Title:
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SOLVING TWO-DIMENSIONAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY LEGENDRE WAVELETS

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Abstract. In this paper, we introduce the two-dimensional Legendre wavelets (2D-LWs), and develop them for solving a class of two-dimensional integro-differential equations (2D-IDEs) of fractional order. We also investigate convergence of the method. Finally, we give some illustrative examples to demonstrate the validity and efficiency of the method.

Keywords: Two-dimensional integro-differential equations, fractional operators, Legendre wavelets, operational matrix.


1. Introduction

Fractional calculus is a generalization of integration and differentiation to arbitrary order. In recent years, a large number of scientists have studied fractional calculus. Fractional integral and differential equations are used to model many practical problems in physics, engineering, mechanics, economics and biology [2,5,6,17,18]. Some numerical methods have been proposed for solving the fractional integro-differential equations in one-dimensional case, such as Adomian decomposition method [13,20], variational iteration and homotopy perturbation method [15] and wavelet method [10,24]. Also, variational iteration, Adomian decomposition, multivariate Pade approximations methods [12,14,22] and wavelet operational method [16] have been used to solve fractional partial differential equations. Recently, a finite difference technique has been developed for solving variable-order fractional integro-differential equations in [23]. On the other hand, although there are many works about two-dimensional integro-differential equations with integer order [7,19], but there are not suitable work on the partial integro-differential equations with fractional order. So,

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in this paper we study this kind of equations. Here, our purpose is to apply the Legendre wavelet method for solving two-dimensional fractional integro-differential equations of the form

\begin{equation}
D_x^\alpha f(x, y) - \lambda \int_0^y \int_0^x k(x, y, u, v)f(u, v)dvdu = g(x, y)
\end{equation}

subject to the initial conditions

\begin{equation}
\frac{\partial^i}{\partial x^i} f(0, y) = 0, \quad i = 0, 1, \ldots, r - 1, \quad r - 1 < \alpha \leq r, \quad r \in \mathbb{N}
\end{equation}

where \( g \in L^2(D) \) and \( k \in L^2(D \times D) \) with \( D = [0, 1] \times [0, 1] \), \( f(x, y) \) is an unknown function to be found and \( D_x^\alpha \) is the partial fractional derivative of order \( \alpha \) with respect to \( x \) in the Caputo sense.

2. Preliminaries

In this section, we present some preliminary results which will be used throughout the paper.

2.1. Digamma function.

**Definition 2.1.** The digamma function, the logarithmic derivative of the gamma function, is defined as

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]

where \( \Gamma(x) \) is the classical gamma function

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.
\]

The derivatives \( \psi', \psi'', \ldots \) are called polygamma functions. Digamma function has the following series representation [1]

\[
\psi(x) = -\gamma + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{x+k} \right) - \frac{1}{x},
\]
where $\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0.57721566\ldots$ is the Euler-Mascheroni’s constant. A simple calculation shows that

\begin{align*}
\psi'(x) &= \sum_{k=1}^{\infty} \left( \frac{1}{(x+k)^2} \right) + \frac{1}{x^2} = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \\
\psi''(x) &= -2 \sum_{k=0}^{\infty} \frac{1}{(x+k)^3}, \\
\psi'''(x) &= 6 \sum_{k=0}^{\infty} \frac{1}{(x+k)^4}.
\end{align*}

2.2. Fractional calculus.

**Definition 2.2.** The partial Rimann-Liouville fractional integral operator $I_x^\alpha$ of order $\alpha > 0$ with respect to $x$ is defined as $[8]

\begin{align*}
I_x^\alpha f(x, y) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} f(\tau, y) d\tau \\
I_0^0 f(x, y) &= f(x, y).
\end{align*}

**Definition 2.3.** The Caputo partial fractional derivative of order $\alpha > 0$ with respect to $x$ is given by $[8]

\begin{align*}
D_x^\alpha f(x, y) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x - \tau)^{n-\alpha-1} \frac{\partial^n f(\tau, y)}{\partial \tau^n} d\tau, \quad n - 1 < \alpha \leq n
\end{align*}

where $n$ is an integer.

Similar to one-dimensional functions, it can be easily shown that:

\begin{align*}
D_x^\alpha I_x^\alpha f(x, y) &= f(x, y).
\end{align*}

(2.1)

2.3. Legendre polynomials. For $n = 0, 1, 2, \ldots$, Legendre polynomials of degree $n$ are given by Rodrigue’s formula

\begin{align*}
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad -1 \leq x \leq 1,
\end{align*}

which satisfies the following recurrence relation

\begin{align*}
P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \quad n \geq 1,
\end{align*}
with \( P_0(x) = 1, \ P_1(x) = x. \)

The following properties hold for the Legendre polynomials [21]:

\[
\begin{align*}
(2.2) \quad & \int_{-1}^{1} P_n(x)P_m(x)dx = 0, \quad n \neq m, \\
(2.3) \quad & \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n + 1}, \\
(2.4) \quad & P_n(-1) = (-1)^n, \quad P_n(1) = 1, \\
(2.5) \quad & P_{n+1}'(x) - P_{n-1}'(x) = (2n + 1)P_n(x), \quad n \geq 1.
\end{align*}
\]

2.4. 2D-LWs and their properties. The two-dimensional Legendre wavelets \( \psi_{n_1m_1n_2m_2}(x, y) \) are defined on the interval \( D \) as [3]

\[
\psi_{n_1m_1n_2m_2}(x, y) = \sum_{n_1=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_2=1}^{\infty} c_{n_1m_1n_2m_2} \psi_{n_1m_1n_2m_2}(x, y),
\]

where

\[
(2.7) \quad c_{n_1m_1n_2m_2} = \int_{0}^{1} \int_{0}^{1} f(x, y)\psi_{n_1m_1n_2m_2}(x, y)dxdy.
\]

We can approximate the function \( f(x, y) \) by the truncated form

\[
f(x, y) \simeq \sum_{n_1=1}^{2^{k_1}-1} \sum_{m_1=0}^{M_1-1} \sum_{n_2=1}^{2^{k_2}-1} \sum_{m_2=1}^{M_2-1} c_{n_1m_1n_2m_2} \psi_{n_1m_1n_2m_2}(x, y).
\]

For simplicity, we write the function \( f(x, y) \) as

\[
f(x, y) \simeq \sum_{i=1}^{\hat{m}_1} \sum_{j=1}^{\hat{m}_2} c_{ij} \psi_{ij}(x, y),
\]
where \( \tilde{m}_1 = 2^{k_1-1}M_1 \), \( \tilde{m}_2 = 2^{k_2-1}M_2 \), \( c_{ij} = c_{n_1,m_1,n_2,m_2} \), \( \psi_{ij} = \psi_{n_1,m_1,n_2,m_2} \) and \( i, j \) are determined by the relations \( i = M_1(n_1 - 1) + m_1 + 1 \) and \( j = M_2(n_2 - 1) + m_2 + 1 \), respectively. The above representation, alternatively, can be written as

\[
f(x, y) \simeq C^T \Psi(x, y)
\]

where \( C \) and \( \Psi(x, y) \) are \( \tilde{m}_1 \tilde{m}_2 \) vectors as

\[
C = [c_{11}, \ldots, c_{1\tilde{m}_2}, c_{21}, \ldots, c_{2\tilde{m}_2}, \ldots, c_{\tilde{m}_1,1}, \ldots, c_{\tilde{m}_1,\tilde{m}_2}]^T
\]

and

\[
(2.8) \quad \Psi(x, y) = [\psi_{11}, \ldots, \psi_{1\tilde{m}_2}, \psi_{21}, \ldots, \psi_{2\tilde{m}_2}, \ldots, \psi_{\tilde{m}_1,1}, \ldots, \psi_{\tilde{m}_1,\tilde{m}_2}]^T.
\]

3. Solution of 2D-IDEs of the fractional order

An \( \tilde{m}_1 \tilde{m}_2 \)-set of the two-dimensional Block-Pulse functions (2D-BPFs) on the interval \( D \) is defined as

\[
b_{ij}(x, y) = \begin{cases} 
1, & \frac{i-1}{\tilde{m}_1} \leq x < \frac{i}{\tilde{m}_1}, \quad \frac{j-1}{\tilde{m}_2} \leq y < \frac{j}{\tilde{m}_2}, \\
0, & \text{otherwise},
\end{cases}
\]

where \( i = 1, 2, \ldots, \tilde{m}_1 \), \( j = 1, 2, \ldots, \tilde{m}_2 \). These functions have disjointness and orthogonality properties [4].

Similar to given operational matrix of BPFs for fractional integration in [9], the 2D-BPFs operational matrix of the fractional integration \( F^\alpha \) can be obtained as

\[
(3.2) \quad F^\alpha_x (B(x, y)) = F^\alpha B(x, y),
\]

where

\[
(3.3) \quad B(x, y) = [b_{11}(x, y), \ldots, b_{1\tilde{m}_2}(x, y), \ldots, b_{\tilde{m}_1,1}(x, y), \ldots, b_{\tilde{m}_1,\tilde{m}_2}(x, y)]^T
\]

and

\[
F^\alpha = \frac{1}{\tilde{m}_1^\alpha} \frac{1}{\Gamma(\alpha + 2)} \left[ \begin{array}{cccccccccccccccc}
1 & 0 & \cdots & 0 & \xi_1 & 0 & \cdots & 0 & \xi_2 & \cdots & \xi_{\tilde{m}_1-1} & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & \xi_1 & 0 & \cdots & 0 & \xi_2 & \cdots & \xi_{\tilde{m}_1-1} & 0 & \\
& \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array} \right]
\]

The nonzero elements of the above matrix are computed as follows

\[
F^\alpha_{k,k} = 1, \quad k = 1, 2, \ldots, \tilde{m}_1 \tilde{m}_2,
\]

\[
F^\alpha_{i,i+j\tilde{m}_2} = \xi_j, \quad i = 1, \ldots, \tilde{m}_1 \tilde{m}_2 - j, \quad j = 1, \ldots, \tilde{m}_1 - 1,
\]
where \( \xi_j = (j + 1)^{\alpha+1} - 2j^{\alpha+1} + (j - 1)^{\alpha+1} \).

The 2D-LWs may be expanded into an \( \hat{m}_1 \hat{m}_2 \)-term of the 2D-BPFs as

\[
\Psi(x, y) = \Phi B(x, y)
\]

in which

\[
\Phi = [\Psi(x_1, y_1), \ldots, \Psi(x_1, y_{\hat{m}_2}), \ldots, \Psi(x_{\hat{m}_1}, y_1), \ldots, \Psi(x_{\hat{m}_1}, y_{\hat{m}_2})]
\]

where the vector \( \Psi(x_i, y_j) \) can be obtained from (2.8) and

\[
x_i = \frac{2i - 1}{2\hat{m}_1}, \quad i = 1, 2, \ldots, \hat{m}_1, \quad y_j = \frac{2j - 1}{2\hat{m}_2}, \quad j = 1, 2, \ldots, \hat{m}_2.
\]

Now, we can get

\[
I_x^\alpha \Psi(x, y) = P^\alpha \Psi(x, y),
\]

where \( P^\alpha \) is called the 2D-LWs operational matrix of fractional integration.

Using (3.2) and (3.4), we get

\[
I_x^\alpha \Psi(x, y) = I_x^\alpha \Phi B(x, y) = \Phi I_x^\alpha B(x, y) = \Phi F^\alpha B(x, y),
\]

and applying (3.6) and (3.7) lead to

\[
P^\alpha \Phi B(x, y) = P^\alpha \Psi(x, y) = \Phi F^\alpha B(x, y).
\]

Therefore, one gets

\[
P^\alpha = \Phi F^\alpha \Phi^{-1}.
\]

To solve (1.1), we can approximate the functions \( D_x^\alpha f(x, y) \) and \( g(x, y) \) by the Legendre wavelets as

\[
D_x^\alpha f(x, y) \simeq C^T \Psi(x, y),
\]

\[
g(x, y) \simeq G^T \Psi(x, y),
\]

where

\[
G = [g_{11}, \ldots, g_{1\hat{m}_2}, g_{21}, \ldots, g_{2\hat{m}_2}, \ldots, g_{\hat{m}_1}, \ldots, g_{\hat{m}_1 \hat{m}_2}]^T,
\]

\[
g_{ij} = \int_0^1 \int_0^1 \psi_{ij}(x, y)g(x, y)dxdy.
\]

Now using (3.6) and (3.8), we obtain

\[
f(x, y) = I_x^\alpha D_x^\alpha f(x, y) \simeq I_x^\alpha (C^T \Psi(x, y)) = C^T P^\alpha \Psi(x, y).
\]

Substituting (3.8), (3.9) and (3.10) into (1.1), implies

\[
C^T \Psi(x, y) - \lambda \int_0^x \int_0^x h(x, y, u, v)C^T P^\alpha \Psi(u, v)dudv = G^T \Psi(x, y).
\]
By taking the collocation points (3.5), we collocate (1.1) at \( \tilde{m}_1 \tilde{m}_2 \) points \((x_i, t_j)\) as follows
\[(3.11)\]
\[C^T \Psi(x_i, y_j) - \lambda \int_0^{y_j} \int_0^{x_i} k(x_i, y_j, u, v) C^T P^\alpha \Psi(u, v) du dv = G^T \Psi(x_i, y_j).\]

By setting
\[s = \frac{2}{x_i} u - 1, \quad t = \frac{2}{t_j} v - 1,\]
the \(y\)-intervals \([0, x_i]\) and \(z\)-intervals \([0, y_j]\) are converted into \(s\) and \(t\) intervals \([-1, 1]\). Thus, (3.11) can be written as
\[(3.12)\]
\[C^T \Psi(x_i, y_j) - \lambda \frac{x_i y_j}{4} \int_{-1}^{1} \int_{-1}^{1} k(x_i, y_j, \frac{x_i}{2}(s + 1), \frac{y_j}{2}(t + 1)) \times C^T P^\alpha \Psi \left( \frac{x_i}{2}(s + 1), \frac{y_j}{2}(t + 1) \right) ds dt = G^T \Psi(x_i, y_j).\]

Finally using the Gaussian integration formula, we obtain
\[(3.13)\]
\[C^T \Psi(x_i, y_j) - \lambda \frac{x_i y_j}{4} \sum_{p=1}^{h_1} \sum_{q=1}^{h_2} w_{1p} w_{2q} k(x_i, y_j, \frac{x_i}{2}(s_q + 1), \frac{y_j}{2}(t_p + 1)) \times C^T P^\alpha \Psi \left( \frac{x_i}{2}(s_q + 1), \frac{y_j}{2}(t_p + 1) \right) = G^T \Psi(x_i, y_j),\]
where \(t_p\) and \(s_q\) are zeros of Legendre polynomials of degrees \(h_1\) and \(h_2\) respectively, and \(w_{1p}\) and \(w_{2q}\) are corresponding weights. By solving this linear system of \(\tilde{m}_1 \tilde{m}_2\) equations, the approximate solution \(f(x, y)\) is obtained from (3.10).

**Remark 3.1.** The cost of computation for (3.13) is approximately \(O(\tilde{m}_1^2 \tilde{m}_2^2)\).

4. **Error analysis**

In this section, we give a lemma and theorem based on [10] about error bound and convergence.

**Lemma 4.1.** Let \(D^\alpha f_{k_1, M_1, k_2, M_2}(x, y)\) be approximation of \(D^\alpha f(x, y)\) which is obtained by Legendre wavelets. Then
\[(4.1)\]
\[D^\alpha f(x, y) - D^\alpha f_{k_1, M_1, k_2, M_2}(x, y) = \sum_{n_1=2^{k_1}-1+1}^{\infty} \sum_{m_1=M_1}^{\infty} \sum_{n_2=2^{k_2}-1+1}^{\infty} \sum_{m_2=M_2}^{\infty} c_{n_1 m_1 n_2 m_2} \psi_{n_1 m_1 n_2 m_2}(x, y).\]
where

\begin{equation}
(4.2)
c_{n_1m_1n_2m_2} = \sqrt{\frac{1}{2^{5k_1+1}(2m_1+1)} \frac{1}{2^{5k_2+1}(2m_2+1)}}
\end{equation}

\begin{equation}
(4.3)
\times \int_{-1}^{1} \int_{-1}^{1} D_x^{k_1+2} D_y^{k_2} f\left(\frac{\hat{n}_1 + u}{2k_1}, \frac{\hat{n}_2 + v}{2k_2}\right) \left(\frac{P_{m_1+2}(u) - P_{m_1}(u)}{2m_1+3}\right) dudv
\end{equation}

\begin{equation}
(4.4)
\frac{P_{m_1}(u) - P_{m_1-2}(u)}{2m_1-1} \left(\frac{P_{m_1+2}(v) - P_{m_2}(v)}{2m_2+3} - \frac{P_{m_2}(v) - P_{m_2-2}(v)}{2m_2-1}\right) dudv
\end{equation}

\begin{equation}
(4.5)
and \ u = 2^{k_1}x - \hat{n}_1, \ v = 2^{k_2}y - \hat{n}_2.
\end{equation}

**Proof.** Since \( D_x^\alpha f_{k_1,k_2,M_2}(x,y) \) is approximation of \( D_x^\alpha f(x,y) \), similar to (3.8), we have

\begin{equation}
(4.6)
D_x^\alpha f_{k_1,k_2,M_2}(x,y) = \sum_{n_1=1}^{2^{k_1-1}M_1-1} \sum_{m_1=0}^{2^{k_2-1}M_2-1} \sum_{n_2=1}^{2^{k_2-1}M_2-1} \sum_{m_2=0}^{2^{k_2-1}M_2-1} c_{n_1m_1n_2m_2} \psi_{n_1m_1n_2m_2}(x,y),
\end{equation}

which leads to (4.1). By setting \( 2^{k_1}x - \hat{n}_1 = u, 2^{k_2}y - \hat{n}_2 = v \) and using (2.7), we have

\[ c_{n_1m_1n_2m_2} = \int_0^1 \int_0^1 D_x^\alpha f(x,y) \psi_{n_1m_1n_2m_2}(x,y) dxdy \]

\[ = \int_{(\hat{n}_1-1)/2^{k_1}}^{(\hat{n}_1+1)/2^{k_1}} D_x^\alpha f(x,y) \sqrt{\frac{2m_1+1}{2}} \sqrt{\frac{2m_2+1}{2}} \]

\[ \times P_{m_1}(2^{k_1}x - \hat{n}_1) P_{m_2}(2^{k_2}y - \hat{n}_2) dxdy \]

\[ = \sqrt{\frac{2m_1+1}{2^{k_1+1}}} \int_{-1}^{1} \int_{-1}^{1} D_x^\alpha f\left(\frac{\hat{n}_1 + u}{2k_1}, \frac{\hat{n}_2 + v}{2k_2}\right) P_{m_1}(u) P_{m_2}(v) dudv. \]
Using (2.4) and (2.5), we have

\[
c_{n_{1}m_{1}n_{2}m_{2}} = \sqrt{\frac{1}{2^{k_{1}+1}(2m_{1} + 1)} \frac{2m_{2} + 1}{2^{k_{2}+1}}} \\
\times \int_{-1}^{1} P_{m_{2}}(v) \int_{-1}^{1} D_{x}^{\alpha} f \left( \frac{\tilde{n}_{1} + u}{2^{k_{1}}}, \frac{\tilde{n}_{2} + v}{2^{k_{2}}} \right) d(P_{m_{1}+1}(u) - P_{m_{1}-1}(u))dv \\
= - \sqrt{\frac{1}{2^{k_{1}+1}(2m_{1} + 1)} \frac{2m_{2} + 1}{2^{k_{2}+1}}} \int_{-1}^{1} P_{m_{2}}(v) \int_{-1}^{1} D_{x}^{\alpha+1} f \left( \frac{\tilde{n}_{1} + u}{2^{k_{1}}}, \frac{\tilde{n}_{2} + v}{2^{k_{2}}} \right) \\
\times (P_{m_{1}+1}(u) - P_{m_{1}-1}(u))dudv \\
= - \sqrt{\frac{1}{2^{k_{1}+1}(2m_{1} + 1)} \frac{2m_{2} + 1}{2^{k_{2}+1}}} \int_{-1}^{1} P_{m_{2}}(v) \int_{-1}^{1} D_{x}^{\alpha+1} f \left( \frac{\tilde{n}_{1} + u}{2^{k_{1}}}, \frac{\tilde{n}_{2} + v}{2^{k_{2}}} \right) \\
\times d \left( \frac{P_{m_{1}+2}(u) - P_{m_{1}}(u)}{2m_{1} + 3} - \frac{P_{m_{1}}(u) - P_{m_{1}-2}(u)}{2m_{1} - 1} \right) dv \\
= \sqrt{\frac{1}{2^{k_{1}+1}(2m_{1} + 1)} \frac{2m_{2} + 1}{2^{k_{2}+1}}} \int_{-1}^{1} P_{m_{2}}(v) \int_{-1}^{1} D_{x}^{\alpha+2} f \left( \frac{\tilde{n}_{1} + u}{2^{k_{1}}}, \frac{\tilde{n}_{2} + v}{2^{k_{2}}} \right) \\
\times \left( \frac{P_{m_{1}+2}(u) - P_{m_{1}}(u)}{2m_{1} + 3} - \frac{P_{m_{1}}(u) - P_{m_{1}-2}(u)}{2m_{1} - 1} \right) dudv,
\]

and by the same way for \( p_{m_{2}}(v) \) we obtain

\[
c_{n_{1}m_{1}n_{2}m_{2}} = \sqrt{\frac{1}{2^{k_{1}+1}(2m_{1} + 1)} \frac{2m_{2} + 1}{2^{k_{2}+1}(2m_{2} + 1)}} \\
\times \int_{-1}^{1} \int_{-1}^{1} D_{x}^{\alpha+2} D_{y}^{2} f \left( \frac{\tilde{n}_{1} + u}{2^{k_{1}}}, \frac{\tilde{n}_{2} + v}{2^{k_{2}}} \right) \left( \frac{P_{m_{1}+2}(u) - P_{m_{1}}(u)}{2m_{1} + 3} - \frac{P_{m_{1}}(u) - P_{m_{1}-2}(u)}{2m_{1} - 1} \right) dvdu.
\]

\[\Box\]

**Theorem 4.2.** Let the assumptions of Lemma 4.1 hold and there exists a positive constant \( \bar{M} \), such that

\[|D_{x}^{\alpha+2} D_{y}^{2} f(x, y)| \leq \bar{M}, \quad \forall (x, y) \in D.\]

Then

\[\| D_{x}^{\alpha} f(x, y) - D_{x}^{\alpha} f_{k_{1}, k_{2}, M_{1}, M_{2}}(x, y) \|_{E} \leq \left( \frac{\bar{M}^{2}}{2^{2k_{1}+2k_{2}} k_{1} k_{2}} \psi''(M_{1} - 1.5) \psi''(M_{2} - 1.5) \right)^{1/2}\]

where \( \| f(x, y) \|_{E} = (\int_{0}^{1} \int_{0}^{1} f^{2}(x, y)dydx)^{1/2} \) and \( \psi(x) \) is the digamma function.
Proof. The orthonormality of the sequence \( \{\psi_{n_1m_1n_2m_2}\} \) on \( D \) implies that
\[
\int_0^1 \int_0^1 \Psi(x,y)\Psi^T(x,y)dx dy = I,
\]
where \( I \) is the identity matrix. Therefore, we get
\[
\left\| D_y^n f(x,y) - D_x^n f_{k_1,k_2,M_1,M_2}(x,y) \right\|_E^2 = \sum_{n_1=2^{k_1-1}+1}^\infty \sum_{m_1=M_1}^\infty \sum_{n_2=2^{k_2-1}+1}^\infty \sum_{m_2=M_2}^\infty c_{n_1m_1n_2m_2}^2.
\]
By Lemma 4.1 and relations (2.2) and (2.3), we have
\[
c_{n_1m_1n_2m_2}^2 \leq \frac{1}{2^{4k_1+1}(2m_1+1)} \frac{1}{2^{4k_2+1}(2m_2+1)} \times \int_{-1}^1 \int_{-1}^1 \left| D_y^n D_x^n f \left( \frac{n_1 + u}{2^{k_1}}, \frac{n_2 + v}{2^{k_2}} \right) \right|^2 du dv \times \int_{-1}^1 \int_{-1}^1 \left| (2m_1 - 1)P_{m_1+2}(u) - (4m_1 + 2)P_{m_1}(u) + (2m_1 + 3)P_{m_1-2}(u) \right|^2 \frac{(2m_1 + 3)(2m_1 - 1)}{(2m_2 + 3)(2m_2 - 1)} du dv \times \left| (2m_2 - 1)P_{m_2+2}(v) - (4m_2 + 2)P_{m_2}(v) + (2m_2 + 3)P_{m_2-2}(v) \right|^2 \frac{(2m_2 + 3)(2m_2 - 1)}{(2m_2 + 3)(2m_2 - 1)} du dv \times \frac{2^{4k_1}(2m_1 + 1)2^{4k_2}(2m_2 + 1)(2m_1 + 3)^2(2m_1 - 1)^2(2m_2 + 3)^2(2m_2 - 1)^2}{(2m_1 + 5)^2 + (4m_1 + 2)^2 + 2 - (2m_1 + 3)^2} \times \frac{2^{4k_1}(2m_1 + 1)2^{4k_2}(2m_2 + 1)(2m_1 + 3)^2(2m_1 - 1)^2(2m_2 + 3)^2(2m_2 - 1)^2}{(2m_2 + 5)^2 + (4m_2 + 2)^2 + 2 - (2m_2 + 3)^2} \times \frac{12(2m_1 + 3)^2}{2m_1 - 3} \times \frac{12(2m_2 + 3)^2}{2m_2 - 3} \frac{144M^2}{144M^2} = \frac{2^{4k_1}(2m_1 + 1)2^{4k_2}(2m_2 + 1)(2m_1 - 1)^2(2m_2 - 3)^2}{(2m_1 - 3)^2(2m_2 - 3)^2}.
\]
Therefore, we get
\[
\sum_{n_1=2^{k_1-1}+1}^\infty \sum_{m_1=M_1}^\infty \sum_{n_2=2^{k_2-1}+1}^\infty \sum_{m_2=M_2}^\infty c_{n_1m_1n_2m_2}^2 < 144M^2 \sum_{n_1=2^{k_1-1}+1}^\infty \frac{1}{(2n_1)^5} \sum_{m_1=M_1}^\infty \frac{1}{(2m_1 - 3)^4} \sum_{n_2=2^{k_2-1}+1}^\infty \frac{1}{(2n_2)^5}.
\]
we obtain
\[
\times \sum_{m_2=M_2}^{\infty} \frac{1}{(2m_2 - 3)^4} \leq \frac{\hat{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1 - 1.5) \psi'''(M_2 - 1.5).
\]
Then, we obtain
\[
\|D^\alpha_x f(x,y) - D^\alpha_x f_{k_1,M_1,k_2,M_2}(x,y)\|_E^2 \leq \frac{\hat{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1 - 1.5) \psi'''(M_2 - 1.5),
\]
which implies
\[
\|D^\alpha_x f(x,y) - D^\alpha_x f_{k_1,M_1,k_2,M_2}(x,y)\|_E \leq \left( \frac{\hat{M}^2}{225} \frac{1}{2^{4k_1}2^{4k_2}} \psi'''(M_1 - 1.5) \psi'''(M_2 - 1.5) \right)^{1/2}.
\]
It completes the proof. □

Note that \(\|D^\alpha_x f(x,y) - D^\alpha_x f_{k_1,M_1,k_2,M_2}(x,y)\|_E = O(N_1^{-1}N_2^{-1})\) with \(N_1 = 2^{2k_1}\) and \(N_2 = 2^{2k_2}\).

**Corollary 4.3.** Under the assumptions of Theorem 4.2, we have
\[
\|D^\alpha_x f(x,y) - D^\alpha_x f_{k_1,M_1,k_2,M_2}(x,y)\|_E \to 0.
\]
Using (2.1), in view of (1.2), we get
\[
\|f(x,y) - f_{k_1,M_1,k_2,M_2}(x,y)\|_E \to 0,
\]
when \(M_1,M_2\) is fixed and \(k_1,k_2 \to \infty\).

5. Numerical results

In this section, we apply the proposed method based on the two-dimensional Legendre wavelets for some examples to show the efficiency and accuracy of the method.

**Example 5.1.** Consider the linear two-dimensional fractional integro-differential equation (See [11])
\[
D^{0.5}_x f(x,y) - \int_0^y \int_0^x (x^2 y + u) f(u,v)du dv = 4y \sqrt{\frac{x}{\pi}} - \frac{1}{2} x^4 y^3 - \frac{1}{3} x^3 y^2
\]
subject to the initial condition \(f(0,y) = 0\). The exact solution of this equation is \(f(x,y) = 2xy\). The values of absolute errors at some points are shown in Table 1 for different values of \(k_1\) and \(k_2\). According to this table, we observe that the results are improved when the numbers \(k_1\) and \(k_2\) increase.

**Example 5.2.** Consider the equation
\[
D^{0.75}_x f(x,y) - \int_0^y \int_0^x (y + v) f(u,v)du dv = \frac{6.4}{\Gamma(0.25)} yx^{5/4} - \frac{5}{18} x^3 y^3
\]
such that \(f(0,y) = 0\), with the exact solution \(f(x,y) = x^2 y\). Here, we also use the approximate \(L_2\)-norm of absolute error as
\[
\|e(x,y)\|_2 = \left( \int_0^1 \int_0^1 e(x,y)^2 dx dy \right)^{1/2}.
\]
Solving 2D-FIDEs by Legendre wavelets

Table 1. Numerical results of example 5.1

<table>
<thead>
<tr>
<th>(x, t)</th>
<th>(k_1 = 3, k_2 = 3)</th>
<th>(k_1 = 4, k_2 = 4)</th>
<th>(k_1 = 5, k_2 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.8)</td>
<td>0.8074e-2</td>
<td>0.1173e-3</td>
<td>0.1250e-3</td>
</tr>
<tr>
<td>(0.2,0.6)</td>
<td>0.2026e-3</td>
<td>0.1805e-3</td>
<td>0.2751e-4</td>
</tr>
<tr>
<td>(0.3,0.8)</td>
<td>0.6235e-3</td>
<td>0.9276e-4</td>
<td>0.1189e-4</td>
</tr>
</tbody>
</table>

The value of error at some points together with \(L_2\)-norm errors are reported in Table 2.

Table 2. Numerical results of Example 5.2

<table>
<thead>
<tr>
<th>(x, t)</th>
<th>(k_1 = 3, k_2 = 3)</th>
<th>(k_1 = 4, k_2 = 4)</th>
<th>(k_1 = 5, k_2 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0.7)</td>
<td>0.5624e-2</td>
<td>0.1404e-2</td>
<td>0.3508e-3</td>
</tr>
<tr>
<td>(0.1,0.3)</td>
<td>0.2124e-2</td>
<td>0.1636e-2</td>
<td>0.1342e-3</td>
</tr>
<tr>
<td>(0.3,0.8)</td>
<td>0.1688e-2</td>
<td>0.1456e-2</td>
<td>0.1099e-3</td>
</tr>
<tr>
<td>(0.4,0.2)</td>
<td>0.1429e-2</td>
<td>0.1087e-2</td>
<td>0.8962e-4</td>
</tr>
<tr>
<td>(0.6,0.6)</td>
<td>0.4400e-2</td>
<td>0.3248e-3</td>
<td>0.2700e-3</td>
</tr>
<tr>
<td>(0.7,0.5)</td>
<td>0.1222e-2</td>
<td>0.8878e-3</td>
<td>0.6759e-4</td>
</tr>
<tr>
<td>(0.8,0.4)</td>
<td>0.8837e-2</td>
<td>0.7061e-3</td>
<td>0.5285e-4</td>
</tr>
<tr>
<td>(0.9,0.9)</td>
<td>0.7037e-2</td>
<td>0.5898e-3</td>
<td>0.4090e-3</td>
</tr>
<tr>
<td>(L_2)-norm</td>
<td>0.3133e-2</td>
<td>0.7745e-3</td>
<td>0.1974e-3</td>
</tr>
</tbody>
</table>

The plots of the absolute errors for Example 5.2 are also shown in Figure 1.

Example 5.3. Consider the equation

\[
D_x^{0.5} f(x, y) - \int_0^y \int_0^x (x\cos(u) + yv)f(u, v)dudv = \frac{2\sin(y)\sqrt{\pi}}{\Gamma(0.5)} + x - x\cos(x) - x^2\sin(x) - x\cos(y) + x^2\sin(x)\cos(y) - \frac{1}{2}x^2y\sin(y) + \frac{1}{2}x^2y^2\cos(y)
\]

with initial condition \(f(0, y) = 0\) and exact solution \(f(x, y) = x\sin(y)\). We solve this problem for \(M_1 = M_2 = 4\) and \(k_1 = k_2 = 2, 3, 4\). The numerical results are reported in Table 3 and plotted in Figure 2. Clearly, the accuracy is very satisfactory. Moreover, higher accuracy can be achieved by taking higher-order approximations.
Figure 1. The plots of the absolute errors of example 5.2 for different values of $k_1, k_2$ and $M_1 = M_2 = 2$
Figure 2. The plots of the absolute errors of example 5.3 for different values of $k_1, k_2$ and $M_1 = M_2 = 4$
Example 5.4. Consider the equation
\[ D_2^\alpha f(x, y) - \int_0^y \int_0^x f(u, v) dudv = e^{2y} + \frac{1}{4} x^2 - \frac{1}{4} x^2 e^{2y} \]
subject to the initial condition \( f(0, y) = 0 \). The exact solution of this problem for \( \alpha = 1 \) is \( f(x, y) = xe^{2y} \). The numerical results are given in Table 4. We see that the numerical solution is very close to the exact solution when \( \alpha = 1 \). Also, results of Table 4 illustrate that, when \( \alpha \), the approximate solution, tends to the exact solution \( \alpha \to 0 \).

### Table 3. Numerical results of example 5.3

<table>
<thead>
<tr>
<th>((x, t))</th>
<th>(k_1 = 2, k_2 = 2)</th>
<th>(k_1 = 3, k_2 = 3)</th>
<th>(k_1 = 4, k_2 = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>0.6417e-3</td>
<td>0.1599e-3</td>
<td>0.5398e-4</td>
</tr>
<tr>
<td>(0.2,0.2)</td>
<td>0.6384e-3</td>
<td>0.2155e-3</td>
<td>0.5185e-4</td>
</tr>
<tr>
<td>(0.3,0.3)</td>
<td>0.8544e-3</td>
<td>0.1566e-3</td>
<td>0.6503e-4</td>
</tr>
<tr>
<td>(0.4,0.4)</td>
<td>0.8798e-3</td>
<td>0.2122e-3</td>
<td>0.7688e-4</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>0.4962e-3</td>
<td>0.2477e-3</td>
<td>0.8809e-4</td>
</tr>
<tr>
<td>(0.6,0.6)</td>
<td>0.7754e-3</td>
<td>0.2971e-3</td>
<td>0.9899e-4</td>
</tr>
<tr>
<td>(0.7,0.7)</td>
<td>0.1021e-2</td>
<td>0.3662e-3</td>
<td>0.1226e-3</td>
</tr>
<tr>
<td>(0.8,0.8)</td>
<td>0.1311e-2</td>
<td>0.4738e-3</td>
<td>0.1599e-3</td>
</tr>
<tr>
<td>(0.9,0.9)</td>
<td>0.5624e-2</td>
<td>0.6344e-3</td>
<td>0.2246e-3</td>
</tr>
</tbody>
</table>

### Table 4. Numerical results of example 5.4

<table>
<thead>
<tr>
<th>((x, t))</th>
<th>(\alpha = 0.7)</th>
<th>(\alpha = 0.8)</th>
<th>(\alpha = 0.9)</th>
<th>(\alpha = 1)</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>0.26601</td>
<td>0.20659</td>
<td>0.15935</td>
<td>0.12214</td>
<td>0.12214</td>
</tr>
<tr>
<td>(0.2,0.2)</td>
<td>0.53239</td>
<td>0.44194</td>
<td>0.36432</td>
<td>0.29836</td>
<td>0.29836</td>
</tr>
<tr>
<td>(0.3,0.3)</td>
<td>0.86665</td>
<td>0.74817</td>
<td>0.64156</td>
<td>0.54663</td>
<td>0.54663</td>
</tr>
<tr>
<td>(0.4,0.4)</td>
<td>1.29938</td>
<td>1.15276</td>
<td>1.01614</td>
<td>0.89021</td>
<td>0.89021</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>1.86431</td>
<td>1.68793</td>
<td>1.51904</td>
<td>1.35913</td>
<td>1.35914</td>
</tr>
<tr>
<td>(0.6,0.6)</td>
<td>2.60016</td>
<td>2.39255</td>
<td>2.18912</td>
<td>1.99205</td>
<td>1.99207</td>
</tr>
<tr>
<td>(0.7,0.7)</td>
<td>3.55669</td>
<td>3.31648</td>
<td>3.07616</td>
<td>2.83857</td>
<td>2.83864</td>
</tr>
<tr>
<td>(0.8,0.8)</td>
<td>4.79588</td>
<td>4.52245</td>
<td>4.24337</td>
<td>3.96219</td>
<td>3.96242</td>
</tr>
<tr>
<td>(0.9,0.9)</td>
<td>6.39578</td>
<td>6.08985</td>
<td>5.77115</td>
<td>5.44400</td>
<td>5.44468</td>
</tr>
</tbody>
</table>

### 6. Conclusion

In this paper, the two-dimensional Legendre wavelets are developed for solving two-dimensional fractional integro-differential equations. The convergence results also are investigated. Numerical results confirm convergency of the method, too. It seems that, the presented method can be applied for solving multi-order partial fractional integro-differential equations.
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References


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