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SOLVING TWO-DIMENSIONAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY LEGENDRE WAVELETS

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Abstract. In this paper, we introduce the two-dimensional Legendre wavelets (2D-LWs), and develop them for solving a class of two-dimensional integro-differential equations (2D-IDEs) of fractional order. We also investigate convergence of the method. Finally, we give some illustrative examples to demonstrate the validity and efficiency of the method.
Keywords: Two-dimensional integro-differential equations, fractional operators, Legendre wavelets, operational matrix.
MSC(2010): 65R20.

1. Introduction

Fractional calculus is a generalization of integration and differentiation to arbitrary order. In recent years, a large number of scientists have studied fractional calculus. Fractional integral and differential equations are used to model many practical problems in physics, engineering, mechanics, economics and biology [2,5,6,17,18]. Some numerical methods have been proposed for solving the fractional integro-differential equations in one-dimensional case, such as Adomian decomposition method [13, 20], variational iteration and homotopy perturbation method [15] and wavelet method [10, 24]. Also, variational iteration, Adomian decomposition, multivariate Pade approximations methods [12,14,22] and wavelet operational method [16] have been used to solve fractional partial differential equations. Recently, a finite difference technique has been developed for solving variable-order fractional integro-differential equations in [23]. On the other hand, although there are many works about two-dimensional integro-differential equations with integer order [7, 19], but there are not suitable work on the partial integro-differential equations with fractional order. So,

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in this paper we study this kind of equations.
Here, our purpose is to apply the Legendre wavelet method for solving two-dimensional fractional integro-differential equations of the form

$$D_x^{\alpha} f(x,y) - \lambda \int_0^y \int_0^x k(x,y,u,v) f(u,v) du dv = g(x,y) \tag{1.1}$$

subject to the initial conditions

$$\frac{\partial^i}{\partial x^i} f(0,y) = 0, \quad i = 0, 1, \ldots, r-1, \quad r-1 < \alpha \leq r, \quad r \in N \tag{1.2}$$

where $g \in L^2(D)$ and $k \in L^2(D \times D)$ with $D = [0,1) \times [0,1)$, $f(x,y)$ is an unknown function to be found and D_x^{α} is the partial fractional derivative of order α with respect to x in the Caputo sense.

2. Preliminaries

In this section, we present some preliminary results which will be used throughout the paper.

2.1. Digamma function.

Definition 2.1. The digamma function, the logarithmic derivative of the gamma function, is defined as

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

where $\Gamma(x)$ is the classical gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

The derivatives $\psi', \psi'', \ldots$ are called polygamma functions.
Digamma function has the following series representation [1]

$$\psi(x) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k}\right) - \frac{1}{x},$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\dfrac{1}{k} - \ln n\right) \cong 0.57721566...$ is the Euler-Mascheroni's constant. A simple calculation shows that

$$\psi'(x) = \sum_{k=1}^{\infty}\left(\frac{1}{(x+k)^2}\right) + \frac{1}{x^2} = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2},$$
$$\psi''(x) = -2\sum_{k=0}^{\infty}\frac{1}{(x+k)^3},$$
$$\psi'''(x) = 6\sum_{k=0}^{\infty}\frac{1}{(x+k)^4}.$$

2.2. **Fractional calculus.**

Definition 2.2. The partial Rimann-Liouville fractional integral operator I_x^α of order $\alpha > 0$ with respect to x is defined as [8]

$$I_x^\alpha f(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau,y)d\tau$$
$$I_x^0 f(x,y) = f(x,y).$$

Definition 2.3. The Caputo partial fractional derivative of order $\alpha > 0$ with respect to x is given by [8]

$$D_x^\alpha f(x,y) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\tau)^{n-\alpha-1}\frac{\partial^n f(\tau,y)}{\partial\tau^n}d\tau, \quad n-1 < \alpha \leq n$$

where n is an integer.

Similar to one-dimensional functions, it can be easily shown that:

$$D_x^\alpha I_x^\alpha f(x,y) = f(x,y).$$

$$I_x^\alpha D_x^\alpha f(x,y) = f(x,y) - \sum_{k=0}^{n-1}\frac{\partial^k f(0^+,y)}{\partial x^k}\frac{x^k}{k!}, \quad x > 0. \tag{2.1}$$

2.3. **Legendre polynomials.** For $n = 0, 1, 2, \ldots$, Legendre polynomials of degree n are given by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad -1 \leq x \leq 1,$$

which satisfies the following recurrence relation

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x), \quad n \geq 1,$$

with $P_0(x) = 1,\ P_1(x) = x$.
The following properties hold for the Legendre polynomials [21]:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \quad n \neq m, \tag{2.2}$$

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}, \tag{2.3}$$

$$P_n(-1) = (-1)^n, \quad P_n(1) = 1, \tag{2.4}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n \geq 1. \tag{2.5}$$

2.4. **2D-LWs and their properties.** The two-dimentional Legendre wavelets $\psi_{n_1m_1n_2m_2}(x,y)$ are defined on the interval D as [3]

$$\begin{cases} \sqrt{\frac{2m_1+1}{2}\frac{2m_2+1}{2}}2^{(k_1+k_2)/2}P_{m_1}(2^{k_1}x-\widehat{n}_1)P_{m_2}(2^{k_2}y-\widehat{n}_2), \\ \qquad\qquad \frac{\widehat{n}_1-1}{2^{k_1}} \leq x < \frac{\widehat{n}_1+1}{2^{k_1}}, \quad \frac{\widehat{n}_2-1}{2^{k_2}} \leq y < \frac{\widehat{n}_2+1}{2^{k_2}}, \\ 0, \qquad\qquad \text{other wise,} \end{cases} \tag{2.6}$$

where $\widehat{n}_1 = 2n_1 - 1,\ \widehat{n}_2 = 2n_2 - 1,\ n_1 = 1, 2, \ldots, 2^{k_1-1},\ n_2 = 1, 2, \ldots, 2^{k_2-1}$ and $k_1,\ k_2$ are assumed positive integers. Also $m_1 = 0, 1, \ldots, M_1 - 1$ and $m_2 = 0, 1, \ldots, M_2 - 1$ are the order of the Legendre polynomials where M_1 and M_2 are fixed positive integers. Furthermore, P_{m_1} and P_{m_2} are the Legendre polynomials of order m_1 and m_2, respectively.
Since the 2D-LWs are orthonormal, so a function $f \in L^2(D)$ may be expressed in terms of the Legendre wavelets as

$$f(x,y) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} c_{n_1m_1n_2m_2}\psi_{n_1m_1n_2m_2}(x,y),$$

where

$$c_{n_1m_1n_2m_2} = \int_0^1\int_0^1 f(x,y)\psi_{n_1m_1n_2m_2}(x,y)dxdy. \tag{2.7}$$

We can approximate the function $f(x,y)$ by the truncated form

$$f(x,y) \simeq \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} c_{n_1m_1n_2m_2}\psi_{n_1m_1n_2m_2}(x,y).$$

For simplicity, we write the function $f(x,y)$ as

$$f(x,y) \simeq \sum_{i=1}^{\widehat{m}_1}\sum_{j=1}^{\widehat{m}_2} c_{ij}\psi_{ij}(x,y),$$

where $\widehat{m}_1 = 2^{k_1-1}M_1$, $\widehat{m}_2 = 2^{k_2-1}M_2$, $c_{ij} = c_{n_1m_1n_2m_2}$, $\psi_{ij} = \psi_{n_1m_1n_2m_2}$ and i, j are determined by the relations $i = M_1(n_1-1)+m_1+1$ and $j = M_2(n_2-1)+m_2+1$, respectively. The above representation, alternatively, can be written as

$$f(x,y) \simeq C^T\Psi(x,y)$$

where C and $\Psi(x,y)$ are $\widehat{m}_1\widehat{m}_2$ vectors as

$$C = [c_{11},\ldots,c_{1\widehat{m}_2},c_{21},\ldots,c_{2\widehat{m}_2},\ldots,c_{\widehat{m}_11},\ldots,c_{\widehat{m}_1\widehat{m}_2}]^T$$

and

$$\Psi(x,y) = [\psi_{11},\ldots,\psi_{1\widehat{m}_2},\psi_{21},\ldots,\psi_{2\widehat{m}_2},\ldots,\psi_{\widehat{m}_11},\ldots,\psi_{\widehat{m}_1\widehat{m}_2}]^T. \tag{2.8}$$

3. Solution of 2D-IDEs of the fractional order

An $\widehat{m}_1\widehat{m}_2$-set of the two-dimensional Block-Pulse functions(2D-BPFs) on the interval D is defined as

$$b_{ij}(x,y) = \begin{cases} 1, & \frac{i-1}{\widehat{m}_1} \le x < \frac{i}{\widehat{m}_1},\ \frac{j-1}{\widehat{m}_2} \le y < \frac{j}{\widehat{m}_2}, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

where $i = 1,2,\ldots,\widehat{m}_1$, $j = 1,2,\ldots,\widehat{m}_2$. These functions have disjointness and orthogonality properties [4].

Similar to given operational matrix of BPFs for fractional integration in [9], the 2D-BPFs operational matrix of the fractional integration(F^α) can be obtained as

$$I_x^\alpha(B(x,y)) = F^\alpha B(x,y), \tag{3.2}$$

where

$$B(x,y) = [b_{11}(x,y),\ldots,b_{1\widehat{m}_2}(x,y),\ldots,b_{\widehat{m}_11}(x,y),\ldots,b_{\widehat{m}_1\widehat{m}_2}(x,y)]^T \tag{3.3}$$

and

$$F^\alpha = \frac{1}{\widehat{m}_1^\alpha}\frac{1}{\Gamma(\alpha+2)} \times \begin{bmatrix} 1 & 0 & \cdots & 0 & \xi_1 & 0 & \cdots & 0 & \xi_2 & \cdots & \xi_{\widehat{m}_1-1} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \xi_1 & 0 & \cdots & 0 & \xi_2 & \cdots & \xi_{\widehat{m}_1-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The nonzero elements of the above matrix are computed as follows

$$F^\alpha_{k,k} = 1,\ k = 1,2,\ldots,\widehat{m}_1\widehat{m}_2,$$

$$F^\alpha_{i,i+j\widehat{m}_2} = \xi_j,\quad i = 1,\ldots,\widehat{m}_1\widehat{m}_2 - j,\ j = 1,\ldots,\widehat{m}_1 - 1,$$

where $\xi_j = (j+1)^{\alpha+1} - 2j^{\alpha+1} + (j-1)^{\alpha+1}$.
The 2D-LWs may be expanded into an $\widehat{m}_1\widehat{m}_2$-term of the 2D-BPFs as

$$\Psi(x,y) = \Phi B(x,y) \tag{3.4}$$

in which

$$\Phi = [\Psi(x_1,y_1),\ldots,\Psi(x_1,y_{\widehat{m}_2}),\ldots,\Psi(x_{\widehat{m}_1},y_1),\ldots,\Psi(x_{\widehat{m}_1},y_{\widehat{m}_2})]$$

where the vector $\Psi(x_i,y_j)$ can be obtained from (2.8) and

$$x_i = \frac{2i-1}{2\widehat{m}_1},\ i=1,2,\ldots,\widehat{m}_1,\quad y_j = \frac{2j-1}{2\widehat{m}_2},\ j=1,2,\ldots,\widehat{m}_2. \tag{3.5}$$

Now, we can get

$$I_x^\alpha \Psi(x,y) = P^\alpha \Psi(x,y), \tag{3.6}$$

where P^α is called the 2D-LWs operational matrix of fractional integration.
Using (3.2) and (3.4), we get

$$I_x^\alpha \Psi(x,y) = I_x^\alpha \Phi B(x,y) = \Phi I_x^\alpha B(x,y) = \Phi F^\alpha B(x,y), \tag{3.7}$$

and applying (3.6) and (3.7) lead to

$$P^\alpha \Phi B(x,y) = P^\alpha \Psi(x,y) = \Phi F^\alpha B(x,y).$$

Therefore, one gets

$$P^\alpha = \Phi F^\alpha \Phi^{-1}.$$

To solve (1.1), we can approximate the functions $D_x^\alpha f(x,y)$ and $g(x,y)$ by the Legendre wavelets as

$$D_x^\alpha f(x,y) \simeq C^T \Psi(x,y), \tag{3.8}$$

$$g(x,y) \simeq G^T \Psi(x,y), \tag{3.9}$$

where

$$\begin{aligned} G &= [g_{11},\ldots,g_{1\widehat{m}_2},g_{21},\ldots,g_{2\widehat{m}_2},\ldots,g_{\widehat{m}_1 1},\ldots,g_{\widehat{m}_1\widehat{m}_2}]^T, \\ g_{ij} &= \int_0^1\int_0^1 \psi_{ij}(x,y)g(x,y)dxdy. \end{aligned}$$

Now using (3.6) and (3.8), we obtain

$$f(x,y) = I_x^\alpha D_x^\alpha f(x,y) \simeq I_x^\alpha (C^T\Psi(x,y)) = C^T P^\alpha \Psi(x,y). \tag{3.10}$$

Substituting (3.8), (3.9) and (3.10) into (1.1), implies

$$C^T\Psi(x,y) - \lambda \int_0^y \int_0^x k(x,y,u,v) C^T P^\alpha \Psi(u,v) dudv = G^T \Psi(x,y).$$

By taking the collocation points (3.5), we collocate (1.1) at $\widehat{m}_1\widehat{m}_2$ points (x_i, t_j) as follows

(3.11)
$$C^T\Psi(x_i, y_j) - \lambda \int_0^{y_j}\int_0^{x_i} k(x_i, y_j, u, v)C^T P^\alpha \Psi(u, v)dudv = G^T\Psi(x_i, y_j).$$

By setting
$$s = \frac{2}{x_i}u - 1, \quad t = \frac{2}{t_j}v - 1,$$

the y-intervals $[0, x_i]$ and z-intervals $[0, y_j]$ are converted into s and t intervals $[-1, 1]$. Thus, (3.11) can be written as

$$C^T\Psi(x_i, y_j) - \lambda\frac{x_iy_j}{4}\int_{-1}^1\int_{-1}^1 k(x_i, y_j, \frac{x_i}{2}(s+1), \frac{y_j}{2}(t+1))$$
$$\times C^T P^\alpha\Psi(\frac{x_i}{2}(s+1), \frac{y_j}{2}(t+1))dsdt = G^T\Psi(x_i, y_j). \tag{3.12}$$

Finally using the Gaussian integration formula, we obtain

$$C^T\Psi(x_i, y_j) - \lambda\frac{x_iy_j}{4}\sum_{p=1}^{h_1}\sum_{q=1}^{h_2} w_{1p}w_{2q}k(x_i, y_j, \frac{x_i}{2}(s_q+1), \frac{y_j}{2}(t_p+1))$$
$$\times C^T P^\alpha\Psi(\frac{x_i}{2}(s_q+1), \frac{y_j}{2}(t_p+1)) = G^T\Psi(x_i, y_j),$$
$$i = 1, 2, \dots, \widehat{m}_1,\ j = 1, 2, \dots, \widehat{m}_2. \tag{3.13}$$

where t_p and s_q are zeros of Legendre polynomials of degrees h_1 and h_2 respectively, and w_{1p} and w_{2q} are corresponding weights. By solving this linear system of $\widehat{m}_1\widehat{m}_2$ equations, the approximate solution $f(x, y)$ is obtained from (3.10).

Remark 3.1. The cost of computation for (3.13) is approximately $O(\widehat{m}_1^2\widehat{m}_2^2)$.

4. **Error analysis**

In this section, we give a lemma and theorem based on [10] about error bound and convergence.

Lemma 4.1. *Let $D_x^\alpha f_{k_1,M_1,k_2,M_2}(x, y)$ be approximation of $D_x^\alpha f(x, y)$ which is obtained by Legendre wavelets. Then*

$$D_x^\alpha f(x, y) - D_x^\alpha f_{k_1,M_1,k_2,M_2}(x, y) =$$
$$\sum_{n_1=2^{k_1-1}+1}^{\infty}\ \sum_{m_1=M_1}^{\infty}\ \sum_{n_2=2^{k_2-1}+1}^{\infty}\ \sum_{m_2=M_2}^{\infty} c_{n_1m_1n_2m_2}\psi_{n_1m_1n_2m_2}(x, y). \tag{4.1}$$

where

(4.2)

$$c_{n_1m_1n_2m_2} = \sqrt{\frac{1}{2^{5k_1+1}(2m_1+1)}\frac{1}{2^{5k_2+1}(2m_2+1)}}$$

(4.3)

$$\times \int_{-1}^{1}\int_{-1}^{1} D_x^{\alpha+2}D_y^2 f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right)\left(\frac{P_{m_1+2}(u)-P_{m_1}(u)}{2m_1+3}\right.$$

(4.4)

$$\left.-\frac{P_{m_1}(u)-P_{m_1-2}(u)}{2m_1-1}\right)\left(\frac{P_{m_2+2}(v)-P_{m_2}(v)}{2m_2+3} - \frac{P_{m_2}(v)-P_{m_2-2}(v)}{2m_2-1}\right) dvdu$$

(4.5)

and $u = 2^{k_1}x - \widehat{n}_1, \ v = 2^{k_2}y - \widehat{n}_2$.

Proof. Since $D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y)$ is approximation of $D_x^\alpha f(x,y)$, similar to (3.8), we have

(4.6)

$$D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y) = \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} c_{n_1m_1n_2m_2}\psi_{n_1m_1n_2m_2}(x,y),$$

which leads to (4.1). By setting $2^{k_1}x - \widehat{n}_1 = u, \ 2^{k_2}y - \widehat{n}_2 = v$ and using (2.7), we have

$$c_{n_1m_1n_2m_2} = \int_0^1\int_0^1 D_x^\alpha f(x,y)\psi_{n_1m_1n_2m_2}(x,y)dxdy$$
$$= \int_{(\widehat{n}_2-1)/2^{k_2}}^{(\widehat{n}_2+1)/2^{k_2}}\int_{(\widehat{n}_1-1)/2^{k_1}}^{(\widehat{n}_1+1)/2^{k_1}} D_x^\alpha f(x,y)\sqrt{\frac{2m_1+1}{2}\frac{2m_2+1}{2}}2^{(k_1+k_2)/2}$$
$$\times P_{m_1}(2^{k_1}x-\widehat{n}_1)P_{m_2}(2^{k_2}y-\widehat{n}_2)dxdy$$
$$= \sqrt{\frac{2m_1+1}{2^{k_1+1}}\frac{2m_2+1}{2^{k_2+1}}}\int_{-1}^{1}\int_{-1}^{1} D_x^\alpha f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right)P_{m_1}(u)P_{m_2}(v)dudv.$$

Using (2.4) and (2.5) , we have

$$c_{n_1m_1n_2m_2} = \sqrt{\frac{1}{2^{k_1+1}(2m_1+1)}\frac{2m_2+1}{2^{k_2+1}}}$$
$$\times \int_{-1}^{1} P_{m_2}(v) \int_{-1}^{1} D_x^{\alpha} f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right) d(P_{m_1+1}(u) - P_{m_1-1}(u))dv$$
$$= -\sqrt{\frac{1}{2^{3k_1+1}(2m_1+1)}\frac{2m_2+1}{2^{k_2+1}}} \int_{-1}^{1} P_{m_2}(v) \int_{-1}^{1} D_x^{\alpha+1} f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right)$$
$$\times (P_{m_1+1}(u) - P_{m_1-1}(u))dudv$$
$$= -\sqrt{\frac{1}{2^{3k_1+1}(2m_1+1)}\frac{2m_2+1}{2^{k_2+1}}} \int_{-1}^{1} P_{m_2}(v) \int_{-1}^{1} D_x^{\alpha+1} f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right)$$
$$\times d\left(\frac{P_{m_1+2}(u) - P_{m_1}(u)}{2m_1+3} - \frac{P_{m_1}(u) - P_{m_1-2}(u)}{2m_1-1}\right) dv$$
$$= \sqrt{\frac{1}{2^{5k_1+1}(2m_1+1)}\frac{2m_2+1}{2^{k_2+1}}} \int_{-1}^{1} P_{m_2}(v) \int_{-1}^{1} D_x^{\alpha+2} f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right)$$
$$\times \left(\frac{P_{m_1+2}(u) - P_{m_1}(u)}{2m_1+3} - \frac{P_{m_1}(u) - P_{m_1-2}(u)}{2m_1-1}\right) dudv,$$

and by the same way for $p_{m_2}(v)$ we obtain

$$c_{n_1m_1n_2m_2} = \sqrt{\frac{1}{2^{5k_1+1}(2m_1+1)}\frac{1}{2^{5k_2+1}(2m_2+1)}}$$
$$\times \int_{-1}^{1}\int_{-1}^{1} D_x^{\alpha+2} D_y^2 f\left(\frac{\widehat{n}_1+u}{2^{k_1}}, \frac{\widehat{n}_2+v}{2^{k_2}}\right) \left(\frac{P_{m_1+2}(u) - P_{m_1}(u)}{2m_1+3}\right.$$
$$\left. - \frac{P_{m_1}(u) - P_{m_1-2}(u)}{2m_1-1}\right) \left(\frac{P_{m_2+2}(v) - P_{m_2}(v)}{2m_2+3} - \frac{P_{m_2}(v) - P_{m_2-2}(v)}{2m_2-1}\right) dvdu.$$

□

Theorem 4.2. *Let the assumptions of Lemma 4.1 hold and there exists a positive constant $\tilde{M}$, such that*

$$|D_x^{\alpha+2} D_y^2 f(x,y)| \leq \tilde{M}, \quad \forall (x,y) \in D.$$

Then

$$\left\| D_x^{\alpha} f(x,y) - D_x^{\alpha} f_{k_1,M_1,k_2,M_2}(x,y) \right\|_E \leq \left(\frac{\tilde{M}^2}{225}\frac{1}{2^{4k_1}2^{4k_2}}\psi'''(M_1-1.5)\psi'''(M_2-1.5)\right)^{1/2}$$

where $\|f(x,y)\|_E = (\int_0^1\int_0^1 f^2(x,y)dydx)^{1/2}$ *and* $\psi(x)$ *is the digamma function.*

Proof. The orthonormality of the sequence $\{\psi_{n_1m_1n_2m_2}\}$ on D implies that $\int_0^1\int_0^1 \Psi(x,y)\Psi^T(x,y)dxdy = I$, where I is the identity matrix. Therefore, we get

$$\Big\| D_x^{\alpha} f(x,y) - D_x^{\alpha} f_{k_1,M_1,k_2,M_2}(x,y)\Big\|_E^2 = \sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty} c_{n_1m_1n_2m_2}^2.$$

By Lemma 4.1 and relations (2.2) and (2.3), we have

$$\begin{aligned}
{c_{n_1m_1n_2m_2}}^2 &\le \frac{1}{2^{5k_1+1}(2m_1+1)}\frac{1}{2^{5k_2+1}(2m_2+1)}\\
&\times \int_{-1}^{1}\int_{-1}^{1}\left|D_x^{\alpha+2}D_y^2 f\left(\frac{\widehat{n}_1+u}{2^{k_1}},\frac{\widehat{n}_2+v}{2^{k_2}}\right)\right|^2 dudv\\
&\times \int_{-1}^{1}\int_{-1}^{1}\left|\frac{(2m_1-1)P_{m_1+2}(u)-(4m_1+2)P_{m_1}(u)+(2m_1+3)P_{m_1-2}(u)}{(2m_1+3)(2m_1-1)}\right|^2\\
&\times \left|\frac{(2m_2-1)P_{m_2+2}(v)-(4m_2+2)P_{m_2}(v)+(2m_2+3)P_{m_2-2}(v)}{(2m_2+3)(2m_2-1)}\right|^2 dudv\\
&< \frac{\tilde{M}^2}{2^{5k_1}(2m_1+1)2^{5k_2}(2m_2+1)(2m_1+3)^2(2m_1-1)^2(2m_2+3)^2(2m_2-1)^2}\\
&\times \left((2m_1-1)^2\frac{2}{2m_1+5}+(4m_1+2)^2\frac{2}{2m_1+1}+(2m_1+3)^2\frac{2}{2m_1-3}\right)\\
&\times \left((2m_2-1)^2\frac{2}{2m_2+5}+(4m_2+2)^2\frac{2}{2m_2+1}+(2m_2+3)^2\frac{2}{2m_2-3}\right)\\
&< \frac{\tilde{M}^2}{2^{5k_1}(2m_1+1)2^{5k_2}(2m_2+1)(2m_1+3)^2(2m_1-1)^2(2m_2+3)^2(2m_2-1)^2}\\
&\times \frac{12(2m_1+3)^2}{2m_1-3}\frac{12(2m_2+3)^2}{2m_2-3}\\
&= \frac{144\tilde{M}^2}{2^{5k_1}(2m_1+1)2^{5k_2}(2m_2+1)(2m_1-3)(2m_1-1)^2(2m_2-3)(2m_2-1)^2}\\
&< \frac{144\tilde{M}^2}{(2n_1)^5(2m_1-3)^4(2n_2)^5(2m_2-3)^4}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty} c_{n_1m_1n_2m_2}^2\\
&< 144\tilde{M}^2\sum_{n_1=2^{k_1-1}+1}^{\infty}\frac{1}{(2n_1)^5}\sum_{m_1=M_1}^{\infty}\frac{1}{(2m_1-3)^4}\sum_{n_2=2^{k_2-1}+1}^{\infty}\frac{1}{(2n_2)^5}
\end{aligned}$$

$$\times \sum_{m_2=M_2}^{\infty} \frac{1}{(2m_2-3)^4} \leq \frac{\tilde{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1-1.5)\psi'''(M_2-1.5).$$

Then, we obtain

$$\left\| D_x^\alpha f(x,y) - D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y) \right\|_E^2 \leq \frac{\tilde{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1-1.5)\psi'''(M_2-1.5),$$

which implies

$$\left\| D_x^\alpha f(x,y) - D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y) \right\|_E \leq \left(\frac{\tilde{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1-1.5)\psi'''(M_2-1.5) \right)^{1/2}.$$

It completes the proof. □

Note that $\|D_x^\alpha f(x,y) - D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y)\|_E = O({N_1}^{-1}{N_2}^{-1})$ with $N_1 = 2^{2k_1}$ and $N_2 = 2^{2k_2}$.

Corollary 4.3. *Under the assumptions of Theorem 4.2, we have*

$$\|D_x^\alpha f(x,y) - D_x^\alpha f_{k_1,M_1,k_2,M_2}(x,y)\|_E \to 0.$$

Using (2.1), in view of (1.2), we get

$$\|f(x,y) - f_{k_1,M_1,k_2,M_2}(x,y)\|_E \to 0,$$

when M_1, M_2 is fixed and $k_1, k_2 \to \infty$.

5. **Numerical results**

In this section, we apply the proposed method based on the two-dimensional Legendre wavelets for some examples to show the efficiency and accuracy of the method.

Example 5.1. Consider the linear two-dimensional fractional integro-differential equation (See [11])

$$D_x^{0.5} f(x,y) - \int_0^y \int_0^x (x^2y+u) f(u,v)dudv = 4y\sqrt{\frac{x}{\pi}} - \frac{1}{2}x^4y^3 - \frac{1}{3}x^3y^2$$

subject to the initial condition $f(0,y) = 0$. The exact solution of this equation is $f(x,y) = 2xy$. The values of absolute errors at some points are shown in Table 1 for different values of k_1 and k_2. According to this table, we observe that the results are improved when the numbers k_1 and k_2 increase.

Example 5.2. Consider the equation

$$D_x^{0.75} f(x,y) - \int_0^y \int_0^x (y+v) f(u,v)dudv = \frac{6.4}{\Gamma(0.25)} yx^{5/4} - \frac{5}{18}x^3y^3$$

such that $f(0,y) = 0$, with the exact solution $f(x,y) = x^2y$. Here, we also use the approximate L_2-norm of absolute error as

$$\| e(x,y) \|_2 = \left(\int_0^1 \int_0^1 e(x,y)^2 dxdy \right)^{1/2}.$$

TABLE 1. Numerical results of example 5.1

(x,t)	$k_1=3, k_2=3$	$k_1=4, k_2=4$	$k_1=5, k_2=5$
(0.1,0.8)	0.8674e-2	0.1173e-3	0.1250e-3
(0.2,0.6)	0.2026e-3	0.1805e-3	0.2751e-4
(0.3,0.8)	0.6235e-3	0.9276e-4	0.1189e-4
(0.4,0.6)	0.2376e-3	0.2710e-4	0.1395e-5
(0.5,0.5)	0.8078e-4	0.7309e-5	0.4065e-5
(0.6,0.5)	0.6571e-4	0.3884e-4	0.1174e-4
(0.7,0.3)	0.1011e-3	0.3548e-4	0.9798e-5
(0.8,0.4)	0.3268e-3	0.9069e-4	0.2406e-4
(0.9,0.9)	0.2395e-2	0.6179e-3	0.1607e-3

The valuse of error at some points together with L_2-norm errors are reported in Table 2.

TABLE 2. Numerical results of Example 5.2

(x,t)	$k_1=3, k_2=3$	$k_1=4, k_2=4$	$k_1=5, k_2=5$
(0,0.7)	0.5624e-2	0.1404e-2	0.3508e-3
(0.1,0.3)	0.2124e-2	0.1636e-3	0.1342e-3
(0.3,0.8)	0.1688e-2	0.1456e-2	0.1099e-3
(0.4,0.2)	0.1429e-2	0.1087e-3	0.8962e-4
(0.6,0.6)	0.4400e-2	0.3248e-3	0.2700e-3
(0.7,0.5)	0.1122e-2	0.8878e-3	0.6759e-4
(0.8,0.4)	0.8837e-3	0.7061e-3	0.5285e-4
(0.9,0.9)	0.7037e-2	0.5898e-3	0.4090e-3
L_2-$norm$	0.3133e-2	0.7745e-3	0.1974e-3

The plots of the absolute errors for Example 5.2 are also shown in Figure 1. We observe that, the higher accuracy can be obtained by taking larger values of k_1 and k_2.

Example 5.3. Consider the equation

$$D_x^{0.5}f(x,y)-\int_0^y\int_0^x(xcos(u)+yv)f(u,v)dudv=\frac{2sin(y)\sqrt{x}}{\Gamma(0.5)}+x-xcos(x)$$
$$-x^2sin(x)-xcos(y)+xcos(x)cos(y)+x^2sin(x)cos(y)-\frac{1}{2}x^2ysin(y)+\frac{1}{2}x^2y^2cos(y)$$

with initial condition $f(0,y)=0$ and exact solution $f(x,y)=xsin(y)$. We solve this problem for $M_1=M_2=4$ and $k_1=k_2=2,3,4$. The numerical results are reported in Table 3 and plotted in Figure 2. Clearly, the accuracy is very satisfactory. Moreover, higher accuracy can be achieved by taking higher-order approximations.

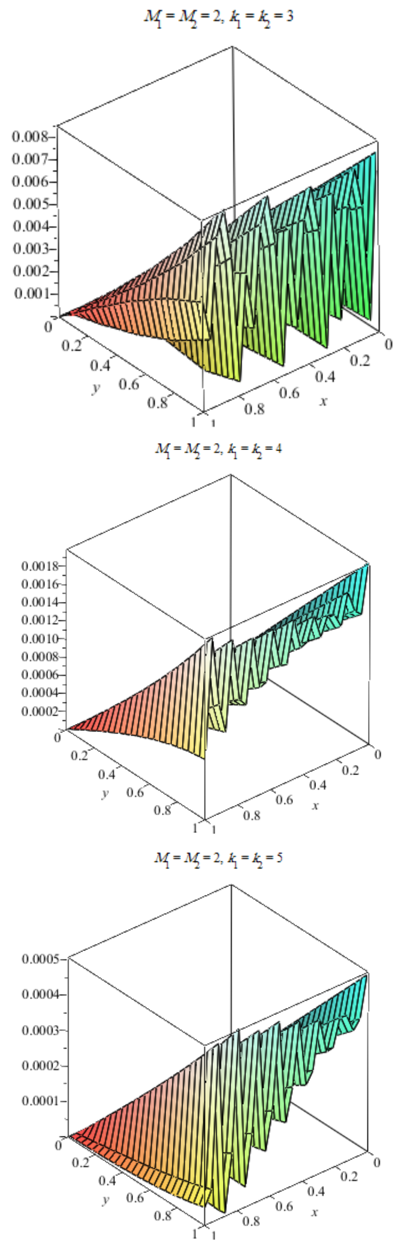

FIGURE 1. The plots of the absolute errors of example 5.2 for different values of k_1, k_2 and $M_1 = M_2 = 2$

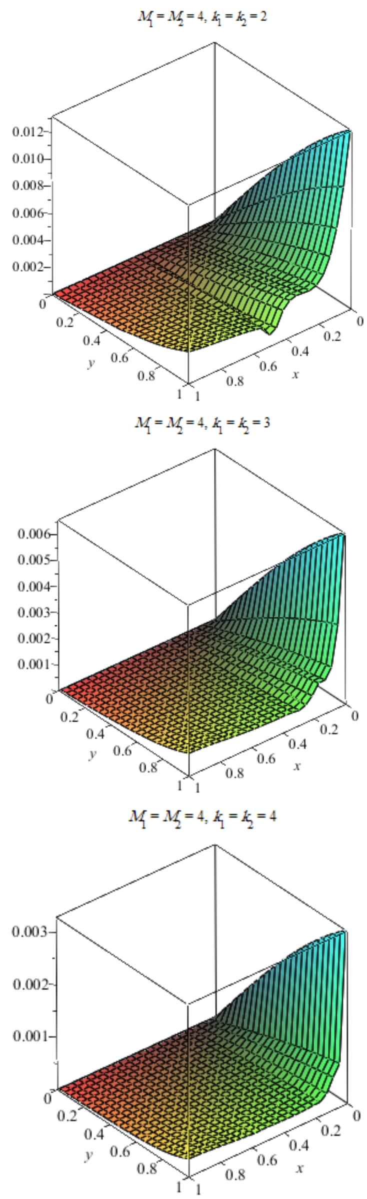

FIGURE 2. The plots of the absolute errors of example 5.3 for different values of k_1, k_2 and $M_1 = M_2 = 4$

TABLE 3. Numerical results of example 5.3

(x,t)	$k_1=2,k_2=2$	$k_1=3,k_2=3$	$k_1=4,k_2=4$
(0.1,0.1)	0.6417e-3	0.1599e-3	0.5398-4
(0.2,0.2)	0.6384e-3	0.2155e-3	0.5185e-4
(0.3,0.3)	0.8544e-3	0.1566e-3	0.6503e-4
(0.4,0.4)	0.8798e-3	0.2122e-3	0.7688e-4
(0.5,0.5)	0.4962e-3	0.2477e-3	0.8809e-4
(0.6,0.6)	0.7754e-3	0.2971e-3	0.9899e-4
(0.7,0.7)	0.1021e-2	0.3662e-3	0.1226e-3
(0.8,0.8)	0.1311e-2	0.4738e-3	0.1599e-3
(0.9,0.9)	0.5624e-2	0.6344e-3	0.2246e-3

Example 5.4. Consider the equation

$$D_x^\alpha f(x,y)-\int_0^y\int_0^x f(u,v)dudv=e^{2y}+\frac{1}{4}x^2-\frac{1}{4}x^2e^{2y}$$

subject to the initial condition $f(0,y)=0$. The exact solution of this problem for $\alpha=1$ is $f(x,y)=xe^{2y}$. The numerical results are given in Table 4. We see that the numerical solution is very closed to the exact solution when $\alpha=1$. Also, results of Table 4 illustrates that, when α, the approximate solution, tends to the exact solution $\alpha\to 0$.

TABLE 4. Numerical results of example 5.4

(x,t)	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	Exact solution
(0.1,0.1)	0.26601	0.20659	0.15935	0.12214	0.12214
(0.2,0.2)	0.53239	0.44194	0.36432	0.29836	0.29836
(0.3,0.3)	0.86665	0.74817	0.64156	0.54663	0.54663
(0.4,0.4)	1.29938	1.15276	1.01614	0.89021	0.89021
(0.5,0.5)	1.86431	1.68793	1.51904	1.35913	1.35914
(0.6,0.6)	2.60016	2.39255	2.18912	1.99205	1.99207
(0.7,0.7)	3.55669	3.31648	3.07616	2.83857	2.83864
(0.8,0.8)	4.79588	4.52245	4.24337	3.96219	3.96242
(0.9,0.9)	6.39578	6.08985	5.77115	5.44400	5.44468

6. **Conclusion**

In this paper, the two-dimensional Legendre wavelets are developed for solving two-dimensional fractional integro-differential equations. The convergence results also are investigated. Numerical results confirm convergency of the method, too. It seems that, the presented method can be applied for solving multi-order partial fractional integro-differential equations.

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