) \\ &\quad +W(t_1-s_1,t_2-s_2)f(s_1,s_2) \\ &= W(t_1-s_1,t_2-s_2)f(s_1,s_2). \end{aligned}$$

So one obtains

$$dg = W(t_1-s_1,t_2-s_2)f(s_1,s_2)(ds_1+ds_2). \tag{2.1}$$

If 2-IACP has a solution and the Bochner line integral of the above assertion from the point (0,0) to point (t_1,t_2) exists (for example if $f(t,t_0) \in L^1(0,a_1,X)$ for each $t_0 \in [0,a_2)$, and $f(t_0,t) \in L^1(0,a_2,X)$ for each $t_0 \in [0,a_1)$), then this Bochner line integral is independent of the path that connecting these two points to each other. So by line integrating of two-sided of the assertion (2.1) from (0,0) to (t_1,t_2) it yields to

$$u(t_1,t_2) = W(t_1,t_2)u_0 + \oint_{(0,0)}^{(t_1,t_2)} W(t_1-s_1,t_2-s_2)f(s_1,s_2)(ds_1+ds_2). \tag{2.2}$$

This proves the uniqueness of the solution. Now we introduced a path that we use in the Bochner line integral

$$v(t_1,t_2) = \oint_{(0,0)}^{(t_1,t_2)} W(t_1-s_1,t_2-s_2)f(s_1,s_2)(ds_1+ds_2).$$

This path contains two line segments

$$\begin{cases} s_1 = t & 0 \le t \le t_1 \\ s_2 = 0, & \end{cases}$$

$$\begin{cases} s_1 = t_1 & \\ s_2 = t & 0 \le t \le t_2. \end{cases}$$

We calculate the Bochner line integral $v(t_1, t_2)$ on this special path and denote it by $V(t_1, t_2)$, so we have

$$V(t_1,t_2) = \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt. \tag{2.3}$$

Now if $V(t_1, t_2)$ exists then for every $u_0 \in X$ the 2-IACP has at most one solution and if it has a solution, then $V(t_1, t_2) = v(t_1, t_2)$ and the solution of 2-IACP is given by (2.2). It is natural to consider the right-hand side of (2.3) as a generalized solution of 2-IACP even if it has not partial derivatives relative to t_1 or t_2, and does not strictly satisfy the equation in the sence of (classical) solution. We therefore give the following definition:

Definition 2.1. Let (H_1, H_2) be the infinitesimal generator of a C_0-two-parameter semigroup $\{W(t_1, t_2)\}_{t_1, t_2 \ge 0}$. Let $u_0 \in X$ and $f(t_1, t) \in L^1(0, a_2; X)$ and $f(t, t_2) \in L^1(0, a_1; X)$ for each $t_1 \in [0, a_1)$ and $t_2 \in [0, a_2)$. The continuous function

$$u(t_1,t_2) = W(t_1,t_2)u_0 + \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt$$

is called the mild solution of the 2-IACP.

We will be interested in imposing further conditions on f so that for $u_0 \in \bigcap_{i=1}^2 D(H_i)$, the mild solution becomes a (classical) one.
Now we show that the continuity of f, in general is not sufficient to ensure the existence of solutions of 2-IACP for $u_0 \in \bigcap_{i=1}^2 D(H_i)$. Indeed, let (H_1, H_2) be the infinitesimal generator of a C_0-two-parameter semigroup $\{W(t_1, t_2)\}_{t_1, t_2 \ge 0}$ and let $x \in X$ be such that $W(t_1, t_2)x$ does not belong to $D(H_1)$ for any $t_1, t_2 \ge 0$. Let $f(s_1, s_2) = W(s_1, s_2)x$. Then $f(s_1, s_2)$ is continuous for $s_1, s_2 \ge 0$. Consider the following 2-IACP

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1,t_2) = H_i u(t_1,t_2) + W(t_1,t_2)x & i = 1, 2 \\ u(0,0) = 0. & \end{cases} \tag{2.4}$$

We claim that (2.4) has no solution even though $u(0,0) = 0 \in \bigcap_{i=1}^{2} D(H_i)$. Indeed, the mild solution of (2.4) is

$$u(t_1,t_2) = \oint_{(0,0)}^{(t_1,t_2)} W(t_1-s_1,t_2-s_2)W(s_1,s_2)x(ds_1+ds_2) = (t_1+t_2)W(t_1,t_2)x.$$

But $(t_1+t_2)W(t_1,t_2)x$ does not have partial derivative relative to t_1 for $t_1 > 0$ and therefore $u(t_1,t_2)$ cannot be the solution of (2.4).

Thus in order to prove the existence of solutions of 2-IACP we have to require more than just the continuity of f. In the following theorem we state a general criterion for the existence of solutions of 2-IACP. In fact we prove that if $u_0 \in \bigcap_{i=1}^{2} D(H_i)$ and f has some conditions then

$$u(t_1,t_2) = W(t_1,t_2)u_0 + \int_0^{t_1} W(t_1-t,t_2)f(t,0)dt + \int_0^{t_2} W(0,t_2-t)f(t_1,t)dt$$

is a solution of 2-IACP. Our technique for proving theorem 2.2 is based on Pazy's technique for the one-parameter case [6].

Theorem 2.2. *Let (H_1,H_2) be the infinitesimal generator of a C_0-two-parameter semigroup $\{W(t_1,t_2)\}_{t_1,t_2\geq 0}$ and let f be a continuous function on $[0,a_1) \times [0,a_2)$ such that $f(t_1,t) \in L^1(0,a_2;X)$ for each $t_1 \in [0,a_1)$ and $f(t,t_2) \in L^1(0,a_1;X)$ for each $t_2 \in [0,a_2)$,* rang$(f) \subseteq \bigcap_{i=1}^{2} D(H_i)$ *and f has continuous partial derivatives and satisfies the following partial differential equation*

$$\frac{\partial}{\partial t_1} f(t_1,t_2) - \frac{\partial}{\partial t_2} f(t_1,t_2) = (H_1 - H_2)f(t_1,t_2).$$

Let

$$V(t_1,t_2) = \int_0^{t_1} W(t_1-t,t_2)f(t,0)dt + \int_0^{t_2} W(0,t_2-t)f(t_1,t)dt$$

for $0 \leq t_1 \leq a_1$, $0 \leq t_2 \leq a_2$.

Then 2-IACP has a (classical) solution u on $[0,a_1) \times [0,a_2)$ for every $u_0 \in \bigcap_{i=1}^{2} D(H_i)$ if one of the following conditions is satisfied:
(i) *$V(t_1,t_2)$ has continuous partial derivatives on $(0,a_1) \times (0,a_2)$.*
(ii) *$V(t_1,t_2) \in \bigcap_{i=1}^{2} D(H_i)$ for $0 < t_1 < a_1$, $0 < t_2 < a_2$ and $H_1V(t_1,t_2)$ and $H_2V(t_1,t_2)$ are continuous on $(0,a_1) \times (0,a_2)$.*
If 2-IACP has a (classical) solution u on $[0,a_1) \times [0,a_2)$ for some $u_0 \in$

$\bigcap_{i=1}^{2} D(H_i)$, *then* $V(t_1,t_2) = v(t_1,t_2)$ *satisfies both* (i) *and* (ii).

Proof. If 2-IACP has a solution u for some $u_0 \in \bigcap_{i=1}^{2} D(H_i)$, then this solution is given by (2.2). Consequently $v(t_1,t_2) = u(t_1,t_2) - W(t_1,t_2)u_0$ has partial derivatives for $t_1 > 0$ and $t_2 > 0$ and we have

$$\frac{\partial}{\partial t_i} v(t_1,t_2) = \frac{\partial}{\partial t_i} u(t_1,t_2) - W(t_1,t_2)H_i u_0, \qquad i = 1,2.$$

Obviously the above derivatives are continuous on $(0,a_1)\times(0,a_2)$. Therefore (i) is satisfied. Also if $u_0 \in \bigcap_{i=1}^{2} D(H_i)$, then $W(t_1,t_2)u_0 \in \bigcap_{i=1}^{2} D(H_i)$ for $t_1,t_2 \geq 0$ and therefore $v(t_1,t_2) = u(t_1,t_2) - W(t_1,t_2)u_0 \in \bigcap_{i=1}^{2} D(H_i)$ for $t_1,t_2 > 0$ and

$$\begin{aligned} H_i v(t_1,t_2) &= H_i u(t_1,t_2) - H_i W(t_1,t_2)u_0 \\ &= \tfrac{\partial}{\partial t_i} u(t_1,t_2) - f(t_1,t_2) - W(t_1,t_2)H_i u_0 \end{aligned}$$

is continuous on $(0,a_1)\times(0,a_2)$. Thus (ii) also is satisfied.
Now we show that if $V(t_1,t_2)$ satisfies one of the conditions (i) or (ii), then $u(t_1,t_2) = W(t_1,t_2)u_0 + V(t_1,t_2)$ is the unique solution of 2-IACP. For $V(t_1,t_2)$ we have

$$\begin{aligned} \tfrac{W(h,0)-I}{h} V(t_1,t_2) = \; & \tfrac{V(t_1+h,t_2)-V(t_1,t_2)}{h} \\ & -\tfrac{1}{h}\int_{t_1}^{t_1+h} W(t_1+h-t,t_2)f(t,0)dt \\ & -\tfrac{1}{h}\int_{0}^{t_2} W(0,t_2-t)f(t_1+h,t)dt \\ & +\tfrac{1}{h}\int_{0}^{t_2} W(h,t_2-t)f(t_1,t)dt \end{aligned} \tag{2.5}$$

By the continuity of f the second term on the right-hand side of (2.5) tends to $W(0,t_2)f(t_1,0)$ when h tends to zero. Also by adding

$$\pm\frac{1}{h}\int_0^{t_2} W(0,t_2-t)f(t_1,t)dt$$

to the last term of the right-hand side of (2.5), and letting h goes to zero we obtain

$$\begin{aligned} H_1 V(t_1,t_2) = \; & \tfrac{\partial}{\partial t_1} V(t_1,t_2) - W(0,t_2)f(t_1,0) \\ & -\int_0^{t_2} W(0,t_2-t)\tfrac{\partial f}{\partial t_1}(t_1,t)dt \\ & +\int_0^{t_2} H_1 W(0,t_2-t)f(t_1,t)dt. \end{aligned}$$

So we have

$$H_1V(t_1,t_2) - \frac{\partial}{\partial t_1}V(t_1,t_2) = -W(0,t_2)f(t_1,0) + \int_0^{t_2} W(0,t_2-t)[H_1 f(t_1,t) - \frac{\partial f}{\partial t_1}(t_1,t)dt]. \tag{2.6}$$

Now we show that the right-hand side of (2.6) is equal to $-f(t_1,t_2)$.

$$\begin{aligned}
&W(0,t_2)f(t_1,0) - \int_0^{t_2} W(0,t_2-t)[H_1 f(t_1,t) - \tfrac{\partial f}{\partial t_1}(t_1,t)dt]\\
&= W(0,t_2)f(t_1,0) - \int_0^{t_2} W(0,t_2-t)[H_2 f(t_1,t) - \tfrac{\partial f}{\partial t_2}(t_1,t)dt]\\
&= W(0,t_2)f(t_1,0) - \int_0^{t_2} H_2W(0,t_2-t)f(t_1,t) - W(0,t_2-t)\tfrac{\partial f}{\partial t_2}(t_1,t)dt\\
&= W(0,t_2)f(t_1,0) + \int_0^{t_2} \tfrac{dW(0,t_2-t)}{dt}f(t_1,t)dt + W(0,t_2-t)\tfrac{df(t_1,t)}{dt}dt\\
&= W(0,t_2)f(t_1,0) + f(t_1,t_2) - W(0,t_2)f(t_1,0) = f(t_1,t_2).
\end{aligned}$$

So we obtain

$$H_1V(t_1,t_2) = \frac{\partial}{\partial t_1}V(t_1,t_2) - f(t_1,t_2). \tag{2.7}$$

On the other hand it is easy to verify that for $h > 0$ the identity

$$\frac{W(0,h)-I}{h}V(t_1,t_2) = \frac{V(t_1,t_2+h)-V(t_1,t_2)}{h} - \frac{1}{h}\int_{t_2}^{t_2+h} W(0,t_2+h-t)f(t_1,t)dt \tag{2.8}$$

holds.

By the continuity of f it is clear that the second term on the right-hand side of (2.8) has the limit $f(t_1,t_2)$ as $h \to 0$. So we have

$$H_2V(t_1,t_2) = \frac{\partial}{\partial t_2}V(t_1,t_2) - f(t_1,t_2). \tag{2.9}$$

If $V(t_1,t_2)$ has continuous partial derivatives on $(0,a_1)\times(0,a_2)$, then it follows from (2.7) and (2.9) that $V(t_1,t_2) \in \bigcap_{i=1}^2 D(H_i)$ for $0 < t_1 < a_1$, $0 < t_2 < a_2$ and since $V(0,0) = 0$ it follows that $u(t_1,t_2) = W(t_1,t_2)u_0 + V(t_1,t_2)$ is the solution of 2-IACP for $u_0 \in \bigcap_{i=1}^2 D(H_i)$. If $V(t_1,t_2) \in \bigcap_{i=1}^2 D(H_i)$ then it follows from (2.4) and (2.7) that $V(t_1,t_2)$ has partial derivatives from the right at t_1 and t_2 and the right partial derivative $\frac{\partial^+}{\partial t_i}V(t_1,t_2)$, $i = 1,2$, of V satisfies the equation $\frac{\partial^+}{\partial t_i}V(t_1,t_2) = H_iV(t_1,t_2) + f(t_1,t_2)$. Since $\frac{\partial^+}{\partial t_i}V(t_1,t_2)$, $i = 1,2$, is continuous, $V(t_1,t_2)$ has continuous partial derivatives at t_1 and t_2

and $\frac{\partial}{\partial t_i}V(t_1,t_2) = H_iV(t_1,t_2) + f(t_1,t_2)$. Since $V(0,0) = 0$, $u(t_1,t_2) = W(t_1,t_2)u_0 + V(t_1,t_2)$ is the solution of 2-IACP for $u_0 \in \bigcap_{i=1}^2 D(H_i)$ and the proof is complete. □

Now we can obtain $f(t_1,t_2)$ from two-parameter initial value problem (1.2). As we mentioned two-parameter initial value problem (1.2) doesn't have a unique solution for each $x \in \bigcap_{i=1}^2 D(H_i)$ in both positive two-cells I_a and $I_{a'}$, so $f(t_1,t_2)$ is not unique for each $x \in \bigcap_{i=1}^2 D(H_i)$ in both positive two-cells I_a and $I_{a'}$. For example, assume that $u_0 \in \bigcap_{i=1}^2 D(H_i)$ and $f(t_1,t_2) = W(t_1,t_2)u_0$. By these assumptions, the conditions of Theorem 2.2 hold and $v(t_1,t_2) = (t_1+t_2)W(t_1,t_2)u_0$. So $u(t_1,t_2) = (t_1+t_2+1)W(t_1,t_2)u_0$ becomes a classical solution of 2-IACP.

Acknowledgment

The authors would like to thank the referees for their useful comments.

References

[1] P. L. Butzer and H. Berens, *Semigroup of Operators and Approximation,* Springer-Verlag, New York, 1967.

[2] K. J. Engle and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* Springer-Verlag, New York, 2000.

[3] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, *Amer. Math. Soc. Colloq.* **31**, Providence R. I., 1957.

[4] M. Janfada and A. Niknam, On the n-Parameters Abstract Cauchy Problem, *Bul. Aus. Math. Soc.* **69** (2004), 383-394.

[5] M. Janfada and A. Niknam, Two-Parameter Dynamical System and Applications, *J. Science of I. R. Iran*, to appear.

[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, *Appl. Math. Sci.* **44**, Springer-Verlag, New York, 1983.

M. Khanehgir
Department of Mathematics
Islamic Azad University of Mashhad
Mashhad P. O. Box 413-91735
Iran
e-mail:khanehgir@mshdiau.ac.ir

M. Janfada
Department of Mathematics
Teacher Training University of Sabzevar
Sabzevar P. O. Box 397
Iran
e-mail:Janfada@sttu.ac.ir

A. Niknam
Department of Mathematics
Ferdowsi University of Mashhad
Mashhad P. O. Box 1159-91775
Iran.
e-mail:Niknam@math.um.ac.ir