SHAPE OPERATOR OF SLANT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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ABSTRACT. In this paper, we establish some relations between the sectional curvature and the shape operator and also between the $k$-Ricci curvature and the shape operator for slant submanifolds in Kenmotsu space forms.

1. Introduction

According to B.Y. Chen, one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Scalar curvature and Ricci curvature are among the main intrinsic invariants, while the squared mean curvature is the main extrinsic invariant. In [4], B.Y. Chen establishes a relationship between sectional curvature function $K$ and the shape operator for submanifolds in real space forms. In [5], he also gives a relationship between Ricci curvature and squared mean curvature.

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A contact version of B.Y. Chen’s inequality and its applications to slant immersions in a Sasakian space form $\tilde{M}(c)$ are given in [3]. But there is another interesting class of almost contact metric manifolds, namely Kenmotsu manifolds [7]. In the present paper, we study slant submanifolds of Kenmotsu space forms and establish relations between the sectional curvature and the shape operator and also between the $k$-Ricci curvature and the shape operator for slant submanifolds in Kenmotsu space forms.

2. Preliminaries

Let $\tilde{M}$ be an almost contact metric manifold [1] with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $\tilde{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

(2.1)

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.2)

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X)$$

(2.3)

for all $X, Y \in T\tilde{M}$.

An almost contact metric manifold is known to be a Kenmotsu manifold [7] if

$$(\tilde{\nabla}_X \phi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

(2.4)

$$\tilde{\nabla}_X \xi = -\phi^2 X = X - \eta(X)\xi, \quad X \in T\tilde{M},$$

(2.5)

for any vector fields $X, Y$ on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to $g$.

We denote by $\Phi$ the fundamental 2-form of $\tilde{M}$, that is, $\Phi(X, Y) = g(\phi X, Y)$, for any vector fields $X, Y$ on $\tilde{M}$. It was proved that the pairing $(\Phi, \eta)$ defines a locally conformal cosymplectic structure, that is

$$d\Phi = 2\Phi \wedge \eta, \quad d\eta = 0.$$

(2.6)
A plane section $\sigma$ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a $\varphi$-section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\tilde{M}$ is of constant $\varphi$-sectional curvature if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the $\varphi$-section $\sigma$ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any $\varphi$-section $\sigma$ of $T_p\tilde{M}$, the function $c(p) = \tilde{K}(\sigma)$ is called the $\varphi$-sectional curvature of $\tilde{M}$. A Kenmotsu manifold $\tilde{M}$ with constant $\varphi$-sectional curvature $c$ is said to be a Kenmotsu space form and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Kenmotsu space form $\tilde{M}(c)$ is given by [7]

\[
4\tilde{R}(X, Y, Z, W) = (c - 3)[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)] \\
+ (c + 1)[g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) - \\
2g(\varphi X, Y)g(\varphi Z, W) + g(X, Z)\eta(Y)\eta(W) - \\
g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) - \\
g(X, W)\eta(Y)\eta(Z)] \\
\] (2.7)

for all $X, Y, Z, W \in T\tilde{M}$.

Let $M$ be an $n$-dimensional Riemannian manifold. The scalar curvature $\tau$ at $p$ is given by $\tau = \sum_{i<j}K_{ij}$, where $K_{ij}$ is the sectional curvature of $M$ associated with a plane section spanned by $e_i$ and $e_j$ at $p \in M$ for any orthonormal basis $\{e_1, \ldots, e_n\}$ for $T_pM$. Now let $M$ be a submanifold of an $m$-dimensional manifold $\tilde{M}$ equipped with a Riemannian metric $g$. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$ for all $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla$, $\nabla^\perp$ and $\tilde{\nabla}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, $M$ and the normal bundle $T^\perp M$ of $M$ respectively, and $\sigma$ is the second fundamental form related to the shape operator $A$ by $g(h(X, Y), N) = g(A_N X, Y)$. Then the equation of Gauss is given by

\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) \\
- g(h(X, Z), h(Y, W)) \\
\] (2.8)

for any vectors $X, Y, Z, W$ tangent to $M$, where $\tilde{R}$ and $R$ are the curvature tensors of $\tilde{M}$ and $M$ respectively.
The relative null space of $M$ at a point $p \in M$ is defined by
$$\mathcal{N}_p = \{ X \in T_pM | \sigma(X, Y) = 0, \text{ for all } Y \in T_pM \}.$$ 
Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space $T_pM$. The mean curvature vector $H$ at $p \in M$ is
$$H = \frac{1}{n} \text{trace}(\sigma) = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i).$$ \hspace{1cm} (2.9)

The submanifold $M$ is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then $M$ is totally umbilical. We put
$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r), \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)),$$ 
where $e_r$ belongs to an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of the normal space $T_p^\perp M$.

Suppose $L$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L$. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of $L$ such that $e_1 = X$. Define the Ricci curvature $\text{Ric}_L$ of $L$ at $X$ by
$$\text{Ric}_L(X) = K_{12} + K_{13} + ... + K_{1k},$$ \hspace{1cm} (2.10)
where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. We simply called such a curvature a $k$-Ricci curvature.

The scalar curvature $\tau$ of the $k$-plane section $L$ is given by
$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.$$ \hspace{1cm} (2.11)

For each integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on an $n$-dimensional Riemannian manifold $M$ is defined by
$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \text{Ric}_L(X), \quad p \in M,$$ \hspace{1cm} (2.12)
where $L$ runs over all $k$-plane sections in $T_pM$ and $X$ runs over all unit vectors in $L$.

From now on, we assume that the dimension of $M$ is $n+1$ and that of the ambient manifold $\tilde{M}$ is $2m+1$. We also assume that the structure vector field $\xi$ is tangent to $M$. 


For a vector field $X$ in $M$, we put
\[ \varphi X = PX + FX, \quad PX \in TM, \quad FX \in T^\perp M. \]
(2.13)

For any local orthonormal basis \( \{e_1, e_2, \ldots, e_{n+1}\} \) for \( T_pM \), we can define the squared norm of \( P \) and \( F \) by
\[ \|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2, \quad \|F\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Fe_j)^2, \]
(2.14)
respectively. It is easy to show that both \( \|P\|^2 \) and \( \|F\|^2 \) are independent of the choice of the orthonormal basis.

A submanifold $M$ of an almost contact metric manifold with $\xi \in TM$ is called a semi-invariant submanifold or a contact CR submanifold [8] if there exists two differentiable distributions $D$ and $D^\perp$ on $M$ such that (i) $TM = D \oplus D^\perp \oplus E$, (ii) the distribution $D$ is invariant by $\varphi$, i.e., $\varphi(D) = D$, and (iii) the distribution $D^\perp$ is anti-invariant by $\varphi$, i.e., $\varphi(D^\perp) \subseteq T^\perp M$.

The submanifold $M$ tangent to $\xi$ is said to be invariant or anti-invariant [8] according to $F = 0$ or $P = 0$. Thus, a contact CR-submanifold is invariant or anti-invariant according to $D^\perp = \{0\}$ or $D = \{0\}$. A proper contact CR-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that $X$ is not proportional to $\xi_p$, we denote the angle between $\varphi X$ and $T_pM$ by $\theta(X)$. Then $M$ is said to be slant [2] and [6] if the angle $\theta(X)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle $\theta$ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A proper slant immersion is neither invariant nor anti-invariant.

3. Sectional curvature and shape operator

B.Y. Chen establishes a sharp relationship between the shape operator and the sectional curvature for submanifolds in real space forms [5]. In this section, we establish a similar inequality between
the shape operator and the sectional curvature for slant submanifolds in a Kenmotsu space form.

Let $M$ be an $(n+1)$-dimensional $\theta$-slant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. Let $p \in M$ and a number

$$b > \frac{c - 3}{4} + \frac{(3n\cos^2\theta - 2n)(c + 1)}{4n(n + 1)}$$

such that the sectional curvature $K \geq b$ at $p$.

Now we choose an orthonormal basis

$$\{e_1, \ldots, e_{n+1} = \xi, e_{n+2}, \ldots, e_{2m+1}\}$$

at $p$ such that $e_{n+2}$ is parallel to the mean curvature vector $H$, and $e_1, \ldots, e_{n+1}$ diagonalize the shape operator $A_{n+2}$. Then we have

$$A_{n+2} = \begin{pmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{n+1}
\end{pmatrix}, \quad (3.1)$$

$$A_r = (\sigma_{ij}^r), \quad \text{trace } A_r = \sum_{i=1}^{n+1} \sigma_{ii}^r = 0,$$

$$i, j = 1, \ldots, n+1; r = n+3, \ldots, 2m+1. \quad (3.2)$$

For $i \neq j$, we put

$$u_{ij} = a_i a_j = u_{ji}. \quad (3.3)$$

By Gauss equation, for $X = Z = e_i, Y = W = e_j$, we have

$$u_{ij} \geq b - \frac{c - 3}{4} - \frac{(3n\cos^2\theta - 2n)(c + 1)}{4n(n + 1)} - \sum_{r=n+3}^{2m+1} \frac{\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2}{4n(n + 1)}. \quad (3.4)$$

**Lemma 3.1.** For $u_{ij}$ we have

(a) For any fixed $i \in \{1, \ldots, n+1\}$, we have

$$\sum_{i \neq j} u_{ij} \geq n \left( b - \frac{c - 3}{4} - \frac{(3n\cos^2\theta - 2n)(c + 1)}{4n(n + 1)} \right).$$
(b) For distinct $i, j, k \in \{1, \ldots, n+1\}$, we have $a_i^2 = u_{ij}u_{ik}/u_{jk}$.

(c) For a fixed $k$, $1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ and for each $B \in S_k \equiv \{B \subset \{1, \ldots, n+1\} : |B| = k\}$, we have

$$
\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k(n-k+1) \left( b - \frac{c+3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right),
$$

where $\bar{B}$ is the complement of $B$ in $\{1, \ldots, n+1\}$.

(d) For distinct $i, j \in \{1, \ldots, n+1\}$, it follows that $u_{ij} > 0$.

**Proof.** (a) From (3.2), (3.3) and (3.4), we have

$$
\sum_{i \neq j} u_{ij} \geq n \left( b - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right)
$$

$$
- \sum_{r=n+3}^{2n+1} \left( \sigma_{ii}^r \left( \sum_{j \neq i} \sigma_{jj}^r \right) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right)
$$

$$
= n \left( b - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right)
$$

$$
- \sum_{r=n+3}^{2n+1} \left( \sigma_{ii}^r (-\sigma_{ii}) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right)
$$

$$
= n \left( b - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right)
$$

$$
+ \sum_{r=n+3}^{2n+1} \sum_{j=1}^{n+1} (\sigma_{ij}^r)^2
$$

$$
\geq n \left( b - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right) > 0.
$$

(b) We have $u_{ij}u_{ik}/u_{jk} = a_i a_j a_i a_k / a_j a_k = a_i^2$.
(c) Let $B = \{1, ..., k\}$ and $\bar{B} = \{k + 1, ..., n + 1\}$. Then

\[
\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k(n - k + 1) \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4n(n + 1)} \right)
\]

\[
- \sum_{r=n+3}^{2m+1} \left( \sum_{j=1}^{k} \sum_{t=k+1}^{n+1} \left[ \sigma_{jj}^r \sigma_{tt}^r - (\sigma_{jt}^r)^2 \right] \right)
\]

\[
= k(n - k + 1) \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4n(n + 1)} \right)
\]

\[
+ \sum_{r=n+3}^{2m+1} \left( \sum_{j=1}^{k} \sum_{t=k+1}^{n+1} (\sigma_{jt}^r)^2 + \sum_{j=1}^{k} (\sigma_{jj}^r) \right)
\]

\[
\geq k(n - k + 1) \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4n(n + 1)} \right).
\]

(d) For $i \neq j$, if $u_{ij} = 0$ then $a_i = 0$ or $a_j = 0$. The statement $a_i = 0$ implies that $u_{il} = a_i a_l = 0$ for all $l \in \{1, ..., n + 1\}, l \neq i$. Then, we get

\[
\sum_{j \neq i} u_{ij} = 0,
\]

which is a contradiction with (a). Thus, for $i \neq j$, it follows that $u_{ij} \neq 0$. We assume that $u_{1 n + 1} < 0$. From (b), for $1 < i < n + 1$, we get $u_{1i} u_{in + 1} < 0$. Without loss of generality, we may assume

\[
u_{12}, \ldots, u_{1t}, u_{1 n + 1}, \ldots, u_{n n + 1} > 0,
\]

\[
u_{1 t+1}, \ldots, u_{1 n + 1}, u_{2 n + 1}, \ldots, u_{n n + 1} < 0,
\]

for some $\left[ \frac{n}{2} + 1 \right] \leq l \leq n$. If $l = n$, then $u_{1 n + 1} + u_{2 n + 1} + \cdots + u_{n n + 1} < 0$, which contradicts to (a). Thus, $l < n$. From (b), we get

\[
u_{n + 1}^2 = \frac{u_{i n + 1} u_{t n + 1}}{u_{i t}} > 0,
\]

(3.6)

where $2 \leq i \leq l$, $l + 1 \leq t \leq n$. By (3.5) and (3.6), we obtain $u_{il} < 0$, which implies that

\[
\sum_{i=1}^{l} \sum_{t=l+1}^{n+1} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n+1} u_{it} + \sum_{i=1}^{l} u_{i n + 1} + \sum_{t=l+1}^{n+1} u_{it} < 0,
\]
which is a contradiction to (c). Thus (d) is proved. \(\square\)

**Theorem 3.2.** Let \(M\) be an \((n+1)\)-dimensional \(\theta\)-slant submanifold in a \((2m+1)\)-dimensional Kenmotsu space form \(\tilde{M}(c)\). If at a point \(p \in M\) there exists a number \(b > \frac{(c-3)}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}\) such that the sectional curvature \(K \geq b\) at \(p\), then the shape operator \(A_H\) at the mean curvature vector satisfies

\[
A_H > n \left( \frac{b - c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right) I_n \quad \text{at } p, \tag{3.7}
\]

where \(I_n\) denotes the identity map identified with \(\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\).

**Proof.** Let \(p \in M\) and a number \(b > \frac{(c-3)}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}\) such that the sectional curvature \(K \geq b\) at \(p\). We choose an orthonormal basis \(\{e_1, \ldots, e_{n+1}, e_{n+2}, \ldots, e_{2m+1}\}\) at \(p\) such that \(e_{n+2}\) is parallel to the mean curvature vector \(H\), and \(e_1, \ldots, e_{n+1}\) diagonalize the shape operator \(A_{n+2}\). Now, from Lemma 3.1 it follows that \(a_1, \ldots, a_{n+1}\) have the same sign. We assume that \(a_j > 0\) for all \(j \in \{1, \ldots, n+1\}\). Then

\[
\sum_{j \neq i} u_{ij} = a_i (a_1 + \cdots + a_{n+1}) - a_i^2 \\
\geq n \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right) \tag{3.8}
\]

From (3.8) and (3.1), we obtain

\[
a_i (n+1) \|H\| \geq n \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right) + a_i^2 \\
> n \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \right),
\]
which implies that
\[
a_i \| H \| > \frac{n}{n + 1} \left( b - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4n(n + 1)} \right).
\]

Hence, we get (3.7). \(\square\)

In particular, from the above theorem we have the following results.

**Corollary 3.3.** Let \( M \) be an \((n + 1)\)-dimensional invariant submanifold in a \((2m + 1)\)-dimensional Kenmotsu space form \( \tilde{M}(c) \) such that \( \xi \in TM \). If at a point \( p \in M \) there exists a number \( b > \frac{c - 3}{4} + \frac{c + 1}{4(n+1)} \) such that the sectional curvature \( K \geq b \) at \( p \), then the shape operator \( A_H \) at the mean curvature vector satisfies
\[
A_H > \frac{n}{n + 1} \left( b - \frac{c - 3}{4} - \frac{c + 1}{4(n+1)} \right) I_n \quad \text{at } p. \tag{3.9}
\]

**Corollary 3.4.** Let \( M \) be an \((n + 1)\)-dimensional anti-invariant submanifold isometrically immersed in a \((2m+1)\)-dimensional Kenmotsu space form \( \tilde{M}(c) \) such that \( \xi \in TM \). If at a point \( p \in M \) there exists a number \( b > \frac{(c-3)}{4} - \frac{c+1}{2(n+1)} \) such that the sectional curvature \( K \geq b \) at \( p \), then the shape operator \( A_H \) at the mean curvature vector satisfies
\[
A_H > \frac{n}{n + 1} \left( b - \frac{c - 3}{4} + \frac{c + 1}{2(n+1)} \right) I_n \quad \text{at } p. \tag{3.10}
\]

4. \( k \)-Ricci curvature and shape operator

In this section, we establish a relation between the shape operator and the \( k \)-Ricci curvature for an \((n + 1)\)-dimensional slant submanifold in a \((2m + 1)\)-dimensional Kenmotsu space form.
Theorem 4.1. Let $M$ be an $(n + 1)$-dimensional $\theta$-slant submanifold in a $(2m + 1)$-dimensional Kenmotsu space form $\tilde{M}(c)$ such that $\xi \in TM$. Then, for any integer $k, 2 \leq k \leq n + 1$, and any point $p \in M$, we have

1. If $\Theta_k(p) \neq \frac{c-3}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}]I_n \quad \text{at } p,$$

where $I_n$ denotes the identity map identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

2. If $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}$, then $A_H \geq 0$ at $p$.

3. A unit vector $X \in T_pM$ satisfies

$$A_H(X) = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}]X$$

if and only if $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}$ and $X \in N_p$.

Proof. Let $\{e_1, \ldots, e_{n+1} = \xi\}$ be an orthonormal basis of $T_pM$. Denote by $L_{i_1 \ldots i_k}$ the $k$-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. It follows from (2.10) and (2.11) that

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \text{Ric}_{L_{i_1 \ldots i_k}}(e_i),$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \tau(L_{i_1 \ldots i_k}).$$

Combining (2.12), (4.3) and (4.4), we find

$$\tau(p) \geq \frac{n(n+1)}{2} \Theta_k(p).$$

From the equation of Gauss, for $X = Z = e_i, Y = W = e_j$, by summing over $\{1, 2, \ldots, n + 1\}$ with respect to $i$ and $j$ ($i \neq j$), we obtain
\[(n + 1)^2 \|H\|^2 =
\|\sigma\|^2 + 2\tau - \frac{n(n + 1)}{4} (c - 3) - \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4}. \tag{4.6}\]

Now we choose an orthonormal basis
\[
\{e_1, \ldots, e_{n+1} = \xi, e_{n+2}, \ldots, e_{2m+1}\}
\]
at \(p\) such that \(e_{n+2}\) is parallel to the mean curvature vector \(H(p)\), and \(e_1, \ldots, e_{n+1}\) diagonalize the shape operator \(A_{n+2}\). Then we have the relations (3.1), (3.2) and (3.4). From (4.6) we get
\[
\frac{(n + 1)^2 \|H\|^2}{4} = 2\tau + \sum_{i=1}^{n+1} a_i^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (\sigma_{ij}^r)^2 - \frac{n(n + 1)}{4} (c - 3)
- \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4}. \tag{4.7}\]

On the other hand, since
\[
0 \leq \sum_{i<j} (a_i - a_j)^2 = n \sum_{i=1}^{n+1} a_i^2 - 2 \sum_{i<j} a_i a_j,
\]
we obtain
\[
(n + 1)^2 \|H\|^2 = \left(\sum_{i=1}^{n+1} a_i\right)^2 = \sum_{i=1}^{n+1} a_i^2 + 2 \sum_{i<j} a_i a_j \leq (n + 1) \sum_{i=1}^{n+1} a_i^2,
\]
which implies
\[
\sum_{i=1}^{n+1} a_i^2 \geq (n + 1) \|H\|^2. \tag{4.8}\]

From (4.7) and (4.8), we have
\[
(n + 1)^2 \|H\|^2 \geq 2\tau + (n + 1) \|H\|^2 - \frac{n(n + 1)}{4} (c - 3)
- \frac{(3n \cos^2 \theta - 2n)(c + 1)}{4}, \tag{4.9}\]
or equivalently
\[
\|H\|^2 \geq \frac{2\tau}{n(n+1)} - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}.
\] (4.10)

From (29) and (34), we have
\[
\|H\|^2(p) \geq \Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}.
\] (4.11)

This shows that \(H(p) = 0\) may occur only when \(\Theta_k(p) \leq \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\). Consequently, if \(H(p) = 0\), statements (1) and (2) hold automatically. Therefore, without loss of generality, we assume \(H(p) \neq 0\). From the Gauss equation we get
\[
a_i a_j = K_{ij} - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}
- \sum_{r=n+3}^{2m+1} \{\sigma_{ij}^r - (\sigma_{ij}^r)^2\}.
\] (4.12)

From (4.12) we have
\[
a_1(a_{i_2} + \cdots + a_{i_k}) = \text{Ric}_{L_{1i_2\cdots i_k}} - \frac{(k-1)(c-3)}{4}
- \frac{(3n\cos^2\theta - 2n)(k-1)(c+1)}{4n(n+1)}
+ \sum_{r=n+3}^{2m+1} \sum_{j=2}^{k} ((\sigma_{ij}^r)^2 - \sigma_{ij}^{r_1}\sigma_{ij}^{r_j}),
\] (4.13)

which yields
\[
a_1(a_{i_2} + \cdots + a_{i_{n+1}}) = \frac{1}{\binom{n}{k-1}} \sum_{2\leq i_2<\cdots<i_{k}\leq n+1} \text{Ric}_{L_{1i_2\cdots i_k}}(e_1) - \frac{n(c-3)}{4}
- \frac{(3n\cos^2\theta - 2n)(c+1)}{4(n+1)}
+ \sum_{r=n+3}^{2m+1} \sum_{j=1}^{n+1} (\sigma_{ij}^r)^2.
\] (4.14)
From (2.12) and (4.14) we have

\[ a_1(a_2 + \cdots + a_{n+1}) \geq n[\Theta_k(p) - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}]. \]  

(4.15)

Then

\[ a_1(a_1 + \cdots + a_{n+1}) = a_1^2 + a_1(a_2 + \cdots + a_{n+1}) \]

\[ \geq a_1^2 + n[\Theta_k(p) - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}] \]

\[ \geq n[\Theta_k(p) - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}]. \]  

(4.16)

Similar inequalities hold when 1 is changed by \( j \in \{2, \cdots, n+1\} \). So we have

\[ a_j(a_1 + \cdots + a_{n+1}) \geq n[\Theta_k(p) - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}]\]  

for \( i \in \{2, \cdots, n+1\} \). Then we can get

\[ A_H > \frac{n}{n+1} [\Theta_k(p) - \frac{c - 3}{4} - \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)}] I_n \text{ at } p. \]  

(4.18)

The equality does not hold because \( H(p) \neq 0 \). So (4.1) is valid.

The statement (2) is obvious.

(3) Let \( X \in T_p M \) be a unit vector satisfying (4.2). By (4.14) and (4.16) we have \( a_1 = 0 \) and \( \sigma_{ij}^r = 0 \) for any \( j \in \{2, \cdots, n+1\}, \ r \in \{n+3, \cdots, 2m+1\} \). It follows that

\[ \Theta_k(p) = \frac{c-3}{4} + \frac{(3n \cos^2 \theta - 2n)(c+1)}{4n(n+1)} \]

and \( X \in \mathcal{N}_p \). The converse is clear. This completes the proof of the Theorem 4.1. \( \square \)

**Corollary 4.2.** Let \( M \) be an \((n+1)\)-dimensional invariant submanifold in a \((2m+1)\)-dimensional Kenmotsu space form \( \tilde{M}(c) \). Then, for any integer \( k, 2 \leq k \leq n+1 \), and any point \( p \in M \), we have
(1) If $\Theta_k(p) \neq \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]I_n \quad \text{at } p.$$  (4.19)

(2) If $\Theta_k(p) = \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$, then $A_H \geq 0$ at $p$.

(3) A unit vector $X \in T_pM$ satisfies

$$A_H(X) = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]X$$  (4.20)

if and only if $\Theta_k(p) = \frac{c-3}{4} + \frac{(c+1)}{4(n+1)}$ and $X \in N_p$.

(4) $A_H = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} - \frac{(c+1)}{4(n+1)}]I_n$ at $p$ if and only if $p$ is a totally geodesic point.

**Corollary 4.3.** Let $M$ be an $(n+1)$-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$. Then, for any integer $k$, $2 \leq k \leq n+1$, and any point $p \in M$, we have

(1) If $\Theta_k(p) \neq \frac{c-3}{4} - \frac{(c+1)}{2(n+1)}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} + \frac{(c+1)}{2(n+1)}]I_n \quad \text{at } p.$$  (4.21)

(2) If $\Theta_k(p) = \frac{c-3}{4} - \frac{(c+1)}{2(n+1)}$, then $A_H \geq 0$ at $p$.

(3) A unit vector $X \in T_pM$ satisfies

$$A_H = \frac{n}{n+1}[\Theta_k(p) - \frac{c-3}{4} + \frac{(c+1)}{2(n+1)}]X$$  (4.22)

if and only if $\Theta_k(p) = \frac{c-3}{4} - \frac{(c+1)}{2(n+1)}$ and $X \in N_p$.

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