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Author(s):

Y. Pakravesh and A. Iranmanesh

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NORMAL EDGE-TRANSITIVE CAYLEY GRAPHS ON THE NON-ABELIAN GROUPS OF ORDER $4p^2$, WHERE p IS A PRIME NUMBER

Y. PAKRAVESH AND A. IRANMANESH*

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Abstract. In this paper, we determine all of connected normal edge-transitive Cayley graphs on non-abelian groups with order $4p^2$, where p is a prime number.

Keywords: Cayley graph, normal edge-transitive, vertex-transitive, edge-transitive.
MSC(2010): Primary: 20D60; Secondary: 05B25.

1. Introduction

Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of Γ. An edge joining the vertices u and v is denoted by $\{u, v\}$. The group of automorphisms of Γ is denoted by $Aut(\Gamma)$, which acts on vertices, edges and arcs of Γ. If $Aut(\Gamma)$ acts transitively on vertices, edges or arcs of Γ, then Γ is called vertex-transitive, edge-transitive or arc-transitive respectively. If Γ is vertex and edge-transitive but not arc-transitive, then Γ is called 1/2-arc-transitive. Let G be a finite group and S be an inverse closed subset of G, i.e., $S = S^{-1}$, such that $1 \notin S$. The Cayley graph $\Gamma = Cay(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} | g \in G, s \in S\}$. This graph is connected if and only if $G =< S >$. For $g \in G$, define the mapping $\rho_g : G \to G$ by $\rho_g(x) = xg, x \in G$. We have $\rho_g \in Aut(\Gamma)$ for every $g \in G$, thus $R(G) = \{\rho_g | g \in G\}$ is a regular subgroup of $Aut(\Gamma)$ isomorphic to G, forcing Γ to be a vertex-transitive graph. Let $\Gamma = Cay(G, S)$ be the Cayley graph of a finite group G on S. Let $Aut(G, S) = \{\sigma \in Aut(G) | S^\sigma = S\}$ and $A = Aut(\Gamma)$. Then the normalizer of $R(G)$ in A is equal to $N_A(R(G)) = R(G) \rtimes Aut(G, S)$, where $\rtimes$ denotes the semi-direct product of two groups. In [13], the graph Γ is called normal if $R(G)$ is a normal subgroup of $Aut(\Gamma)$. Therefore, according to

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*Corresponding author.

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[4], $\Gamma = Cay(G,S)$ is normal if and only if $A := Aut(\Gamma) = R(G) \rtimes Aut(G,S)$, and in this case $A_1 = Aut(G,S)$, where A_1 is the stabilizer of the identity element of G under A. The normality of Cayley graphs has been extensively studied from different points of views by many authors. In [12] all disconnected normal Cayley graphs are obtained.

Definition 1.1. A Cayley graph Γ is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of Γ, respectively. If Γ is normal edge-transitive, but not normal arc-transitive, then it is called a normal 1/2-arc-transitive Cayley graph.

Edge-transitivity of Cayley graphs of small valency have received attention in the literature. A relation between regular maps and edge-transitive Cayley graphs of valency 4 is studied in [9], and in [7] Li et al., characterized edge-transitive Cayley graphs of valency four and odd order. Houlis in [6], classified normal edge-transitive Cayley graphs of groups Z_{pq}, where p and q are distinct primes. In [1], normal edge-transitive Cayley graphs on some abelian groups of valency at most 5 are studied. And in [3], edge-transitive Cayley graphs of valency 4 on non-abelian simple groups are studied. The normal edge-transitivity of dihedral group of order $2n$ is studied in [11]. In this paper, we investigate the normal edge-transitive Cayley graphs on the non-abelian groups of order $4p^2$.

2. Preliminaries

Keeping fixed terminologies used in Section 1, we mention a few results whose proofs can be found in the literature. The following result is proved in [13] and [4].

Proposition 2.1. *Let $\Gamma = Cay(G,S)$. Then the following hold:*
1. $N_A(R(G)) = R(G) \rtimes Aut(G,S)$;
2. $R(G) \trianglelefteq A$ if and only if $A = R(G) \rtimes Aut(G,S)$;
3. Γ is normal if and only if $A_1 = Aut(G,S)$.

The following proposition is very useful for our work (see [10]).

Proposition 2.2. *Let $\Gamma = Cay(G,S)$ be a connected Cayley graph (undirected) on S. Then Γ is normal edge-transitive if and only if $Aut(G,S)$ is either transitive on S, or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S such that $S = T \cup T^{-1}$.*

In the action of $Aut(G,S)$ on S, every element of each orbit has the same order. Therefore, we have following proposition (see [2])

Proposition 2.3. *Let $\Gamma = Cay(G,S)$ and H be the subset of all involutions of the group G. If $< H > \neq G$ and Γ is connected normal edge-transitive, then its valency is even.*

For a general graph $\Gamma = (V, E)$, if v is a vertex in Γ, then $\Gamma(v)$ denotes the set of the neighbors of v, i.e., $\Gamma(v) = \{u \in V | \{u, v\} \in E\}$. The following result which can be deduced from a result in [5], characterize normal arc-transitive Cayley graphs in terms of the action of $Aut(G, S)$ on S.

Proposition 2.4. *Let $\Gamma = Cay(G, S)$ be a connected Cayley graph (undirected) on S. Then Γ is normal arc-transitive if and only if $Aut(G, S)$ acts transitively on S.*

We can extract the following corollary from Proposition 2.2 and 2.4 and the fact that if G is an abelian group, then $\sigma : G \to G$ defined by $\sigma(x) = x^{-1}$, for all $x \in G$, is an automorphism.

Corollary 2.5. *If Γ is a Cayley graph of an abelian group, then Γ is not a normal $1/2$-arc-transitive Cayley graph.*

The following result is obtained in [11].

Proposition 2.6. *Let $\Gamma = Cay(G, S)$ be a connected normal edge-transitive Cayley graph of the dihedral group D_{2n}. Then $Aut(D_{2n}, S)$ is transitive on S.*

Corollary 2.7. *If $\Gamma = Cay(G, S)$ is a Cayley graph of a dihedral group D_{2n}, then Γ is not a normal $1/2$-arc-transitive Cayley graph.*

The following result is mentioned in [10].

Proposition 2.8. *Let Γ be a connected Cayley graph of a non-abelian simple group with valency 3. If Γ is normal edge-transitive, then it is normal.*

The following result is mentioned in [2].

Proposition 2.9. *Let S be a generating set of group G. Then the action of $Aut(G, S)$ on S is faithful.*

In [8], the groups of order $4p^2$ are classified. When $p \equiv 1 \pmod 4$, -1 is a quadratic residue modulo p and also modulo p^2. Let λ be an integer so that $\lambda^2 \equiv -1 \pmod{p^2}$. The non-abelian group of order $4p^2$ is isomorphic to one of the following groups which are given by generators and relations:

1. $G_1 =< a,\ b | a^{p^2} = b^4 = 1,\ b^{-1}ab = a^{-1} >$
2. $G_2 =< a,\ b | a^{p^2} = b^4 = 1,\ b^{-1}ab = a^{\lambda} >$ $(p \equiv 1 \pmod 4)$
3. $G_3 =< a,\ b,\ c | a^{p^2} = b^2 = c^2 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = a^{\lambda} > \cong D_{4p^2}$
4. $G_4 =< a,\ b,\ c,\ d | a^p = b^p = c^2 = d^2 = 1,\ ab = ba,\ ad = da, bc = cb,\ dbd = b^{-1},\ cd = dc > \cong < a,\ b,\ c | a^{2p} = b^p = c^2 = 1,\ ab = ba,\ ac = ca,\ cbc = b^{-1} >$
5. $G_5 =< a,\ b,\ c,\ d | a^p = b^p = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ dad = a^{-1},\ bc = cb,\ dbd = b^{-1},\ cd = dc > \cong < a,\ b,\ c | a^{2p} = b^p = c^2 = 1,\ ab = ba,\ cac = a^{-1},\ cbc = b^{-1} >$
6. $G_6 =< a,\ b,\ c,\ d | a^p = b^p = c^2 = d^2 = 1,\ ab = ba,\ cac = a^{-1},\ ad = da,\ bc = cb,\ dbd = b^{-1},\ cd = dc > \cong D_{2p} \times D_{2p}$

7. $G_7 =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = b^{-1} >$
8. $G_8 =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ c^{-1}ac = a^{-1},\ c^{-1}bc = b^{-1} >$
9. $G_9 =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ c^{-1}ac = b^{-1},\ c^{-1}bc = a >$ where $p \not\equiv 1 \pmod 4$. When $p \equiv 1 \pmod 4$ this group is presented by $G'_9 =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ c^{-1}ac = a^{\lambda},\ c^{-1}bc = b^{-\lambda} >$
10. $G_{10} =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = b^{\lambda} >$
11. $G_{11} =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ c^{-1}ac = a^{-1},\ c^{-1}bc = b^{\lambda} >$
12. $G_{12} =< a,\ b,\ c|a^p = b^p = c^4 = 1,\ ab = ba,\ c^{-1}ac = a^{\lambda},\ c^{-1}bc = b^{\lambda} >$

Since the dihedral group $G_3 = D_{4p^2}$ is studied in [11], we study the other groups and obtain connected normal edge-transitive Cayley graphs on this groups.

3. Normal edge-transitive Cayley graphs on group G_1

Elements of G_1 can be written uniquely in the form $a^i b^j,\ 0 \le i < p^2$, $0 \le j < 4$. The order of elements of G_1 are as follows:

$$O(a^i) = \frac{p^2}{(i,p^2)} = \begin{cases} p & \text{if } (i,p^2) = p \\ p^2 & \text{if } (i,p^2) = 1 \end{cases}$$

$$O(a^i b^2) = \begin{cases} 2p & \text{if } (i,p^2) = p \\ 2p^2 & \text{if } (i,p^2) = 1 \end{cases}$$

where $1 \le i < p^2$. We have $O(b^2) = 2$, $O(a^i b^k) = 4,\ 0 \le i < p^2,\ k = 1,3$. Using the above facts, we can find $Aut(G_1)$.

Let U_n be the set of units in Z_n, $n \ge 1$. Then U_n is a group under multiplication mod n.

Lemma 3.1. *For prime number p, $Aut(G_1) \cong Z_{p^2} \rtimes (U_{p^2} \times Z_2)$, and it has the following orbits on G_1 : $\{1\}, \{a^i | 1 \le i < p^2, (i,p^2) = 1\}$, $\{b^2\}, \{a^{mp} | 1 \le m < p\}, \{a^i b^k | 0 \le i < p^2, k = 1,3\}, \{a^i b^2 | 0 \le i < p^2, (i,p^2) = 1\}$ and $\{a^{mp} b^2 | 1 \le m < p\}$.*

Proof. Any $\sigma \in Aut(G_1)$ is determined by its effect on a and b. Taking orders into account, we have $\sigma(a) = a^i$, where $1 \le i < p^2, (i,p^2) = 1$ and $\sigma(b) = a^j b^k, 0 \le j < p^2, k = 1,3$. It can be verified that $\sigma = f_{i,j,k}$ defined as above can be extended to an automorphism of G_1. Therefore, $Aut(G_1) = \{f_{i,j,k} | 1 \le i < p^2, (i,p^2) = 1, 0 \le j < p^2, k = 1,3\}$ is a group of order $2p^2\phi(p^2) = 2p^2(p^2 - p)$. We have $f_{i,j,k} o f_{i',j',k'} = f_{ii',ij'+j,kk'}$ and $f^{-1}_{i,j,k} = f_{i_0,-ji_0,k_0}$ where i_0 and k_0 are numbers such that $i_0 i \equiv 1 \pmod{p^2}$ and $k_0 k \equiv 1 \pmod 4$, hence if we define $A = \{f_{1,j,1} | 0 \le j < p^2\}$ and $B = \{f_{i,0,k} | 1 \le i < p^2, (i,p^2) = 1, k = 1,3\}$, then $Aut(G_1) = A \times B$, $A \cap B = id$ and $A \trianglelefteq Aut(G_1)$. So $Aut(G_1) \cong Z_{p^2} \rtimes (U_{p^2} \times Z_2)$ and the lemma is proved. □

Lemma 3.2. *If $Cay(G_1, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| > 2$ is even.*

Proof. By Proposition 2.2, elements in S have the same order. Since $< S >= G_1$, the set S cannot contain elements of order p^2, $2p$, 2, p or $2p^2$, and should contain elements of order 4 only. By Proposition 2.3, $|S| > 2$ and it is even. □

Lemma 3.3. *Let $i \neq j$. The set $S = \{a^ib, a^ib^3, a^jb, a^jb^3\}$ generates G_1 if and only if $0 \leq i, j < p^2$, $i \not\equiv j \pmod p$, Moreover, in this case, $Aut(G_1, S) \cong Z_2 \times Z_2$.*

Proof. Generating condition of S comes from the relations $(a^kb)^{-1} = a^kb^3$ for $0 \leq k < p^2$, $a^iba^jb^3 = a^{i-j}$. If $i \not\equiv j \pmod p$, then we can conclude that $a \in< S >$ and so $b \in< S >$. Now let $S = \{x, x^{-1}, y, y^{-1}\}$ and $G = Aut(G_1, S)$. Then by Proposition 2.9, G acts on S faithfully, and so is a subgroup of S_4. But G does not have elements of order 3 or 4, because if $f \in G$ has order 3, then it should fix an element on S such as s, thus $f(s^{-1}) = s^{-1}$, contradiction with the order of f. Also if f is an element of order 4, then its cycle structure on S have the form $(x\ y\ x^{-1}\ y^{-1})$ or $(x\ y^{-1}\ x^{-1}\ y)$, where $x = a^ib$, $y = a^jb$ and $f = f_{r,s,k}$ (as mentioned in Lemma 3.1, $(r, p^2) = 1, 1 \leq r < p^2, 0 \leq s < p^2, k = 1, 3)$) and we may assume $i > j$. In the first case, we have $ri + s \equiv j \pmod{p^2}$, $k \equiv 3 \pmod 4$ and $rj + s \equiv i \pmod{p^2}$, $3k \equiv 3 \pmod 4$. But in this case we obtain $k \equiv 3 \pmod 4$ and $3k \equiv 3 \pmod 4$ that is impossible. In the second case, we have $k \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$ that is impossible. Therefore, G is a subgroup of S_4 which does not have any element of order 3 or 4, but at least it has two elements of order 2 such as $f_{-1,i+j,3}$ and $f_{1,0,3}$, imply that $G \cong Z_2 \times Z_2$. □

Lemma 3.4. *Let $\Gamma = Cay(G_1, S)$ be a Cayley graph of valency 4. The Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^ib, a^ib^3, a^jb, a^jb^3\}$ where $0 \leq i, j < p^2$, $i \not\equiv j \pmod p$. Moreover, in this case, Γ is not a normal Cayley graph, i.e., there is a connected normal edge-transitive Cayley graph which is not normal Cayley graph.*

Proof. It is enough to show that $Aut(G_1, S)$ acts transitively on S. The elements $f_{-1,i+j,1}, f_{1,0,3}, f_{-1,i+j,3}$ are in $Aut(G_1, S)$ and send a^ib to a^jb, a^ib^3, a^jb^3, respectively. So $Aut(G_1, S)$ acts transitively on S and Γ is a connected normal edge-transitive Cayley graph. The set S is equivalent to $S' = \{b, b^3, ab, ab^3\}$, since $(S')^{f_{j-i,i,1}} = S$. For the second part, it is enough to check the case S'. We have $\Gamma(b) = \{b^2,\ ab^2,\ a,\ 1\} = \Gamma(b^3)$ thus $\sigma = (b\ b^3) \in (Aut\Gamma)_1$, but $f_{1,0,3},\ f_{-1,1,1},\ f_{-1,1,3} \in Aut(G_1, S)$ and Lemma 3.3 show that $\sigma \notin Aut(G_1, S)$, i.e., $(Aut\Gamma)_1 \neq Aut(G_1, S)$ and by Proposition 2.1, Γ is not a normal Cayley graph. □

In the next theorem, we present the main condition under which the Cayley graph of group G_1 becomes connected normal edge-transitive.

Theorem 3.5. *$\Gamma = Cay(G_1, S)$ is a connected normal edge-transitive Cayley graph if and only if its valency is even, greater than two, $S \subseteq \{a^i b, a^j b^3 | \ 0 \leq i, j < p^2, \ i \not\equiv j \pmod p\}$, $S = S^{-1}$ and $Aut(G_1, S)$ acts transitively on S.*

Proof. If Γ is a connected normal edge-transitive Cayley graph, then by Proposition 2.3, its valency should be even. Since $< S >= G_1$, then by Lemma 3.3 and Lemma 3.4, $S \subseteq \{a^i b, a^j b^3 | i \not\equiv j \pmod p\}$, the graph is undirected, $S = S^{-1}$. Hence $S \subseteq \{a^i b, a^j b^3 | \ for some \ 0 \leq i, j < p^2, \ i \not\equiv j \pmod p\}$. From Proposition 2.2, either $Aut(G_1, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $Aut(G_1, S)$ on S. But we observe $f_{1,0,3} \in Aut(G_1, S)$, which implies both of $a^i b$ and $(a^i b)^{-1} = a^i b^3$ belong to the same orbit for $0 \leq i < p^2$ in which $a^i b \in S$, and that contradiction with the assumption $S = T \cup T^{-1}$. Hence $Aut(G_1, S)$ acts transitively on S. □

Corollary 3.6. *If Γ is a connected Cayley graph of the group G_1, then Γ is not normal $1/2$-arc-transitive.*

Theorem 3.7. *Let $\Gamma = Cay(G_1, S)$ be a normal edge-transitive Cayley graph of valency $2d$. Then either $d = p^2$ or $d | p(p-1)$ and $d \neq p$. Moreover, for each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. By Theorem 3.5, $S \subseteq \{a^i b, a^j b^3 | \ 0 \leq i, j < p^2, \ i \not\equiv j \pmod p\}$. Set $U = \{a^i b, a^i b^3 | \ 0 \leq i < p^2\}$, in this case, U is an orbit of the action of $Aut(G_1)$ on G_1 and so $Cay(G_1, U)$ is a connected normal edge-transitive graph of valency $2p^2$. Now suppose $S \subseteq \{a^i b, a^j b^3 | \ 0 \leq i, j < p^2, \ i \not\equiv j \pmod p\}$, $< S >= G_1$ and Γ is a Cayley graph of valency $2d$. Since $Aut(G_1, S) \leq Aut(G_1)$ and $Aut(G_1, S)$ is transitive on S (Theorem 3.5), we have $|S| = 2d \mid |Aut(G_1, S)| \mid |Aut(G_1)| = 2p^3(p-1)$, implying $d | p^3(p-1)$. On the other hand, we have $d \leq p^2$, hence either $d = p^2$ or $d | p(p-1)$ proving the first assertion of the theorem. To prove the existence and uniqueness part in the theorem, if $d = p^2$, then as mentioned above, $Cay(G_1, U)$ is the unique normal maximal edge-transitive Cayley graph of valency $2p^2$. Now suppose $d | p(p-1), d > 1$. The stabilizer of b under $A = Aut(G_1)$ is the group $A_b = \{f_{i,0,1} | 1 \leq i < p^2, i \not\equiv 0 \pmod p\} \cong U_{p^2}$. Let t be a generator of U_{p^2}, so that $A_b =< f_{t,0,1} >$. Since $d | p(p-1)$, the group U_{p^2} contains a unique subgroup of order d, and if we set $u = t^{\frac{p-1}{d}}$, then $< f_{u,0,1} >$ is a subgroup of A_b with order d. Now consecutive effects of $f_{u,0,1}$ on ab yields the set $T = \{ab, a^u b, ..., a^{u^{(d-1)}} b\}$ whose size is d and is invariant under $f_{u,0,1}$. Let us set $T^{-1} = \{x^{-1} | x \in T\} = \{ab^3, a^u b^3, ..., a^{u^{(d-1)}} b^3\}$ and $S = T \cup T^{-1}$. We claim that $Cay(G_1, S)$ is a connected normal edge-transitive Cayley graph. By the argument used in Lemma 3.4, where $d \neq p$, we have $< S >= G_1$. It is easy to see that $f_{u,0,3}$ interchanges elements of T and T^{-1}, also the automorphism group of $Cay(G_1, S)$ is $< f_{u,0,1}, f_{u,0,3} >$, implying $Cay(G_1, S)$ is connected normal edge-transitive of valency $2d$. □

4. Normal edge-transitive Cayley graphs on group G_2

We consider the group G_2, which is defined in the section 2 and we will prove that its Cayley graph on some set can be connected normal 1/2-arc-transitive Cayley graph. Recall that we assume p is an odd prime. The existence of λ satisfying the condition $(\lambda)^2 \equiv -1 \pmod{p^2}$ implies that $4|(p-1)$, hence p must be a prime of the form $p = 1 + 4k$. The order of non-identity elements of G_2 are as follows:

$$O(a^i) = \frac{p^2}{(i, p^2)} = \begin{cases} p & \text{if } (i, p^2) = p \\ p^2 & \text{if } (i, p^2) = 1 \end{cases}$$

We have $O(a^i b^2) = 2$, $O(a^i b^k) = 4$, $0 \leq i < p^2$, $k = 1, 3$. Using the above facts, we can find $Aut(G_2)$. Thus if $\sigma \in Aut(G_2)$, then $\sigma(a) = a^i$ and either $\sigma(b) = a^j b$ or $\sigma(b) = a^j b^3$ for $1 \leq i < p^2$, $(i, p^2) = 1$ and $0 \leq j < p^2$, but we also have $\sigma(b^{-1}ab) = \sigma(a)$, thus in the latter case we obtain a contradiction. Therefore, we have:

$$Aut(G_2) = \{g_{i,j} |\ g_{i,j}(a) = a^i,\ g_{i,j}(b) = a^j b,\ 1 \leq i < p^2, (i, p^2) = 1$$
$$and\ 0 \leq j < p^2\} \cong Z_{p^2} \rtimes U_{p^2}$$

and it has the following orbits on $G_2 : \{1\}, \{a^i | 1 \leq i < p^2, (i, p^2) = 1\}, \{a^{mp} | 1 \leq m < p\}, \{a^i b^3 | 0 \leq i < p^2\}, \{a^i b | 0 \leq i < p^2\}$ and $\{a^i b^2 | 0 \leq i < p^2, (i, p^2) = 1\}$.

Theorem 4.1. *$\Gamma = Cay(G_2, S)$ is a connected normal edge-transitive Cayley graph if and only if it has even valency, $S = T \cup T^{-1}$, where $T \subseteq \{a^i b, a^j b |\ 0 \leq i, j < p^2, i \not\equiv j \pmod{p}\}$ and $Aut(G_2, S)$ acts transitively on T. Moreover, if $\Gamma = Cay(G_2, S)$ is a normal edge-transitive Cayley graph of valency $2d$, Then either $d = p^2$ or $d | p(p-1)$ and $d \neq p$. Moreover, for each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. At first we assume that Γ is a connected normal edge-transitive Cayley graph. The fact that Γ has even valency follows from Proposition 2.3. By Proposition 2.2, in the action of $Aut(G_2, S)$ on S, we can deduce either S is an orbit or $S = T \cup T^{-1}$, where T is an orbit. We have $(a^i b^2)^{-1} = a^i b^2$, thus if $a^i b^2 \in S$ for some $0 \leq i < p^2$, the case $S = T \cup T^{-1}$ cannot occur, i.e., $Aut(G_2, S)$ acts transitively on S, but Γ is connected, i.e., $< S >= G$, therefore S should contain some element other than $a^j b^2, 0 \leq j < p^2$, say x, such that its order is not 2. Hence, there is no $g_{r,s} \in Aut(G_2, S) \subseteq Aut(G_2)$ such that $g_{r,s}(x) = a^i b^2$, a contradiction. Suppose $y = a^i \in S$ for some $1 \leq i < p^2$. Since Γ is connected, i.e., $< S >= G_2$, S should contain an element x, where $x = a^j b$ or $x = a^j b^3$ for some $0 \leq j < p^2$. But since $(x)^{-1} \neq y$, without loss of generality, we can assume x and y are contained in the same orbit. But there is no $g_{r,s} \in Aut(G_2, S) \subseteq Aut(G_2)$ such that $g_{r,s}(x) = y$, a contradiction. Therefore, S contains only elements of types $a^i b$ and $a^j b^3$ for $0 \leq i, j < p^2, i \not\equiv j \pmod p$. But $S = S^{-1}$ and for each $0 \leq j < p^2$, there is some $0 \leq i < p^2$, where $(a^j b)^{-1} = a^i b^3$, hence S contains not only $a^i b$ but also $a^j b^3$ for $0 \leq i, j < p^2$. Therefore, $Aut(G_2)$ and consequently $Aut(G_2, S)$ is not transitive on S, hence $S = T \cup T^{-1}$, where $T \subseteq \{a^i b, a^j b \mid 0 \leq i, j < p^2, i \not\equiv j \pmod p\}$, and $Aut(G_2, S)$ acts transitively on T. The second part of the theorem is similar to the proof of Theorem 3.7. □

Example 4.2. Let $\Gamma = Cay(G_2, S)$ be a Cayley graph of valency 4. Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b,\ a^j b,\ a^{-i\lambda} b^3, a^{-j\lambda} b^3\}$ for some $0 \leq i, j < p^2,\ i \not\equiv j \pmod p$ and in this case $|Aut(G_2, S)| = 2$. By Theorem 4.1, it is sufficient to put $T = \{a^i b, a^j b\}$ and consider $g_{-1.i+j} \in Aut(G_2, S)$.

5. Normal edge-transitive Cayley graphs on group G_4

The order of non-identity elements of G_4 are as follows:

$$O(a^i b^j) = \begin{cases} p & \text{if } 0 \leq i < 2p, i \ is \ even, 0 \leq j < p \\ 2p & \text{if } 0 \leq i < 2p, i \ is \ odd, 0 \leq j < p \\ 2 & \text{if } i = p, j = 0 \end{cases}$$

and

$$O(a^i b^j c) = \begin{cases} 2 & \text{if } i = 0 \ or \ p, 0 \leq j < p \\ 2p & \text{if } 1 \leq i < 2p, i \neq p, 0 \leq j < p \end{cases}$$

Using the above facts, we can find $Aut(G_4)$. If $\sigma \in Aut(G_4)$, then $\sigma(a) \in \{a^i b^j,\ a^k b^j c \mid i \ is \ odd, 0 < i, k < 2p, 0 \leq j < p\}$, $\sigma(b) \in \{a^i b^j \mid i \ is \ even, 0 \leq i < 2p, 0 \leq j < p\}$ and $\sigma(c) \in \{a^i b^j c \mid i = 0 \ or \ p, 0 \leq j < p\}$, but we also have $\sigma(ab) = \sigma(ba)$, $\sigma(ac) = \sigma(ca)$ and $\sigma(cbc) = \sigma(b^{-1})$. According to this relations, we have:

$$\begin{aligned} Aut(G_4) = \{f_{i,j,l,k} \mid\ & f_{i,j,l,k}(a) = a^i,\ f_{i,j,l,k}(b) = b^j, f_{i,j,l,k}(c) = a^l b^k c, \\ & 1 \leq i < 2p, (i, 2p) = 1, 1 \leq j < p,\ k = 0, p,\ 0 \leq k < p\} \\ \cong\ & (Z_2 \times Z_p) \rtimes (U_{2p} \times U_p) \end{aligned}$$

and it has the following orbits on $G_4 : \{1\}, \{a^i \mid 1 \leq i < 2p, (i, 2p) = 1\}$, $\{a^i \mid 1 \leq i < 2p, (i, 2p) = 2\}$, a^p, $\{b^k \mid 1 \leq k < p\}, \{a^i b^j c \mid 1 \leq i < 2p, i \neq p, 0 \leq j < p\}, \{a^i b^j c \mid i = 0, p, 0 \leq j < p\}, \{a^i b^j \mid 1 \leq i < 2p, (i, 2p) = 1, 1 \leq j < p\}$, $\{a^i b^j \mid 1 \leq i < 2p, (i, 2p) = 2, 1 \leq j < p\}$ and $\{a^p b^j \mid 1 \leq j < p\}$.

Theorem 5.1. *$\Gamma = Cay(G_4, S)$ is a connected normal edge-transitive Cayley graph if and only if its valency is even, greater than two, $S \subseteq \{a^i b^j c, a^k b^l c \mid$*

$0 < i,k < 2p, 0 \leq j,l < p, j \neq l, i+k$ is odd$\}$, $S = S^{-1}$ and $Aut(G_4, S)$ acts transitively on S.

Proof. At first, we assume that Γ is a connected normal edge-transitive Cayley graph. The fact that Γ has even valency follows from Proposition 2.3. Since $G_4 =< S >$ and the elements of S have same order, so S consist of elements of order $2p$. Thus if $a^ib^j \in S$ for some $0 \leq i < 2p$ and $0 \leq j < p$, S should contain some element other than $a^ib^j \in S$, say x, such that its order is $2p$. If $x = a^kb^l$, for some k,l, then $< S >< G_4$, a contradiction. So x must be a^kb^lc but in this case, there is no $f_{i,j,l,k} \in Aut(G_4, S) \subseteq Aut(G_4)$ such that $f_{i,j,l,k}(x) = a^ib^j$, a contradiction. If a^ib^jc and a^kb^lc are in S for some $0 \leq i,k < 2p$ and $0 \leq j,l < p$, then we have $(a^ib^jc)^2 = a^{2i}$ and $(a^ib^jca^kb^lc)^p = a^{p(i+j)}$, if $i+j$ is odd, then $a^{p+2} \in S$ implying $a \in S$ and by $i \neq j$, we can conclude that $b \in S$ and so $c \in S$. So we have $S \subseteq \{a^ib^jc, a^kb^lc | 0 \leq i,k < 2p, 0 \leq j,l < p, j \neq l, i+k \text{ is odd}\}$. By Proposition 2.2, in the action of $Aut(G_4, S)$ on S, we can deduce either S is an orbit or $S = T \cup T^{-1}$, where T is an orbit. But we observe $f_{-1,1,0,0} \in Aut(G_4, S)$, which implies both of a^ib^jc and $(a^ib^jc)^{-1} = a^{-i}b^jc$ belong to the same orbit for $0 \leq i < 2p$ and $0 \leq j < p$ in which $a^ib^jc \in S$, and that contradiction with the assumption $S = T \cup T^{-1}$. Hence $Aut(G_4, S)$ acts transitively on S. □

Lemma 5.2. *Let $\Gamma = Cay(G_4, S)$ be a Cayley graph of valency 4. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^ib^jc,\ a^{i+p}b^lc,\ a^{-i}b^jc,\ a^{p-i}b^lc\}$ or $S = \{a^ib^jc,\ a^kb^lc,\ a^{-i}b^jc,\ a^{-k}b^lc\}$ when $p \equiv 1 \pmod 4$ for some $0 < i,k < 2p,\ 0 \leq j,l < p, j \neq l$. In the first case, $Aut(G_4, S) \cong Z_2 \times Z_2$.*

Proof. By Theorem 5.1, we have $S = \{a^ib^jc,\ a^kb^lc,\ a^{-i}b^jc,\ a^{-k}b^lc\}$ such that $i+j$ is odd number and $j \neq l$. In this case, $G_4 =< S >$ and then Γ is connected. Now $Aut(G_4, S)$ must be transitive on S. Let $f_{m,n,t,s} \in Aut(G_4, S)$ and $f_{m,n,t,s}(a^ib^jc) = a^kb^lc$. Since $i+k$ is odd number (suppose i is odd number and k is even number) we must have $t = p$, then we have one of the following cases:

case 1: $f_{m,n,t,s}(a^kb^lc) = a^ib^jc$.

In this case, we have $im + t \equiv k \pmod{2p}$ and $km + t \equiv i \pmod{2p}$, from this relations we conclude that $k = p+i$, $f_{1,-1,p,l+j}$, $f_{-1,1,0,0}$ and $f_{-1,-1,p,l+j}$ are in $Aut(G_4, S)$ implying that $Aut(G_4, S)$ is transitive on S. We have $Aut(G_4, S) \leq S_4$ and $Aut(G_4, S)$ has no elements of order 3 or 4 but have 3 elements of order 2, then $Aut(G_4, S) \cong Z_2 \times Z_2$.

case 2: $f_{m,n,t,s}(a^kb^lc) = a^{-i}b^jc$.

In this case, we have $im + t \equiv k \pmod{2p}$ and $km + t \equiv -i \pmod{2p}$, from this relations we conclude that $m^2 \equiv -1 \pmod p$ and this equation has answer when $p \equiv 1 \pmod 4$. In this case, $f_{m,-1,p,l+j}$ is in $Aut(G_4, S)$ forcing $Aut(G_4, S)$ is transitive on S.

case 3: $f_{m,n,t,s}(a^kb^lc) = a^{-k}b^jc$.

In this case we must have $km+p \equiv -k \pmod{2p}$, but k is even number and then $km+p$ is odd number implying that $km+p \equiv -k \pmod{2p}$ can not be occur. □

6. Normal edge-transitive Cayley graphs on group G_5

The order of non-identity elements of G_5 are as follows:

$$O(a^ib^j) = \begin{cases} p & \text{if } 0 \leq i < 2p, i \textit{ is even}, 0 \leq j < p \\ 2p & \text{if } 0 \leq i < 2p, i \textit{ is odd}, 0 \leq j < p, i \neq p \\ 2 & \text{if } i = p, j = 0 \end{cases}$$

and

$$O(a^ib^jc) = 2,\ 0 \leq i < 2p, 0 \leq j < p.$$

Using the above facts, we can find $Aut(G_5)$.

$$\begin{gathered} Aut(G_5) = \{f_{k,t,l,n,i,j} |\ f_{k,t,l,n,i,j}(a) = a^kb^t,\ f_{k,t,l,n,i,j}(b) = a^lb^n, \\ f_{k,t,l,n,i,j}(c) = a^ib^jc, 0 \leq k,l,i < 2p, (k,2p) = 1, (l,2p) = 2, \\ 0 \leq n,t,j < p, \text{ and for any } 0 \leq m < 2p \\ l \not\equiv mk \pmod{p} \textit{ or } n \not\equiv mt \pmod{p}\} \end{gathered}$$

and it has the following orbits on G_5 : $\{1\}, \{a^p\}, \{a^pb^k | 1 \leq k < p\}, \{a^ib^jc | 0 \leq i < 2p, 0 \leq j < p\}, \{a^ib^j | 1 \leq i < 2p, (i,2p) = 1, 0 \leq j < p\}$, $\{a^ib^j | 0 \leq i < 2p, (i,2p) = 2, 0 \leq j < p\}$.

Lemma 6.1. *If $Cay(G_5,S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 2.*

Proof. Proof of this lemma is similar to the proof of Lemma 3.2. □

Lemma 6.2. *Let $0 \leq i,k,t < 2p, 0 \leq j,l,n < p$. Then $S = \{a^ib^jc, a^kb^lc, a^tb^nc\}$ generates G_5 if and only if,*
1. $j = l = n$ do not occur.
2. i,k,t are not all even or all odd.
3. If there exist m such that $(j-l)m \equiv j-n \pmod{p}$, then $(i-k)m \not\equiv i-t \pmod{p}$.

Proof. Proof of this lemma is similar to the proof of Lemma 3.3. □

Example 6.3. Let $S = \{a^ib^jc, a^ic, b^jc, c\}$. Then $\Gamma = Cay(G_5,S)$ is a connected normal edge-transitive Cayley of valency 4 and $Aut(G_5,S) \cong Z_2 \times Z_2$.

By Lemma 6.2, we have $G_5 =< S >$ so Γ is connected. The set S is equivalent to $S' = \{abc, ac, bc, c\}$ since $S'^{f_{i,0,0,j,0,0}} = S$. Therefore, it is sufficient to check it for S'. $f_{1,0,0,-1,0,1}, f_{-1,0,0,-1,1,1}, f_{-1,0,0,1,1,0}$ are all in $Aut(G_5,S')$ and send abc to ac, c, bc respectively, so $Cay(G_5,S')$ is a connected normal edge-transitive Cayley of valency 4. Since $Aut(G_5, S)$ has no elements of order 3 or 4 while it has 3 elements of order 2, so $Aut(G_5, S) \cong Z_2 \times Z_2$.

Theorem 6.4. *$\Gamma = Cay(G_5, S)$ is a connected normal edge-transitive Cayley graph if and only if $S \subseteq \{a^i b^j c, a^k b^l c, a^t b^n c | 0 \leq i, k, t < 2p, 0 \leq j, l, n < p\}$, S satisfy the conditions of Lemma 6.2, $S = S^{-1}$ and $Aut(G_5, S)$ acts transitively on S.*

Proof. Proof of this theorem is similar to the proof of Theorem 5.1. □

7. Normal edge-transitive Cayley graphs on group G_6

Elements of G_6 can be written uniquely in the form $\{a^i b^j c^k d^l,\ 0 \leq i < p,\ 0 \leq j < p,\ 0 \leq k, l < 2\}$. The order of elements of G_6 are as follows: $O(a^i b^j) = p, O(a^i b^j cd) = O(a^i c) = O(b^j d) = 2$ where $0 \leq i, j < p$. For $1 \leq i < p,\ 0 \leq j < p$, we have $O(a^j b^i c) = O(a^i b^j d) = 2p$. Using the above facts, we can find $Aut(G_6)$. Any $\sigma \in Aut(G_6)$ is determined by its effect on a, b, c and d. For $1 \leq i, j < p,\ 0 \leq l, k < p$, we have:

$$Aut(G_6) = \{f_{i,j,l,k} |\ f_{i,j,l,k}(a) = a^i,\ f_{i,j,l,k}(b) = b^j, f_{i,j,l,k}(c) = a^l c, f_{i,j,l,k}(d) = b^k d\} \cup \{g_{i,j,l,k} |\ g_{i,j,l,k}(a) = b^i,\ g_{i,j,l,k}(b) = a^j, g_{i,j,l,k}(c) = b^l d, f_{i,j,l,k}(d) = a^k c\}$$

Lemma 7.1. *If $Cay(G_6, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order $2p$ or 2.*

Next, we can express the main condition under which the Cayley graph of group G_6 becomes connected normal edge-transitive.

Theorem 7.2. *$\Gamma = Cay(G_6, S)$ is a connected normal edge-transitive Cayley graph if and only if, $S \subseteq \{a^i b^j c, a^k b^l c\}$ or $S \subseteq \{a^i c, a^j c, b^k d, b^l d,\ 1 \leq j, k < p, 0 \leq i, l < p\}$, $S = S^{-1}$ and $Aut(G_6, S)$ acts transitively on S.*

Proof. If Γ is a connected normal edge-transitive Cayley graph, then by Lemma 7.1, elements of S have order 2 or $2p$. First assume elements of S hava order 2 and let $a^i b^j cd \in S$. Since $(a^i b^j cd)^{Aut(G_6)} = a^t b^m cd$ for some $1 \leq t, m < p$, so $S \subseteq \{a^i b^j cd\}$. But in this case $< S >< G_6$ and $Cay(G_6, S)$ can not be connected. Therefore assume that $a^i c, b^k d \in S$. Since $< S >= G_6$, so S must contain some element other than $a^i c, b^k d \in S$, such that its order is 2. If $S = \{a^i c, b^k d, b^l d\}$, then $b, d \in S$ but $a \notin S$, therefore S must have another element of the form $a^j c$, hence $S \subseteq \{a^i c, a^j c, b^k d, b^l d\}$, $1 \leq j, k < p, 0 \leq i, l < p$. From Proposition 2.2, either $Aut(G_6, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $Aut(G_6, S)$ on S. But we observe that $T = T^{-1}$, therefore, $Aut(G_6, S)$ acts transitively on S. Now assume that elements of S have order $2p$ and let $a^i b^j c \in S$. Since $< S >= G_6$, so S must contain some element other than $a^i b^j c \in S$, such that its order is $2p$. If $S = \{a^i b^j c, a^k b^l d\}$ such that $j, k \neq 0$, then $G_6 =< S >$ and therefore $S \subseteq \{a^i b^j c, a^k b^l d\}$. From Proposition 2.2, either $Aut(G_6, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $Aut(G_6, S)$ on S. But we know that that $T = T^{-1}$, hence $Aut(G_6, S)$ acts transitively on S. But we observe $f_{-1,-1,2i,2l} \in Aut(G_6, S)$, which implies both of a^ib^jc, a^kb^ld and $(a^ib^jc)^{-1} = a^ib^{-j}c, (a^kb^ld)^{-1} = a^{-k}b^ld$ belong to the same orbit for $0 \leq i, j, k, l < p$ in which $a^ib^jc, a^kb^ld \in S$, and that contradiction with the assumption $S = T \cup T^{-1}$. Hence $Aut(G_6, S)$ acts transitively on S. □

Corollary 7.3. *If Γ is a connected Cayley graph of the group G_6, then Γ is not normal 1/2-arc-transitive.*

Lemma 7.4. *Let $\Gamma = Cay(G_6, S)$ be a Cayley graph of valency 4 and elements of S has order 2. The Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^ic, a^jc, b^kd, b^ld\}$ where $0 \leq i, j, k, l < p, i \neq j, l \neq k$. Moreover, in this case $Aut(G_6, S) \cong D_8$.*

Proof. By Theorem 7.2, it is enough to show that $Aut(G_6, S)$ acts transitively on S. The set S is equivalent to $S' = \{c,\ d, ac, bd\}$, since $S'^{f_{j-i,k-l,i,l}} = S$. Therefore, it is sufficient to check it for S'. But $g_{1,1,0,0},\ f_{-1,-1,1,1}, g_{-1,-1,1,1} \in Aut(G_6, S)$ and send c to d, ac, bd, respectively. Moreover, $Aut(G_6, S')$ has no elements of order 3 and has 2 elements of order 4 ($g_{1,-1,0,1}, g_{-1,1,1,0}$) and 5 elements of order 2 ($f_{-1,-1,1,1}, g_{-1,-1,1,1}, g_{1,1,0,0}, f_{-1,1,1,0}, f_{1,-1,0,1}$), so $Aut(G_6, S') \cong D_8$. □

Lemma 7.5. *Let $\Gamma = Cay(G_6, S)$ be a Cayley graph of valency 4 and elements of S has order 2p. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^ib^jc, a^ib^{-j}c, a^kb^ld, a^{-k}b^ld\}$ where $0 \leq i, l < p, 1 \leq j, k < p$. Moreover, in this case $Aut(G_6, S) \cong D_8$.*

Proof. Proof of this lemma is similar to the proof of Lemma 7.4. □

Example 7.6. If $S = \{a^ib^jc^k | 1 \leq i < p, 0 \leq j < p, k = 1, 3\}$, then $Cay(G_6, S)$ is a connected normal edge-transitive Cayley graph of valency $2p(p-1)$.

8. Normal edge-transitive Cayley graphs on group G_7

Elements of G_7 can be written uniquely in the form $\{a^ib^jc^k,\ 0 \leq i < p,\ 0 \leq j < p,\ 0 \leq k < 4\}$. The order of elements of G_7 are as follows:
$O(a^ib^j) = p$ where $0 \leq i, j < p$. For $1 \leq i < p,\ 0 \leq j < p,\ k = 1, 3$, we have $O(c^2) = 2$, $O(b^jc^k) = 4$, $O(a^ib^jc^k) = 4p$ and $O(a^ib^jc^2) = 2p$. Using the above facts, we can find $Aut(G_7)$.

Lemma 8.1. *$Aut(G_7) \cong (Z_2 \times Z_p) \rtimes (U_p \times U_p)$, and it has the following orbits on G_7 : $\{1\}, \{a^i | 1 \leq i < p\}$, $\{b^j | 1 \leq i < p\}$, $\{a^ib^j | 1 \leq i, j < p\}$, $\{a^ic^2 | 1 \leq i < p\}$, $\{a^ib^jc^k | 1 \leq i < p, 0 \leq j < p, k = 1, 3\}$, $\{a^ib^jc^2 | 1 \leq i < p, 1 \leq j < p\}$, $\{b^jc^2 | 1 \leq j < p\}$ and $\{b^jc^k | 0 \leq j < p, k = 1, 3\}$.*

Proof. Any $\sigma \in Aut(G_7)$ is determined by its effect on a, b and c. For $1 \leq i < p,\ 0 \leq j < p,\ k = 1, 3,\ \sigma(a) \in \{a^i b^j\}$, $\sigma(b) \in \{a^i b^j\}$ and $\sigma(c) \in \{b^j c^k\}$, but we also have $\sigma(c^{-1}bc) = \sigma(b^{-1}), \sigma(ab) = \sigma(ba)$ and $\sigma(ac) = \sigma(ca)$, thus according to this relations we have:

$$Aut(G_7) = \{h_{i,j,l,k} |\ h_{i,j,l,k}(a) = a^i,\ h_{i,j,l,k}(b) = b^j, h_{i,j,l,k}(c) = b^l c^k | \\ 1 \leq i, j < p,\ 0 \leq l < p,\ k = 1, 3\}$$

and we have:

$$(h_{i,j,l,k})o(h_{i',j',l',k'}) = h_{ii',jj',jl'+l,kk'}$$

$$h^{-1}_{i,j,l,k} = h_{i_1,j_1,-j_1 l,k}.$$

In the above relations, i_1 and j_1 are the numbers such that $ii_1 \equiv 0 \pmod p$ and $jj_1 \equiv 0 \pmod p$.
Define $A = \{h_{1,1,l,k} |\ k = 1, 3,\ 0 \leq l < p\}$ and $B = \{h_{i,j,0,1} |\ 1 \leq i, j < p\}$, so $A \trianglelefteq Aut(G_4)$, $A \cap B = id$ and $Aut(G_7) = AB$. Therefore we have:

$$Aut(G_7) \cong (Z_2 \times Z_p) \rtimes (U_p \times U_p).$$

□

Lemma 8.2. *If $Cay(G_7, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order $4p$ and $|S| > 2$ is even. Moreover, for odd prime number p, $S = \{a^i b^j c, a^k b^l c, a^{-k} b^l c^3, a^{-i} b^j c^3 | 1 \leq i, k < p, 0 \leq j, l < p\}$ is generate G_7 if and only if $j \neq l$.*

Proof. Proof of the first part is similar to Lemma 3.2. Generating condition of S comes from the relations, $(a^i b^j c)^4 = a^{4i}$, $(a^i b^j c)^2 = a^{2i} c^2$ and $b^l c b^j c = b^{l-j} c^2$. □

Lemma 8.3. *Let $\Gamma = Cay(G_7, S)$ be a Cayley graph of valency 4. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^{\pm i} b^l c, a^{-i} b^j c^3, a^{\mp i} b^l c^3\}$ where $1 \leq i, j < p, 0 \leq l < p,\ j \neq l$. Moreover, in this case $Aut(G_7, S) \cong Z_2 \times Z_2$.*

Proof. According to Lemma 8.2, S have elements of order $4p$. First set $S' = \{ac,\ abc,\ a^{-1}c^3,\ a^{-1}bc^3\}$. In this case $h_{1,-1,1,1},\ h_{-1,1,0,3},\ h_{-1,-1,1,3} \in Aut(G_7, S)$ and $(ac)^{h_{1,-1,1,1}} = abc$, $(ac)^{h_{-1,1,0,3}} = a^{-1}c^3$ and $(ac)^{h_{-1,-1,1,3}} = a^{-1}bc^3$, so $Aut(G_7, S')$ is transitive on S and $Cay(G_7, S')$ is connected normal edge-transitive Cayley graph. Moreover, $Aut(G_7, S') \cong Z_2 \times Z_2$. Similarly, if $S'' = \{ac,\ a^{-1}bc,\ a^{-1}c^3,\ abc^3\}$, then $Cay(G_7, S'')$ is connected normal edge-transitive Cayley graph. Now $h_{i,l-j,j,1} \in Aut(G_7)$, $(S')^{h_{i,l-j,j,1}} = \{a^i b^j c, a^i b^l c, a^{-i} b^j c^3, a^{-i} b^l c^3\}$ and $(S'')^{h_{i,l-j,j,1}} = \{a^i b^j c, a^{-i} b^l c, a^{-i} b^j c^3, a^i b^l c^3\}$. So $\Gamma = Cay(G_7, S)$ is connected normal edge-transitive Cayley graph. Now set $S = \{x,\ y,\ x^{-1},\ y^{-1}\}$, if $x = a^i b^j c$, then since $G_7 =< S >$, y must be either $a^k b^l c$ or $a^k b^l c^3$ such that $j \neq l$. Suppose $\Gamma = Cay(G_7, S)$ is a connected normal edge-transitive Cayley graph. By

Proposition 2.2, in the action of $Aut(G_7, S)$ on S, we can deduce either S is an orbit or $S = T \cup T^{-1}$, where T is an orbit. In the last case we have $T = \{x,\ y\}$ or $T = \{x,\ y^{-1}\}$, but $h_{-1,1,0,3} \in Aut(G_7, S)$, $T^{h_{-1,1,0,3}} = T^{-1}$, therefore $Aut(G_7, S)$ must be transitive on S. Since $Aut(G_7, S)$ is transitive on S, so there exist $h_{m,n,t,s} \in Aut(G_7, S)$, such that $h_{m,n,t,s}(x) = y$ and $h_{m,n,t,s}(y) = x$ *or* x^{-1} *or* y^{-1}. First suppose $h_{m,n,t,s}(x) = y$ and $h_{m,n,t,s}(y) = x$. In this case we must have $im \equiv k \pmod p$ and $km \equiv i \pmod p$, therefore $p|(i-k)(m+1)$. Since $1 \leq i, k, m < p$, we have $i = k$ or $m = -1$. So $S = \{a^ib^jc, a^ib^lc, a^{-i}b^jc^3, a^{-i}b^lc^3\}$ or $S = \{a^ib^jc, a^{-i}b^lc, a^{-i}b^jc^3, a^ib^lc^3\}$.
Now suppose $h_{m,n,t,s}(x) = y$ and $h_{m,n,t,s}(y) = x^{-1}$, in this case we must have $s \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$ that is impossible.
In the last case suppose $h_{m,n,t,s}(x) = y$ and $h_{m,n,t,s}(y) = y^{-1}$, in this case we must have $s \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$ that is impossible. So we have $S = \{a^ib^jc, a^{\pm i}b^lc, a^{-i}b^jc^3, a^{\mp i}b^lc^3\}$ and $Aut(G_7, S) \cong Z_2 \times Z_2$. □

Next, we can express the main condition under which the Cayley graph of group G_7 becomes connected normal edge-transitive.

Theorem 8.4. $\Gamma = Cay(G_7, S)$ *is a connected normal edge-transitive Cayley graph if and only if its valency is even, greater than two,* $S \subseteq \{a^ib^jc, a^kb^lc^3|$ $1 \leq i, k < p, 0 \leq j, l < p,\ j \neq l\}$, $S = S^{-1}$ *and* $Aut(G_7, S)$ *acts transitively on* S.

Proof. If Γ is a connected normal edge-transitive Cayley graph, then by Proposition 2.3, its valency should be even. By Lemma 8.2 and Lemma 8.3, $S \subseteq \{a^ib^jc, a^kb^lc^3|1 \leq i, k < p, 0 \leq j, l < p,\ j \neq l\}$ and $S = S^{-1}$. From Proposition 2.2, either $Aut(G_7, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $Aut(G_7, S)$ on S. But we observe $h_{-1,1,0,3} \in Aut(G_7, S)$, which implies both of a^ib^jc and $(a^ib^jc)^{-1} = a^{-i}b^jc^3$ belong to the same orbit for $1 \leq i < p, 0 \leq j < p$ in which $a^ib^jc \in S$, and that contradicts the assumption $S = T \cup T^{-1}$. Hence $Aut(G_7, S)$ acts transitively on S. □

Corollary 8.5. *If* Γ *is a connected Cayley graph of the group* G_7*, then* Γ *is not normal* $1/2$*-arc-transitive.*

Example 8.6. If $S = \{a^ib^jc^k|1 \leq i < p, 0 \leq j < p, k = 1, 3\}$, then $Cay(G_7, S)$ is a connected normal edge-transitive Cayley graph of valency $2p(p-1)$.

Theorem 8.7. *Let* $\Gamma = Cay(G_7, S)$ *be a normal edge-transitive Cayley graph of valency* $2d$*. Then* $d = p$ *or* $d|p(p-1)$ *or* $d|(p-1)^2$*. Moreover, for each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency* $2d$*.*

Proof. By Theorem 8.4, $S \subseteq \{a^ib^jc, a^kb^lc^3|1 \leq i, k < p, 0 \leq j, l < p,\ j \neq l\}$ and by Example 8.6, $Cay(G_7, U)$ is a connected normal edge-transitive graph of valency $2p(p-1)$, where $U = \{a^ib^jc^k|1 \leq i < p, 0 \leq j < p, k = 1, 3\}$. Now suppose $S \subseteq \{a^ib^jc, a^kb^lc^3|1 \leq i, k < p, 0 \leq j, l < p,\ j \neq l\}$ and Γ is a Cayley graph of valency $2d$. Since $Aut(G_7, S) \leq Aut(G_7)$ and $Aut(G_7, S)$ is transitive on S (Theorem 8.4), we have $|S| = 2d \mid |Aut(G_7, S)| \mid |Aut(G_7)| = 2p(p-1)^2$, implying $d|p(p-1)^2$. On the other hand, $d \leq p(p-1)$, hence $d = p$ or $d|(p-1)$ or $d|p(p-1)$ or $d \nmid (p-1)$ but $d|(p-1)^2$, proving the first assertion of the theorem. To prove the existence and uniqueness part in the theorem, if $d = p(p-1)$, then as mentioned above, $Cay(G_7, U)$ is the unique normal maximal edge-transitive Cayley graph of valency $2p(p-1)$. Now suppose $d = p$. Consecutive application of $h_{1,1,1,1}$ on ac yields the set $T = \{ac, abc, ab^2c, ..., ab^{(p-1)}c\}$ whose size is p and is invariant under $h_{1,1,1,1}$. Let us set $T^{-1} = \{x^{-1} | x \in T\} = \{a^{-1}c^3, a^{-1}b^3, a^{-1}b^2c^3, ..., a^{-1}b^{(p-1)}c^3\}$ and $S = T \cup T^{-1}$. By the argument used in Lemma 8.2, we have $< S >= G_7$. It is easy to see that $h_{-1,1,0,3}$ interchanges elements of T and T^{-1}, also $Aut(G_7, S)$ is transitive on S, implying $Cay(G_7, S)$ is connected normal edge-transitive of valency $2p$. Now let $d|(p-1), d > 1$. The stabilizer of a and c under $A = Aut(G_7)$ is the group $A_{a,c} = \{h_{1,i,0,1} | 1 \leq i < p\} \cong U_p$. Let t be a generator of U_p, so that $A_{a,c} =< h_{1,t,0,1} >$. Since $d|(p-1)$, the group U_p contains a unique subgroup of order d, and if we set $u = t^{\frac{p-1}{d}}$, then $< h_{1,u,0,1} >$ is a subgroup of $A_{a,c}$ with order d. Now consecutive application of $h_{1,u,0,1}$ on abc yields the set $T = \{abc, ab^uc, ..., ab^{u^{d-1}}c\}$ whose size is d and is invariant under $h_{1,u,0,1}$. Let us set $T^{-1} = \{x^{-1} | x \in T\}$ and $S = T \cup T^{-1}$. We claim that $Cay(G_7, S)$ is a connected normal edge-transitive Cayley graph. Similar to the above, $< S >= G_7$ and $h_{-1,1,0,3}$ interchanges elements of T and T^{-1}, also the automorphism group of $Cay(G_7, S)$ is $< h_{1,u,0,1}, h_{-1,1,0,3} >$, implying $Cay(G_7, S)$ is connected normal edge-transitive of valency $2d$. Now let $d \nmid p-1$ and $d \neq p$ but $d|p(p-1)$, in this case we have $d = tp$ such that $t \mid p-1$. Set $E = \{h_{u,1,i,1} | \ u^t \equiv 1 \pmod p), 0 \leq i < p\} \leq Aut(G_7)$, consecutive effects of $h_{u,1,i,1}$ on ac yields the set $T = \{ac, a^uc, ..., a^{u^{t-1}}c, abc, a^ubc, ..., a^{u^{t-1}}bc, ..., a^{u^{t-1}}b^{p-1}c\}$ whose size is d and is invariant under E and set $S = T \cup T^{-1}$, similar to the above, $Cay(G_7, S)$ is connected normal edge-transitive of valency $2d$. Finally let $d \nmid (p-1)$ but $d|(p-1)^2$, in this case $d = ms$ such that $m \mid p-1$ and $s \mid p-1$. Set $H =< h_{u,1,0,1}, h_{1,t,0,1} >$ such that $t^m \equiv 1 \pmod p$ and $u^s \equiv 1 \pmod p$. Consecutive application of H on abc yields the set $T = \{abc, a^ubc, ..., a^{u^{s-1}}bc, ab^tc, a^ub^tc, ..., a^{u^{s-1}}b^tc, , ..., ab^{t^{k-1}}c, ..., a^{u^{s-1}}b^{t^{k-1}}c\}$ whose size is d and is invariant under H and set $S = T \cup T^{-1}$, hence $Cay(G_7, S)$ is connected normal edge-transitive of valency $2d$. □

9. **Normal edge-transitive Cayley graphs on group G_8**

Elements of G_8 can be written uniquely in the form $\{a^ib^jc^k, \ 0 \leq i < p, \ 0 \leq j < p, \ 0 \leq k < 4\}$. The order of elements of G_8 are as follows:
for $0 \leq i < p, \ 0 \leq j < p, \ k = 1, 3$, we have $O(a^ib^j) = p$, $O(c^2) = 2$, $O(a^ib^jc^k) = 4$ and $O(a^ib^jc^2) = 2p$. Using the above facts, we can find $Aut(G_8)$. Let us choose $0 \leq i, j, k, l < p$ with following properties:

1. If there exist $1 \le s < p$ such that $si \equiv k \pmod{p}$), then $sj \not\equiv l \pmod{p}$.
2. $i = j = 0$ and $k = l = 0$ do not occur.
With this condition we set $f_{i,j,k,l,m,n,t}(a) = a^i b^j$, $f_{i,j,k,l,m,n,t}(b) = a^k b^l$ and $f_{i,j,k,l,m,n,t}(c) = a^m b^n c^t$, then we have:

$$Aut(G_8) = \{f_{i,j,k,l,m,n,t} |\ 0 \le i,j,k,l,m,n < p,\ t = 1,3\}$$

and for $0 \le i,j < p$, it has the following orbits on G_8 : $\{1\}, \{a^i b^j\}$, $\{a^i b^j c^2 |$ i and j are not zero in same time$\}$, $\{c^2\}$ and $\{a^i b^j c^k | k = 1,3\}$.

Lemma 9.1. *If $Cay(G_8, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| \ge 6$ is even.*

Proof. By Proposition 2.2, elements of S have the same order. Since $< S >= G_8$, the set S cannot contain elements of order p, $2p$ or 2, and should contain elements of order 4 only. By Proposition 2.3, $|S|$ is even. Now set $|S| = 4$, then $S = \{a^i b^j c, a^k b^l c, a^i b^j c^3, a^k b^l c^3 | 0 \le i,j,k,l < p\}$. We have:

$$(a^i b^j c)^m = \begin{cases} c^2 & \text{m=2} \\ a^i b^j c & \text{m=1} \\ a^i b^j c^3 & \text{m=3} \end{cases}$$

$a^i b^j c a^k b^l c = a^{i-k} b^{j-l} c^2$ and $a^i b^j c a^k b^l c^3 = a^{i-k} b^{j-l}$. So according to the above relations, $< S >< G_8$.
Now set $|S| = 6$, then $S = \{a^i b^j c, a^k b^l c, a^t b^f c, a^i b^j c^3, a^k b^l c^3, a^t b^f c^3 | 0 \le i,j,k,l, t,f < p\}$. In this case $< S >= G_8$ if and only if $0 \le i,j,k,l,t,f < p$ and if there exist $1 \le s < p$ such that $s(k-i) \equiv (t-i) \pmod{p}$, then $s(l-j) \not\equiv (f-j)$ $(\text{mod } p)$. Because we have $a^k b^l c a^i b^j c^3 = a^{k-i} b^{l-j}$ and $a^t b^f c a^i b^j c^3 = a^{t-i} b^{f-j}$. According to the above relations, $b^{s(l-j)-(f-j)} \in < S >$ and so $b \in < S >$, also $a \in < S >$ and $c \in < S >$. □

Theorem 9.2. *Let $\Gamma = Cay(G_8, S)$ be a Cayley graph of valency 6. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^k b^l c, a^t b^f c, a^i b^j c^3, a^k b^l c^3, a^t b^f c^3 | 0 \le i,j,k,l,t,f < p\}$, where if there exist $1 \le s < p$ such that $s(k-i) \equiv (t-i) \pmod{p}$, then $s(l-j) \not\equiv (f-j) \pmod{p}$.*

Proof. Set $S' = \{c, ac, bc, c^3, ac^3, bc^3\}$. First we prove that $Cay(G_8, S')$ is connected normal edge-transitive. By Lemma 9.1, $< S' >= G_8$ and so $Cay(G_8, S')$ is connected. The automorphisms $f_{0,1,1,0,0,0,1}$, $f_{-1,0,-1,1,1,0,1}$, $f_{1,0,0,1,0,0,3}$, $f_{0,1,1,0,0,0,3}$ and $f_{-1,0,-1,1,1,0,3}$ are all in $Aut(G_8, S')$ and transfer ac to bc, c, ac^3, bc^3 and c^3 respectively. So $Aut(G_8, S')$ is transitive on S' and $Cay(G_8, S')$ is connected normal edge-transitive Cayley graph. Now according to condition of theorem, $f_{k-i,l-j,t-i,f-j,i,j,1}$ is in $Aut(G_8)$ and $S'^{f_{k-i,l-j,t-i,f-j,i,j,1}} = S$ and the proof is completed. □

10. Normal edge-transitive Cayley graphs on group G_9 and G'_9

The order of elements of G_9 are as follows:
for $0 \leq i < p,\ 0 \leq j < p,\ k = 1, 3$, we have $O(a^i b^j) = p$, $O(a^i b^j c^k) = 4$ and $O(a^i b^j c^2) = 2$. Using the above facts, we can find $Aut(G_9)$. Set $f_{i,j,k,l}(a) = a^i b^j$, $f_{i,j,k,l}(b) = a^{-j} b^i$, $f_{i,j,k,l}(c) = a^k b^l c$, $g_{i,j,k,l}(a) = a^i b^j$, $g_{i,j,k,l}(b) = a^j b^{-i}$, $g_{i,j,k,l}(c) = a^k b^l c^3$, then we have:

$$Aut(G_9) = \{f_{i,j,k,l}, g_{i,j,k,l} |\ 0 \leq i, j, k, l < p,\ and\ i\ ,\ j\ are\ not\ both\ zero\}$$

$|Aut(G_9)| = 2p^2(p^2-1)$ and for $0 \leq i, j < p, k = 1, 3$, $Aut(G_9)$ has the following orbits on $G_9 : \{1\}, \{a^i b^j\}, \{a^i b^j c^2\}$ and $\{a^i b^j c^k\}$.

Lemma 10.1. *If $Cay(G_9, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| \geq 4$ is even.*

Lemma 10.2. *Let $\Gamma = Cay(G_9, S)$ be a Cayley graph of valency 4. Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^k b^l c, a^{-j} b^i c^3, a^{-l} b^k c^3\}$ where $0 \leq i, j, k, l < p$.*

Proof. At first we prove that $< S >= G_9$. We have $a^i b^j c a^{-l} b^k c^3 = a^{i-k} b^{j-l}$ and $a^{-l} b^k c^3 a^i b^j c = a^{j-l} b^{k-i}$, so there exist integer m such that $m(k-i) \equiv j-l \pmod p$ and $m(l-j) \not\equiv k-i \pmod p$. So we have $(a^{i-k} b^{j-l})^m a^{j-l} b^{k-i} = b^{m(j-l)+(k-i)}$, therefore $b \in < S >$ and also $a \in < S >$ and $c \in < S >$.
Now set $S' = \{c, bc, a^{-1}c^3, c^3\}$. Then $< S' >= G_9$ and $Cay(G_9, S')$ is connected. Also we have $f_{-1,0,0,1}, g_{0,-1,0,0}, g_{0,1,-1,0} \in Aut(G_9, S')$ and $c^{f_{-1,0,0,1}} = bc$, $c^{g_{0,-1,0,0}} = c^3$ and $c^{g_{0,1,-1,0}} = a^{-1}c^3$, implying that $Aut(G_9, S')$ is transitive on S' and so $Cay(G_9, S')$ is connected normal edge-transitive Cayley graph. Now we have $f_{l-j,i-k,i,j} \in Aut(G_9)$ and $S'^{f_{l-j,i-k,i,j}} = S$, therefore $\Gamma = Cay(G_9, S)$ is a connected normal edge-transitive Cayley graph. □

Theorem 10.3. *$\Gamma = Cay(G_9, S)$ is a connected normal edge-transitive Cayley graph if and only if its valency is even, greater than two, $S \subseteq \{a^i b^j c^t | 0 \leq i, j < p, t = 1, 3\}$, $S = S^{-1}$ and $Aut(G_9, S)$ acts transitively on S. Moreover, if Γ is a normal edge-transitive Cayley graph of valency $2d$, then $d = p^2$, $d = p$, $d|(p-1)$, $d|p(p-1)$ or $d \nmid (p-1)$ but $d|(p^2-1)$. For each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. If Γ is a connected normal edge-transitive Cayley graph, then by Proposition 2.3, its valency should be even. By Lemma 10.1, $S \subseteq \{a^i b^j c^t | 0 \leq i, j < p, t = 1, 3\}$, since the graph is undirected, $S = S^{-1}$, and by Lemma 10.2, if $|S| > 2$, then $< S >= G_9$. From Proposition 2.2, either $Aut(G_9, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $Aut(G_9, S)$ on S. But we observe that $g_{-1,0,i-j,i-j} \in Aut(G_9, S)$, which implies both of $a^i b^j c$ and $(a^i b^j c)^{-1} = a^{-j} b^i c^3$ belong to the same orbit for $0 \leq i, j < p$

in which $a^ib^jc \in S$, and that contradiction with the assumption $S = T \cup T^{-1}$. Hence $Aut(G_9, S)$ acts transitively on S.
If $U = \{a^ib^jc^t|0 \leq i, j < p, t = 1, 3\}$, then U is an orbit of $Aut(G_9)$, therefore in this case, $Aut(G_9, U)$ is a connected normal edge-transitive Cayley graph of valency $2p^2$. Now suppose $S \subseteq \{a^ib^jc^t|0 \leq i, j < p, t = 1, 3\}$, $< S >= G_9$ and Γ is a Cayley graph of valency $2d$. Since $Aut(G_9, S) \leq Aut(G_9)$ and $Aut(G_9, S)$ is transitive on S, we have $|S| = 2d \mid |Aut(G_9, S)| \mid |Aut(G_9)| = 2p^2(p^2 - 1)$, implying $d|p^2(p^2 - 1)$. On the other hand, we have $d \leq p^2$, hence $d = p^2$ or $d = p$ or $d|(p - 1)$ or $d|p(p - 1)$ or $d \nmid (p - 1)$ but $d|(p^2 - 1)$, proving the first assertion of the theorem. To prove the existence and uniqueness part in the theorem, if $d = p^2$, then as mentioned above, $Cay(G_9, U)$ is the unique normal maximal edge-transitive Cayley graph of valency $2p^2$. Now suppose $d = p$ and set $S = \{c, ac, a^2c, ..., a^{p-1}c, c^3, bc^3, ..., b^{p-1}c^3\}$ whose size is $2p$. We have $f_{-1,0,i,0}, g_{0,1,0,i} \in Aut(G_9, S)$, $c^{f_{-1,0,i,0}} = a^ic$ and $c^{g_{0,1,0,i}} = b^ic$. So $Aut(G_9, S)$ is transitive on S, implying $Cay(G_9, S)$ is connected normal edge-transitive of valency $2p$.
Now let $d|(p-1), d > 1$. Define $E = \{f_{i,0,0,0}|1 \leq i < p\}$. Then E is a subgroup of $Aut(G_9)$ and $E \cong U_p$. Let t be a generator of U_p, so that $E =< f_{t,0,0,0} >$. Since $d|(p - 1)$, the group U_p contains a unique subgroup of order d, and if we set $u = t^{\frac{p-1}{d}}$, then $< f_{u,0,0,0} >$ is a subgroup of E with order d. Now consecutive application of $f_{u,0,0,0}$ on ac yields the set $T = \{ac, a^uc, ..., a^{u^{d-1}}c\}$ whose size is d and is invariant under $f_{u,0,0,0}$. Let us set $T^{-1} = \{x^{-1}|x \in T\}$ and $S = T \cup T^{-1}$. We claim that $Cay(G_9, S)$ is a connected normal edge-transitive Cayley graph. Similar to the above $< S >= G_9$ and $g_{0,1,0,0}$ interchanges elements of T and T^{-1}. Also the automorphism group of $Cay(G_9, S)$ is $< f_{u,0,0,0}, g_{0,1,0,0} >$, implying $Cay(G_9, S)$ is connected normal edge-transitive of valency $2d$.
Now let $d \nmid p - 1$ and $d \neq p$ but $d|p(p - 1)$. In this case, we have $d = tp$ such that $t \mid p - 1$. Let s be a number such that $s^t \equiv 1 \pmod p$ and set $T = \{ac, a^sc, ..., a^{s^{t-1}}c, abc, a^sbc, ..., a^{s^{t-1}}bc, ..., a^{s^{t-1}}b^{p-1}c\}$ whose size is d and is invariant under $F = \{f_{s,0,0,i}, f_{1,0,0,1}|\ s^t \equiv 1 \pmod p), 0 \leq i < p\} \leq Aut(G_9, T)$. Set $S = T \cup T^{-1}$, similar to the above, $Cay(G_9, S)$ is connected normal edge-transitive of valency $2d$.
Finally let $d|(p^2 - 1)$. The stabilizer of c under $A = Aut(G_9)$ is the abelian group $A_c = \{f_{i,j,0,0}|0 \leq i, j < p, i, j \textit{ are not both zero}\}$ and $|A_c| = p^2 - 1$. Since $d|(p^2 - 1)$, the group A_c contains a unique element of order d, say σ. Now consecutive application of σ on ac yields the set T whose size is d and is invariant under σ. Let us set $T^{-1} = \{x^{-1}|x \in T\}$ and $S = T \cup T^{-1}$. In this case $Cay(G_9, S)$ is a connected normal edge-transitive Cayley graph of valency $2d$. □

Next we consider G'_9.
Elements of G'_9 can be written uniquely in the form $\{a^ib^jc^k,\ 0 \leq i < p,\ 0 \leq j < p,\ 0 \leq k < 4\}$. The order of elements of G'_9 are as follows:
for $0 \leq i,j < p,\ k = 1,3$, We have $O(a^ib^j) = p$, $O(a^ib^jc^k) = 4$ and $O(a^ib^jc^2) = 2$. Using the above facts, we can find $Aut(G'_9)$. Set $f_{i,j,k,l}(a) = a^i$, $f_{i,j,k,l}(b) = b^j$, $f_{i,j,k,l}(c) = a^kb^lc$, $g_{i,j,k,l}(a) = b^i$, $g_{i,j,k,l}(b) = a^j$, $g_{i,j,k,l}(c) = a^kb^lc^3$, then we have:

$$Aut(G'_9) = \{f_{i,j,k,l}, g_{i,j,k,l} |\ 0 \leq k,l < p,\ 1 \leq i,j < p\}$$

$|Aut(G'_9)| = 2p^2(p-1)^2$ and for $0 \leq i,j < p, k = 1,3$, $Aut(G'_9)$ has the following orbits on $G'_9 : \{1\}$,$\{a^i|1 \leq i < p\}$,$\{b^j|1 \leq j < p\}$, $\{a^ib^j|1 \leq i,j < p\}$, $\{a^ib^jc^2|0 \leq i,j < p\}$ and $\{a^ib^jc^k|0 \leq i,j < p, k = 1,3\}$.

Lemma 10.4. *If $Cay(G'_9, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| \geq 4$ is even.*

Lemma 10.5. *Let $\Gamma = Cay(G'_9, S)$ be a Cayley graph of valency 4. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^ib^jc, a^kb^lc, a^{-i\lambda}b^{j\lambda}c^3, a^{-k\lambda}b^{l\lambda}c^3\}$ where $0 \leq i,j,k,l < p$ and $j \neq l, i \neq k$.*

Proof. At first we prove that $< S >= G'_9$. We have $a^ib^jca^{-k\lambda}b^{l\lambda}c^3 = a^{i-k}b^{j-l}$ and $a^ib^jca^kb^lc = a^{i-k\lambda}b^{j+l\lambda}c^2$, $(a^ib^jc)^2 = a^{i-i\lambda}b^{j+j\lambda}c^2$,
$a^{i-k\lambda}b^{j+l\lambda}c^2a^{i-i\lambda}b^{j+j\lambda}c^2 = a^{i\lambda-k\lambda}b^{l\lambda-j\lambda}$. So $b^{2(l-j)\lambda} \in< S >$, since $j \neq l$, then $b \in< S >$. From condition $i \neq k$ and above relations, we conclude that $a \in< S >$ and $c \in< S >$.
Now let m and n be the integers such that $mn \equiv 1 \pmod p$, $m\lambda(i-k) \equiv j-l \pmod p$ and $n\lambda(l-j) \equiv i-k \pmod p$, then $f_{-1,-1,i+k,j+l}, g_{m,n,-k\lambda-jn,l\lambda-im} \in Aut(G'_9, S)$. We have $(a^ib^jc)^{f_{-1,-1,i+k,j+l}} = a^kb^lc$, $(a^ib^jc)^{g_{m,n,-k\lambda-jn,l\lambda-im}} = a^{-k\lambda}b^{l\lambda}c^3$ and $(a^ib^jc)^{f_{-1,-1,i+k,j+l}og_{m,n,-k\lambda-jn,l\lambda-im}} = a^{-i\lambda}b^{j\lambda}c^3$. Therefore $Aut(G'_9, S)$ is transitive on S and Γ is a connected normal edge-transitive Cayley graph. □

11. Normal edge-transitive Cayley graphs on group G_{10}

The order of elements of G_{10} are as follows:
$O(a^ib^j) = p(0 \leq i,j < p)$ and for $1 \leq i < p,\ 0 \leq j < p,\ k = 1,3$, we have $O(b^jc^k) = 4$, $O(b^jc^2) = 2$, $O(a^ib^jc^k) = 4p$ and $O(a^ib^jc^2) = 2p$. Using the above facts, we can find $Aut(G_{10})$.

Lemma 11.1. *For odd prime p, $Aut(G_{10}) \cong Z_p \rtimes (U_p \times U_p)$, and it has the following orbits on $G_{10} : \{1\}$, $\{a^i|1 \leq i < p\}$,$\{b^j|1 \leq j < p\}$, $\{a^ib^j|1 \leq i,j < p\}$,$\{b^jc|0 \leq j < p\}$, $\{b^jc^2|0 \leq j < p\}$,$\{b^jc^3|0 \leq j < p\}$, $\{a^ib^jc|1 \leq i < p, 0 \leq j < p\}$, $\{a^ib^jc^2|1 \leq i < p, 0 \leq j < p\}$ and$\{a^ib^jc^3|1 \leq i < p, 0 \leq j < p\}$.*

Proof. Any $\sigma \in Aut(G_{10})$ is determined by its effect on a, b and c. Taking orders into account and by relations $\sigma(ab) = \sigma(ba)$, $\sigma(ac) = \sigma(ca)$ and

$\sigma(c^{-1}bc) = \sigma(b^{\lambda})$, we have $\sigma(a) = a^i$, $\sigma(b) = b^j$ and $\sigma(c) = b^k c$, where $1 \leq i, j < p,\ 0 \leq k < p$. It can be verified that $\sigma = f_{i,j,k}$ defined as above, can be extended to an automorphism of G_{10}. Therefore, $Aut(G_{10}) = \{f_{i,j,k} | 1 \leq i, j < p, 0 \leq k < p\}$ is a group of order $p(p-1)^2$. We have $f_{i,j,k} o f_{i',j',k'} = f_{ii',jj',jk'+k}$ and $f^{-1}_{i,j,k} = f_{i_1,j_1,-kj_1}$, hence if we define $A = \{f_{1,1,k} | 0 \leq k < p\}$ and $B = \{f_{i,j,0} | 1 \leq i, j < p\}$, then $Aut(G_{10}) = A \times B$, $A \cap B = id$ and $A \trianglelefteq Aut(G_{10})$. So $Aut(G_{10}) \cong Z_p \rtimes (U_p \times U_p)$ and the lemma is proved. □

Lemma 11.2. *If $Cay(G_{10}, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order $4p$. Moreover, $|S| > 2$ is even.*

Lemma 11.3. $S = \{a^i b^j c, a^k b^l c^3, a^{-i} b^{-j\lambda} c^3, a^{-k} b^{l\lambda} c\}$ *generates G_{10} if and only if $1 \leq i, k < p,\ 0 \leq j, l < p, j \not\equiv l\lambda \pmod{p}$.*

Proof. Generating condition of S comes from the relations $(a^i b^j c)^4 = a^{4i}$, $a^i b^j c a^k b^l c^3 = a^{i+k} b^{j-l\lambda}$. Since p is odd, then we can conclude that $a \in < S >$ and so $b \in < S >$ and $c \in < S >$. □

Theorem 11.4. $\Gamma = Cay(G_{10}, S)$ *is a connected normal edge-transitive Cayley graph if and only if it has even valency, $S = T \cup T^{-1}$, where $T \subseteq \{a^i b^j c, a^k b^l c | 1 \leq i, k < p, 0 \leq j, l < p, j \neq l\}$ and $Aut(G_{10}, S)$ acts transitively on T. Moreover, if Γ is a normal edge-transitive Cayley graph of valency $2d$, then $d = p$, $d|(p-1)$, $d|p(p-1)$ or $d \nmid (p-1)$ but $d|(p-1)^2$. For each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. Assume that Γ is a connected normal edge-transitive Cayley graph. The fact that Γ has even valency follows from Proposition 2.3. By Proposition 2.2, in the action of $Aut(G_{10}, S)$ on S, we can deduce either S is an orbit or $S = T \cup T^{-1}$, where T is an orbit. By Lemma 11.2 and Lemma 11.3, S contains only elements of types $a^i b^j c$ and $a^k b^l c^3$ for $1 \leq i, k < p, 0 \leq j, l < p, j \neq l\lambda$. But in the action of $Aut(G_{10})$ on G_{10}, $a^i b^j c$ and $a^k b^l c^3$ belongs to the two separated orbits and since $Aut(G_{10}, S) \leq Aut(G_{10})$ then $Aut(G_{10}, S)$ is not transitive on S. Therefore $S = T \cup T^{-1}$, where $T \subseteq \{a^i b^j c, a^k b^l c | 1 \leq i, k < p, 0 \leq j, l < p, j \neq l\}$, and $Aut(G_{10}, S)$ acts transitively on T. The second part of the theorem is similar to the proof of Theorem 10.3. □

Lemma 11.5. *Let $\Gamma = Cay(G_{10}, S)$ be a Cayley graph of valency 4. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^{\pm i} b^l c, a^{-i} b^{-j\lambda} c^3, a^{\mp i} b^{-l\lambda} c^3\}$ where $1 \leq i < p,\ 0 \leq j, l < p, j \neq l$.*

Proof. By Lemma 11.3, $< S >= G_{10}$ and by Theorem 11.4, $S = T \cup T^{-1}$, where $T = \{a^i b^j c, a^k b^l c |\ for\ some\ 1 \leq i, k < p, 0 \leq j, l < p, j \neq l\}$ and $Aut(G_{10}, S)$ acts transitively on T. Since $Aut(G_{10}, S)$ acts transitively on T. Then there exist $f_{m,n,t} \in Aut(G_{10}, S)$ such that $(a^i b^j c)^{f_{m,n,t}} = a^k b^l c$ and $(a^k b^l c)^{f_{m,n,t}} = a^i b^j c$. Therefore, $im \equiv k \pmod{p}$ and $km \equiv i \pmod{p}$ implying $k = \pm i$. In the case $k = i$, $f_{1,-1,l+j} \in Aut(G_{10}, S)$ and T is invariant under $f_{1,-1,l+j}$. In the case $k = -i$, $f_{-1,-1,l+j} \in Aut(G_{10}, S)$ and T is invariant under $f_{-1,-1,l+j}$. Therefore in both cases, Γ is a connected normal edge-transitive Cayley graph. □

12. **Normal edge-transitive Cayley graphs on group G_{11}**

The order of elements of G_{11} are as follows:
For $0 \leq i, j < p, k = 1, 3$ we have $O(a^i b^j) = p$, $O(a^i b^j c^k) = 4$ and for $1 \leq i < p,\ 0 \leq j < p$, we have $O(b^j c^2) = 2$, and $O(a^i b^j c^2) = 2p$. Using the above facts, we can find $Aut(G_{11})$.

Lemma 12.1. *For odd prime p, $Aut(G_{11}) \cong (Z_p \times Z_p) \rtimes (U_p \times U_p)$, and it has the following orbits on G_{11} : $\{1\}$, $\{a^i | 1 \leq i < p\}$,$\{b^j | 1 \leq j < p\}$, $\{a^i b^j | 1 \leq i, j < p\}$, $\{b^j c^2 | 0 \leq j < p\}$, $\{a^i b^j c | 0 \leq i, j < p\}$, $\{a^i b^j c^2 | 1 \leq i < p, 0 \leq j < p\}$ and $\{a^i b^j c^3 | 0 \leq i, j < p\}$.*

Proof. Any $\sigma \in Aut(G_{11})$ is determined by its effect on a, b and c. Taking orders into account and by relations $\sigma(ab) = \sigma(ba)$, $\sigma(c^{-1}ac) = \sigma(a^{-1})$ and $\sigma(c^{-1}bc) = \sigma(b^{\lambda})$, we have $\sigma(a) = a^i$, $\sigma(b) = b^j$ and $\sigma(c) = a^k b^l c$, where $1 \leq i, j < p$, $0 \leq k, l < p$. It can be verified that $\sigma = f_{i,j,k,l}$ defined as above can be extended to an automorphism of G_{11}. Therefore, $Aut(G_{11}) = \{f_{i,j,k,l} | 1 \leq i, j < p, 0 \leq k, l < p\}$ is a group of order $(p(p-1))^2$. We have $f_{i,j,k,l} o f_{i',j',k',l'} = f_{ii',jj',ik'+k,jl'+l}$ and $f^{-1}_{i,j,k,l} = f_{i_1,j_1,-ki_1,-lj_1}$, hence if we define $A = \{f_{1,1,k,l} | 0 \leq k, l < p\}$ and $B = \{f_{i,j,0,0} | 1 \leq i, j < p\}$, then $Aut(G_{11}) = A \times B$, $A \cap B = id$ and $A \trianglelefteq Aut(G_{11})$. So $Aut(G_{11}) \cong (Z_p \times Z_p) \rtimes (U_p \times U_p)$ and the lemma is proved. □

Lemma 12.2. *If $Cay(G_{11}, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| > 2$ is even.*

Lemma 12.3. *$S = \{a^i b^j c, a^k b^l c, a^i b^{-j\lambda} c^3, a^k b^{-l\lambda} c^3\}$ generates G_{11} if and only if $0 \leq i, j, l, k < p$, $j \neq l$ and $k \neq i$.*

Proof. Proof of this lemma is similar to the proof of Lemma 11.3. □

Theorem 12.4. *$\Gamma = Cay(G_{11}, S)$ is a connected normal edge-transitive Cayley graph if and only if it has even valency, $S = T \cup T^{-1}$, where $T \subseteq \{a^i b^j c, a^k b^l c |\ 0 \leq i, j, l, k < p, k \neq i, j \neq l\}$ and $Aut(G_{11}, S)$ acts transitively on T. Moreover, if Γ is a normal edge-transitive Cayley graph of valency $2d$, then $d = p^2$, $d|(p-1)$, $d|p(p-1)$ or $d \nmid (p-1)$ but $d|(p-1)^2$. For each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. Proof of this theorem is similar to the proof of Theorem 11.4. □

Lemma 12.5. *Let $\Gamma = Cay(G_{11}, S)$ be a Cayley graph of valency 4. Then Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^k b^l c, a^i b^{-j\lambda} c^3, a^k b^{-l\lambda} c^3\}$ where $0 \leq i, j, l, k < p$, $j \neq l$ and $k \neq i$.*

Proof. By Lemma 12.3, $< S >= G_{11}$ and by Theorem 12.4, $S = T \cup T^{-1}$, where $T = \{a^i b^j c, a^k b^l c| \ for\ some\ 1 \le i,k < p, 0 \le j,l < p, j \ne l, i \ne k\}$ and we have $f_{-1,-1,i+k,j+l} \in Aut(G_{11})$ such that $(a^i b^j c)^{f_{-1,-1,i+k,j+l}} = a^k b^l c$. So $Aut(G_{11}, S)$ acts transitively on T and Γ is a connected normal edge-transitive Cayley graph. □

13. Normal edge-transitive Cayley graphs on group G_{12}

The order of elements of G_{12} are as follows:
For $0 \le i,j < p, k = 1,3$ we have $O(a^i b^j) = p$, $O(a^i b^j c^k) = 4$ and $O(a^i b^j c^2) = 2$. Using the above facts, we can find $Aut(G_{12})$. Any $\sigma \in Aut(G_{12})$ is determined by its effect on a, b and c. Using the above facts, we can find $Aut(G_{12})$. Let us choose $0 \le i,j,k,l < p$ with following properties:
1. If there exist $1 \le s < p$ such that $si \equiv k \pmod p$, then $sj \not\equiv l \pmod p$.
2. $i = j = 0$ and $k = l = 0$ do not occur.
With this conditions, we set $f_{i,j,k,l,m,n}(a) = a^i b^j$, $f_{i,j,k,l,m,n,t}(b) = a^k b^l$ and $f_{i,j,k,l,m,n,t}(c) = a^m b^n c$, then we have:

$$Aut(G_{12}) = \{f_{i,j,k,l,m,n}|\ 0 \le i,j,k,l,m,n < p\}$$

and it has the following orbits on G_{12} : $\{1\}, \{a^i b^j\}$, $\{a^i b^j c^2\}$, $\{a^i b^j c\}$ and $\{a^i b^j c^3\}$ where $0 \le i,j < p$.

Lemma 13.1. *If $Cay(G_{12}, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| \ge 6$ is even.*

Proof. By Proposition 2.2, elements in S have the same order. Since $< S >= G_{12}$, the set S cannot contain elements of order p or 2, and should contain elements of order 4 only. By Proposition 2.3, $|S|$ is even. Now set $|S| = 4$, then $S = \{a^i b^j c, a^k b^l c, a^{-i\lambda} b^{-j\lambda} c^3, a^{-k\lambda} b^{-l\lambda} c^3 | 0 \le i,j,k,l < p\}$. We have:

$$(a^i b^j c)^m = \begin{cases} a^{i-i\lambda} b^{j-j\lambda} c^2 & \text{m=2} \\ a^{-i\lambda} b^{-j\lambda} c^3 & \text{m=3} \\ 1 & \text{m=4} \end{cases}$$

$a^i b^j c a^k b^l c = a^{i-k\lambda} b^{j-l\lambda} c^2$ and $a^i b^j c a^{-k\lambda} b^{-l\lambda} c^3 = a^{i-k} b^{j-l}$. So according to above relations $< S >< G_{12}$.
Now set $|S| = 6$, then $S = \{a^i b^j c, a^k b^l c, a^t b^f c, a^{-i\lambda} b^{-j\lambda} c^3, a^{-k\lambda} b^{-l\lambda} c^3, a^{-t\lambda} b^{-f\lambda} c^3 | 0 \le i,j,k,l,t,f < p\}$. In this case $< S >= G_{12}$ if and only if $0 \le i,j,k,l,t,f < p$ and if there exist $1 \le s < p$ such that $s(k-i) \equiv (t-i) \pmod p$, then $s(l-j) \not\equiv (f-j) \pmod p$. Because we have $a^k b^l c a^{-i\lambda} b^{-j\lambda} c^3 = a^{k-i} b^{l-j}$ and $a^t b^f c a^{-i\lambda} b^{-j\lambda} c^3 = a^{t-i} b^{f-j}$. According to above relations, $b^{s(l-j)-(f-j)} \in < S >$ and so $b \in < S >$, also $a \in < S >$ and $c \in < S >$. □

Theorem 13.2. *Let $\Gamma = Cay(G_{12}, S)$ be a Cayley graph of valency 6. Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b^j c, a^k b^l c,$*

$a^tb^fc, a^ib^jc^3, a^kb^lc^3, a^tb^fc^3 | 0 \leq i, j, k, l, t, f < p\}$ *such that if there exist* $1 \leq s < p$ *such that* $s(k-i) \equiv (t-i) \pmod{p}$*, then* $s(l-j) \not\equiv (f-j) \pmod{p}$.

Proof. Set $S' = \{c, ac, bc, c^3, a^{-\lambda}c^3, b^{-\lambda}c^3\}$. With Lemma 13.1, $< S' >= G_{12}$ and so $Cay(G_{12}, S')$ is connected. Set $T = \{c, ac, bc\}$, the automorphisms $f_{0,1,1,0,0,0}$ and $f_{-1,0,-1,1,1,0}$ are in $Aut(G_{12}, S')$ and transfers ac to bc, c. So $Aut(G_{12}, S')$ is transitive on T, therefore $Cay(G_{12}, S')$ is connected normal edge-transitive Cayley graph. Now according to the conditions of theorem, $f_{k-i,l-j,t-i,f-j,i,j}$ is in $Aut(G_{12})$ and $S'^{f_{k-i,l-j,t-i,f-j,i,j}} = S$ and the proof is completed. □

According to the above results, we can state the following theorem.

Theorem 13.3. *Let* Γ *be a connected Cayley graph of order* $4p^2$*, where* p *is a prime number. Then* Γ *is normal* $\frac{1}{2}$*-arc-transitive if and only if* Γ *is a normal edge-transitive Cayley graph of a group isomorphic to one of the groups* G_2*,* G_{10}*,* G_{11} *or* G_{12}*.*

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(Yaghub Pakravesh) Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University P.O. Box 14115-137, Tehran, Iran.
E-mail address: y_pakravesh@yahoo.com

(Ali Iranmanesh) Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University P.O. Box 14115-137, Tehran, Iran.
E-mail address: iranmanesh@modares.ac.ir