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SHAPE OPERATOR OF SLANT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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ABSTRACT. In this paper, we establish some relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for slant submanifolds in Kenmotsu space forms.

1. **Introduction**

According to B.Y. Chen, one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Scalar curvature and Ricci curvature are among the main intrinsic invariants, while the squared mean curvature is the main extrinsic invariant. In [4], B.Y. Chen establishes a relationship between sectional curvature function K and the shape operator for submanifolds in real space forms. In [5], he also gives a relationship between Ricci curvature and squared mean curvature.

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A contact version of B.Y. Chen's inequality and its applications to slant immersions in a Sasakian space form $\tilde{M}(c)$ are given in [3]. But there is another interesting class of almost contact metric manifolds, namely Kenmotsu manifolds [7]. In the present paper, we study slant submanifolds of Kenmotsu space forms and establish relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for slant submanifolds in Kenmotsu space forms.

2. **Preliminaries**

Let $\widetilde{M}$ be an almost contact metric manifold [1] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\widetilde{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$.

An almost contact metric manifold is known to be a Kenmotsu manifold [7] if

$$(\widetilde{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

$$\widetilde{\nabla}_X \xi = -\phi^2 X = X - \eta(X)\xi, \qquad X \in T\widetilde{M}, \tag{2.5}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

We denote by Φ the fundamental 2-form of $\widetilde{M}$, that is, $\Phi(X,Y) = g(\varphi X, Y)$, for any vector fields X, Y on $\widetilde{M}$. It was proved that the pairing (Φ, η) defines a locally conformal cosymplectic structure, that is

$$d\Phi = 2\Phi \wedge \eta, \quad d\eta = 0. \tag{2.6}$$

A plane section σ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a φ-section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\tilde{M}$ is of constant φ-sectional curvature if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ-section σ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any φ-section σ of $T_p\tilde{M}$, the function c defined by $c(p) = \tilde{K}(\sigma)$ is called the φ-sectional curvature of $\tilde{M}$. A Kenmotsu manifold $\tilde{M}$ with constant φ-sectional curvature c is said to be a Kenmotsu space form and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Kenmotsu space form $\tilde{M}(c)$ is given by [7]

$$\begin{aligned}
&4\tilde{R}(X,Y,Z,W) = (c-3)[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] \\
&+(c+1)[g(\varphi X,W)g(\varphi Y,Z) - g(\varphi X,Z)g(\varphi Y,W) - \\
&2g(\varphi X,Y)g(\varphi Z,W) + g(X,Z)\eta(Y)\eta(W) - \\
&g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z) - \\
&g(X,W)\eta(Y)\eta(Z)] \qquad (2.7)
\end{aligned}$$

for all $X, Y, Z, W \in T\tilde{M}$.

Let M be an n-dimensional Riemannian manifold. The scalar curvature τ at p is given by $\tau = \sum_{i<j} K_{ij}$, where K_{ij} is the sectional curvature of M associated with a plane section spanned by e_i and e_j at $p \in M$ for any orthonormal basis $\{e_1, ..., e_n\}$ for T_pM. Now let M be a submanifold of an m-dimensional manifold $\tilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$. Then the equation of Gauss is given by

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(h(X,W),h(Y,Z)) \\
&- g(h(X,Z),h(Y,W)) \qquad (2.8)
\end{aligned}$$

for any vectors X, Y, Z, W tangent to M, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively.

The relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM |\, \sigma(X,Y) = 0, \text{ for all } Y \in T_pM\}.$$

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector H at $p \in M$ is

$$H = \frac{1}{n}\text{trace}(\sigma) = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i). \tag{2.9}$$

The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is totally umbilical. We put

$$\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r), \qquad \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)),$$

where e_r belongs to an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of the normal space $T_p^{\perp}M$.

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

Define the Ricci curvature Ric_L of L at X by

$$\text{Ric}_L(X) = K_{12} + K_{13} + ... + K_{1k}, \tag{2.10}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We simply called such a curvature a k-Ricci curvature.

The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1 \le i < j \le k} K_{ij}. \tag{2.11}$$

For each integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \text{Ric}_L(X), \quad p \in M, \tag{2.12}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

From now on, we assume that the dimension of M is $n+1$ and that of the ambient manifold $\tilde{M}$ is $2m+1$. We also assume that the structure vector field ξ is tangent to M.

For a vector field X in M, we put

$$\varphi X = PX + FX, \qquad PX \in TM,\ FX \in T^{\perp}M. \tag{2.13}$$

For any local orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM, we can define the squared norm of P and F by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2, \quad \|F\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Fe_j)^2, \tag{2.14}$$

respectively. It is easy to show that both $\|P\|^2$ and $\|F\|^2$ are independent of the choice of the orthonormal basis.

A submanifold M of an almost contact metric manifold with $\xi \in TM$ is called a semi-invariant submanifold or a contact CR submanifold [8] if there exists two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that (i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \mathcal{E}$, (ii) the distribution $\mathcal{D}$ is invariant by φ, i.e., $\varphi(\mathcal{D}) = \mathcal{D}$, and (iii) the distribution $\mathcal{D}^{\perp}$ is anti-invariant by φ, i.e., $\varphi(\mathcal{D}^{\perp}) \subseteq T^{\perp}M$.

The submanifold M tangent to ξ is said to be invariant or anti-invariant [8] according to $F = 0$ or $P = 0$. Thus, a contact CR-submanifold is invariant or anti-invariant according to $\mathcal{D}^{\perp} = \{0\}$ or $\mathcal{D} = \{0\}$. A proper contact CR-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that X is not proportional to ξ_p, we denote the angle between φX and T_pM by $\theta(X)$. Then M is said to be slant [2] and [6] if the angle $\theta(X)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A proper slant immersion is neither invariant nor anti-invariant.

3. **Sectional curvature and shape operator**

B.Y. Chen establishes a sharp relationship between the shape operator and the sectional curvature for submanifolds in real space forms [5]. In this section, we establish a similar inequality between the shape operator and the sectional curvature for slant submanifolds in a Kenmotsu space form.

Let M be an $(n+1)$-dimensional θ-slant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. Let $p \in M$ and a number

$$b > \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$$

such that the sectional curvature $K \geq b$ at p.

Now we choose an orthonormal basis

$$\{e_1, \ldots, e_{n+1} = \xi, e_{n+2}, \ldots, e_{2m+1}\}$$

at p such that e_{n+2} is parallel to the mean curvature vector H, and $e_1, \ldots, e_{n+1}$ diagonalize the shape operator A_{n+2}. Then we have

$$A_{n+2} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+1} \end{pmatrix}, \tag{3.1}$$

$$A_r = \left(\sigma_{ij}^r\right), \ \operatorname{trace} A_r = \sum_{i=1}^{n+1} \sigma_{ii}^r = 0,$$
$$i, j = 1, ..., n+1; r = n+3, ..., 2m+1. \tag{3.2}$$

For $i \neq j$, we put

$$u_{ij} = a_i a_j = u_{ji}. \tag{3.3}$$

By Gauss equation, for $X = Z = e_i$, $Y = W = e_j$, we have

$$u_{ij} \geq b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)} - \sum_{r=n+3}^{2m+1} \left(\sigma_{ii}^r \sigma_{jj}^r - \left(\sigma_{ij}^r\right)^2\right). \tag{3.4}$$

Lemma 3.1. *For u_{ij} we have*
(a) *For any fixed $i \in \{1, ..., n+1\}$, we have*

$$\sum_{i \neq j} u_{ij} \geq n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right).$$

(b) *For distinct* $i, j, k \in \{1, ..., n+1\}$, *we have* $a_i^2 = u_{ij}u_{ik}/u_{jk}$.
(c) *For a fixed* k, $1 \leq k \leq \left[\dfrac{n+1}{2}\right]$ *and for each* $B \in S_k \equiv \{B \subset \{1, ..., n+1\} : |B| = k\}$, *we have*

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k\,(n-k+1)\left(b - \frac{c+3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right),$$

where $\bar{B}$ *is the complement of* B *in* $\{1, ..., n+1\}$.
(d) *For distinct* $i, j \in \{1, ..., n+1\}$, *it follows that* $u_{ij} > 0$.

Proof. (a) From (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
\sum_{i \neq j} u_{ij} &\geq n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) \\
&\quad - \sum_{r=n+3}^{2m+1}\left(\sigma_{ii}^r\left(\sum_{j \neq i}\sigma_{jj}^r\right) - \sum_{j \neq i}\left(\sigma_{ij}^r\right)^2\right) \\
&= n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) \\
&\quad - \sum_{r=n+3}^{2m+1}\left(\sigma_{ii}^r\left(-\sigma_{ii}^r\right) - \sum_{j \neq i}\left(\sigma_{ij}^r\right)^2\right) \\
&= n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) \\
&\quad + \sum_{r=n+3}^{2m+1}\sum_{j=1}^{n+1}\left(\sigma_{ij}^r\right)^2 \\
&\geq n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) > 0.
\end{aligned}$$

(b) We have $u_{ij}u_{ik}/u_{jk} = a_i a_j a_i a_k / a_j a_k = a_i^2$.

(c) Let $B=\{1,...,k\}$ and $\bar{B}=\{k+1,...,n+1\}$. Then

$$\begin{aligned}\sum_{j\in B}\sum_{t\in\bar{B}}u_{jt} &\geq k\,(n-k+1)\left(b-\frac{c-3}{4}-\frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}\right)\\ &\quad-\sum_{r=n+3}^{2m+1}\left(\sum_{j=1}^{k}\sum_{t=k+1}^{n+1}\left[\sigma^r_{jj}\sigma^r_{tt}-(\sigma^r_{jt})^2\right]\right)\\ &= k\,(n-k+1)\left(b-\frac{c-3}{4}-\frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}\right)\\ &\quad+\sum_{r=n+3}^{2m+1}\left(\sum_{j=1}^{k}\sum_{t=k+1}^{n+1}\left(\sigma^r_{jt}\right)^2+\sum_{j=1}^{k}\left(\sigma^r_{jj}\right)\right)\\ &\geq k\,(n-k+1)\left(b-\frac{c-3}{4}-\frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}\right).\end{aligned}$$

(d) For $i\neq j$, if $u_{ij}=0$ then $a_i=0$ or $a_j=0$. The statement $a_i=0$ implies that $u_{il}=a_ia_l=0$ for all $l\in\{1,...,n+1\}$, $l\neq i$. Then, we get

$$\sum_{j\neq i}u_{ij}=0,$$

which is a contradiction with (a). Thus, for $i\neq j$, it follows that $u_{ij}\neq 0$. We assume that $u_{1\ n+1}<0$. From (b), for $1<i<n+1$, we get $u_{1i}u_{i\ n+1}<0$. Without loss of generality, we may assume

$$\begin{gathered}u_{12},\dots,u_{1l},,u_{l+1\ n+1},\dots,u_{n\ n+1}>0,\\ u_{1\ l+1},\dots,u_{1\ n+1},u_{2\ n+1},\dots,u_{l\ n+1}<0,\end{gathered}\tag{3.5}$$

for some $\left[\frac{n}{2}+1\right]\leq l\leq n$. If $l=n$, then $u_{1\ n+1}+u_{2\ n+1}+\cdots+u_{n\ n+1}<0$, which contradicts to (a). Thus, $l<n$. From (b), we get

$$a^2_{n+1}=\frac{u_{i\ n+1}u_{t\ n+1}}{u_{i\ t}}>0,\tag{3.6}$$

where $2\leq i\leq l$, $l+1\leq t\leq n$. By (3.5) and (3.6), we obtain $u_{it}<0$, which implies that

$$\sum_{i=1}^{l}\sum_{t=l+1}^{n+1}u_{it}=\sum_{i=2}^{l}\sum_{t=l+1}^{n}u_{it}+\sum_{i=1}^{l}u_{i\ n+1}+\sum_{t=l+1}^{n+1}u_{1t}<0,$$

which is a contradiction to (c). Thus (d) is proved. □

Theorem 3.2. *Let M be an $(n+1)$-dimensional θ-slant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$. If at a point $p \in M$ there exists a number $b > \frac{(c-3)}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$ such that the sectional curvature $K \geq b$ at p, then the shape operator A_H at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) I_n \quad \text{at } p, \tag{3.7}$$

where I_n denotes the identity map identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. Let $p \in M$ and a number $b > \frac{(c-3)}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$ such that the sectional curvature $K \geq b$ at p. We choose an orthonormal basis $\{e_1, \dots, e_{n+1}, e_{n+2}, \dots, e_{2m+1}\}$ at p such that e_{n+2} is parallel to the mean curvature vector H, and $e_1, \dots, e_{n+1}$ diagonalize the shape operator A_{n+2}. Now, from Lemma 3.1 it follows that $a_1, ..., a_{n+1}$ have the same sign. We assume that $a_j > 0$ for all $j \in \{1, \dots, n+1\}$. Then

$$\begin{aligned} \sum_{j \neq i} u_{ij} &= a_i\,(a_1 + \cdots + a_{n+1}) - a_i^2 \\ &\geq n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right). \end{aligned} \tag{3.8}$$

From (3.8) and (3.1), we obtain

$$\begin{aligned} a_i\,(n+1)\,\|H\| &\geq n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right) + a_i^2 \\ &> n\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right), \end{aligned}$$

which implies that

$$a_i \|H\| > \frac{n}{n+1}\left(b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right).$$

Hence, we get (3.7). □

In particular, from the above theorem we have the following results.

Corollary 3.3. *Let M be an $(n+1)$-dimensional invariant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. If at a point $p \in M$ there exists a number $b > \frac{c-3}{4} + \frac{c+1}{4(n+1)}$ such that the sectional curvature $K \geq b$ at p, then the shape operator A_H at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}\left(b - \frac{c-3}{4} - \frac{c+1}{4(n+1)}\right) I_n \quad \text{at } p. \tag{3.9}$$

Corollary 3.4. *Let M be an $(n+1)$-dimensional anti-invariant submanifold isometrically immersed in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. If at a point $p \in M$ there exists a number $b > \frac{(c-3)}{4} - \frac{c+1}{2(n+1)}$ such that the sectional curvature $K \geq b$ at p, then the shape operator A_H at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}\left(b - \frac{c-3}{4} + \frac{c+1}{2(n+1)}\right) I_n \quad \text{at } p. \tag{3.10}$$

4. k-Ricci curvature and shape operator

In this section, we establish a relation between the shape operator and the k-Ricci curvature for an $(n+1)$-dimensional slant submanifold in a $(2m+1)$-dimensional Kenmotsu space form.

Theorem 4.1. *Let M be an $(n+1)$-dimensional θ-slant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. Then, for any integer $k, 2 \leq k \leq n+1$, and any point $p \in M$, we have*

(1) *If $\Theta_k(p) \neq \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}]I_n \quad \text{at } p, \tag{4.1}$$

where I_n denotes the identity map identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

(2) *If $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$, then $A_H \geq 0$ at p.*

(3) *A unit vector $X \in T_pM$ satisfies*

$$A_H(X) = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}]X \tag{4.2}$$

if and only if $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$ and $X \in \mathcal{N}_p$.

Proof. Let $\{e_1, ...e_{n+1} = \xi\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. It follows from (2.10) and (2.11) that

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \mathrm{Ric}_{L_{i_1 \dots i_k}}(e_i), \tag{4.3}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}). \tag{4.4}$$

Combining (2.12), (4.3) and (4.4), we find

$$\tau(p) \geq \frac{n(n+1)}{2} \Theta_k(p). \tag{4.5}$$

From the equation of Gauss, for $X = Z = e_i$, $Y = W = e_j$, by summing over $\{1, 2, \cdots, n+1\}$ with respect to i and j $(i \neq j)$, we obtain

$$(n+1)^2 \|H\|^2 = \|\sigma\|^2 + 2\tau - \frac{n(n+1)}{4}(c-3) - \frac{(3n\cos^2\theta - 2n)(c+1)}{4}. \tag{4.6}$$

Now we choose an orthonormal basis

$$\{e_1, \dots, e_{n+1} = \xi, e_{n+2}, \dots, e_{2m+1}\}$$

at p such that e_{n+2} is parallel to the mean curvature vector $H(p)$, and $e_1, \dots, e_{n+1}$ diagonalize the shape operator A_{n+2}. Then we have the relations (3.1), (3.2) and (3.4). From (4.6) we get

$$(n+1)^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n+1} a_i^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (\sigma_{ij}^r)^2 - \frac{n(n+1)}{4}(c-3) - \frac{(3n\cos^2\theta - 2n)(c+1)}{4}. \tag{4.7}$$

On the other hand, since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = n \sum_{i}^{n+1} a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$(n+1)^2 \|H\|^2 = (\sum_{i=1}^{n+1} a_i)^2 = \sum_{i=1}^{n+1} a_i^2 + 2 \sum_{i<j} a_i a_j \le (n+1) \sum_{i=1}^{n+1} a_i^2,$$

which implies

$$\sum_{i=1}^{n+1} a_i^2 \ge (n+1) \|H\|^2. \tag{4.8}$$

From (4.7) and (4.8), we have

$$(n+1)^2 \|H\|^2 \ge 2\tau + (n+1)\|H\|^2 - \frac{n(n+1)}{4}(c-3) - \frac{(3n\cos^2\theta - 2n)(c+1)}{4}, \tag{4.9}$$

or equivalently

$$\|H\|^2 \geq \frac{2\tau}{n(n+1)} - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}. \tag{4.10}$$

From (29) and (34), we have

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}. \tag{4.11}$$

This shows that $H(p) = 0$ may occur only when $\Theta_k(p) \leq \frac{c-3}{4} + \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}$. Consequently, if $H(p) = 0$, statements (1) and (2) hold automatically. Therefore, without loss of generality, we assume $H(p) \neq 0$. From the Gauss equation we get

$$a_i a_j = K_{ij} - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)} - \sum_{r=n+3}^{2m+1} \{\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2\}. \tag{4.12}$$

From (4.12) we have

$$a_1(a_{i_2} + \cdots + a_{i_k}) = \mathrm{Ric}_{L_{1i_2\cdots i_k}} - \frac{(k-1)(c-3)}{4} - \frac{(3n\cos^2\theta - 2n)(k-1)(c+1)}{4n(n+1)} + \sum_{r=n+3}^{2m+1}\sum_{j=2}^{k}\{(\sigma_{1i_j}^r)^2 - \sigma_{11}^r\sigma_{i_ji_j}^r\}, \tag{4.13}$$

which yields

$$a_1(a_2 + \cdots + a_{n+1}) = \frac{1}{\binom{n}{k-1}} \sum_{2\leq i_2<\cdots<i_k\leq n+1} \mathrm{Ric}_{L_{1i_2\cdots i_k}}(e_1) - \frac{n(c-3)}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4(n+1)} + \sum_{r=n+3}^{2m+1}\sum_{j=1}^{n+1}(\sigma_{1_j}^r)^2. \tag{4.14}$$

From (2.12) and (4.14) we have

$$a_1(a_2+\cdots+a_{n+1}) \geq n[\Theta_k(p) - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}]. \tag{4.15}$$

Then

$$\begin{aligned} a_1(a_1+\cdots+a_{n+1}) &= a_1^2 + a_1(a_2+\cdots+a_{n+1}) \\ &\geq a_1^2 + n[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}] \\ &\geq n[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}]. \end{aligned} \tag{4.16}$$

Similar inequalities hold when 1 is changed by $j \in \{2,\cdots,n+1\}$. So we have

$$a_j(a_1+\cdots+a_{n+1}) \geq n[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}] \tag{4.17}$$

for $i \in \{2,\cdots,n+1\}$. Then we can get

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}]I_n \quad \text{at } p. \tag{4.18}$$

The equality does not hold because $H(p) \neq 0$. So (4.1) is valid. The statement (2) is obvious.

(3) Let $X \in T_pM$ be a unit vector satisfying (4.2). By (4.14) and (4.16) we have $a_1 = 0$ and $\sigma_{1j}^r = 0$ for any $j \in \{2,\cdots,n+1\}$, $r \in \{n+3,\cdots,2m+1\}$. It follows that $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n\cos^2\theta-2n)(c+1)}{4n(n+1)}$ and $X \in \mathcal{N}_p$. The converse is clear. This completes the proof of the Theorem 4.1. □

Corollary 4.2. *Let M be an $(n+1)$-dimensional invariant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n+1$, and any point $p \in M$, we have*

(1) *If* $\Theta_k(p) \neq \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$, *then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]I_n \quad at\ p. \tag{4.19}$$

(2) *If* $\Theta_k(p) = \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$, *then* $A_H \geq 0$ *at* p.

(3) *A unit vector* $X \in T_pM$ *satisfies*

$$A_H(X) = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]X \tag{4.20}$$

if and only if $\Theta_k(p) = \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$ *and* $X \in \mathcal{N}_p$.

(4) $A_H = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]I_n$ *at* p *if and only if* p *is a totally geodesic point.*

Corollary 4.3. *Let* M *be an* $(n+1)$*-dimensional anti-invariant submanifold in a* $(2m+1)$*-dimensional Kenmotsu space form* $\tilde{M}(c)$. *Then, for any integer* $k, 2 \leq k \leq n+1$, *and any point* $p \in M$, *we have*

(1) *If* $\Theta_k(p) \neq \frac{c-3}{4} - \frac{c+1}{2(n+1)}$, *then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} + \frac{c+1}{2(n+1)}]I_n \quad at\ p. \tag{4.21}$$

(2) *If* $\Theta_k(p) = \frac{c-3}{4} - \frac{c+1}{2(n+1)}$, *then* $A_H \geq 0$ *at* p.

(3) *A unit vector* $X \in T_pM$ *satisfies*

$$A_H = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} + \frac{c+1}{2(n+1)}]X \tag{4.22}$$

if and only if $\Theta_k(p) = \frac{c-3}{4} - \frac{c+1}{2(n+1)}$ *and* $X \in \mathcal{N}_p$.

References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer Verlag, 1976.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Slant submanifolds in Kenmotsu manifolds, *Glasgow Math. J.* **42** (2)(2000), 125–138.
[3] A. Carriazo, A contact version of B.-Y. Chen's inequality and its applications to slant immersions, *Kyungpook Math. J.* **39** (2)(1999), 465-476.
[4] B. Y. Chen, Mean curvature and shape operator of isometric immersions in real space form, *Glasgow Math. J.* **38** (1996), 87-97.
[5] B.Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasgow Math. J.* **41** (1999), 33-41.
[6] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Roumanie* **39** (1996), 183-186.
[7] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tôhoku Math. J.* **24** (1972), 93-103.
[8] K. Yano and M. Kon, *Structures on Manifolds*, Ser. Pure Math. 3., World Scientific, 1984.

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