Title:
On a Picone’s identity for the $A_{p(x)}$-Laplacian and its applications

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(Communicated by Asadollah Aghajani)

Abstract. We present a Picone’s identity for the $A_{p(x)}$-Laplacian, which is an extension of the classic identity for the ordinary Laplace. Also, some applications of our results in Sobolev spaces with variable exponent are suggested.

Keywords: Picone’s identity, $A_{p(x)}$-Laplacian, nonlinear elliptic problems.


1. Introduction and preliminaries

Since the pioneering work of M. Picone [15], efforts have been made to establish Picone identities for differential equations of various type (see [7,13,19]). Picone identities play an important role in the oscillation theory for ordinary or partial differential equations or systems. Let us recall the classical Picone’s identity:

For differentiable functions $v > 0$ and $u \geq 0$, we have

$$|\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \nabla v = |\nabla u|^2 - \nabla (\frac{u^2}{v}) \nabla v \geq 0. \quad (1.1)$$

Later Allegreto-Huang [1] presented a Picone’s identity for the $p$-Laplacian, which is an extension of (1.1). As an immediate consequence, they obtained a wide array of applications including the simplicity of the eigenvalues, Sturmian comparison principles, oscillation theorems and Hardy inequalities to name a few.

Recently in [18] the author established a generalization to Picone’s identity in the nonlinear framework. They showed, as an application of their results, that the Morse index of the zero solution to a semilinear elliptic boundary value
problem is 0 and also established a linear relationship between the components of the solution of a nonlinear elliptic system.

They proved that for differentiable functions \( v > 0 \) and \( u \geq 0 \) we have

\[
(1.2) \quad |\nabla u|^2 + \frac{|\nabla u|^2}{f'(v)} + \left( \frac{u \sqrt{f'(v)} \nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}} \right)^2 = |\nabla u|^2 - \nabla \left( \frac{u^2}{f(v)} \right) \cdot \nabla v \geq 0
\]

where \( f(y) \neq 0 \) and \( f'(y) \geq 1 \) for all \( y \neq 0 \); \( f(0) = 0 \).

Moreover, \( |\nabla u|^2 - \nabla (u^2/f(v)) \cdot \nabla v = 0 \) holds if and only if \( u = cv \) for an arbitrary constant \( c \). In a recent paper [3] Bal generalized the main result of [18] for the \( p \)-Laplace operator. The aim of this paper is to prove a generalized analogue of Picone’s identity for the \( \mathcal{A}_{p(x)} \)-Laplacian and, using this, we deal with the problems of the type

\[
(1.3) \quad \begin{cases}
-\mathcal{A}_{p(x)} u = m(x)|u|^{p(x)-2}u, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

where \( p \in C(\overline{\Omega}) \) such that \( 2 \leq p(x) < \infty \), \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( m \in L^\infty(\Omega) \) is a non-negative weight function and \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function. Here \( \mathcal{A}_{p(x)} u := \text{div} \left( \omega(x,u) \frac{\omega^{p(x)-2}}{2} A\nabla u \right) \) and will be called \( \mathcal{A}_{p(x)} \)-Laplace operator where \( A: \overline{\Omega} \to \mathbb{R}^{N \times N} \) is a symmetric matrix function with \( a_{ij}(x) \in L^\infty(\Omega) \bigcap C^1(\overline{\Omega}) \) and satisfy:

\[
(1.4) \quad \langle \mathcal{A}\xi, \xi \rangle = \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq |\xi|^2, \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^N,
\]

where \( \langle ., . \rangle \) denotes the scalar product on \( \mathbb{R}^N \), and \( \alpha: \Omega \times \mathbb{R} \to (0,\infty) \) is a Carathéodory function and for almost every \( x \in \Omega \) and every \( t \in \mathbb{R} \) we have \( \lambda \leq \alpha(x,t) \leq \Lambda \) for some positive \( \lambda \) and \( \Lambda \). The \( \mathcal{A}_{p(x)} \)-Laplace operator represents a generalization of the \( p(x) \)-Laplace operator, i.e. \( \Delta_{p(x)} u = \text{div} (|\nabla u|^{p(x)-2} \nabla u) \), which is obtained in the case when \( A = Id \) and \( \alpha \equiv 1 \).

In the last few decades, special attention has been paid to \( p(x) \)-Laplace type operators since they can model with sufficient accuracy the phenomena arising from the study of electrorheological fluids [16, 17], image restoration [6], mathematical biology [11], dielectric breakdown, electrical resistivity and polycrystal plasticity [4, 5]. In a similar context, we note that a collection of results obtained in the field of partial differential equations involving \( p(x) \)-Laplace type operators can be found in the survey paper by Harjulehto et al. [12]. Finally, we recall that in the case when \( p(x) \) is a constant function, problems involving \( \mathcal{A}_p \)-Laplace type operators have been widely studied. In this regard we point out the papers by Alvino et al. [2] and El Khalil et al. [8].
2451 Rasouli and the references therein. In a recent paper [14], the authors established the existence of solutions for a partial differential equation involving the $A_{p(x)}$-Laplace operator by using the Schauder’s fixed point theorem combined with adequate variational arguments.

To discuss problem (1.3), we need some theory on $W^{1,p(x)}(Ω)$ which is called variable exponent Sobolev space. Firstly we state some basic properties of spaces $W^{1,p(x)}_{0}(Ω)$ which will be used later (for details, see [9, 10]). Denote by $S(Ω)$ the set of all measurable real functions defined on $Ω$. Two functions in $S(Ω)$ are considered as the same element of $S(Ω)$ when they are equal almost everywhere. Write

$$C_{+}(Ω) = \{h : h \in C(Ω), h(x) > 1 \text{ for any } x \in Ω\},$$

$$h^- := \min_{Ω} h(x), \quad h^+ := \max_{Ω} h(x) \quad \text{for every } h \in C_{+}(Ω).$$

Define

$$L^{p(x)}(Ω) = \{u \in S(Ω) : \int_{Ω} |u(x)|^{p(x)} \, dx < +\infty \text{ for } p \in C_{+}(Ω)\}$$

with the norm

$$|u|_{L^{p(x)}(Ω)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_{Ω} \frac{|u(x)|^{p(x)}}{\lambda} \, dx \leq 1\},$$

and

$$W^{1,p(x)}(Ω) = \{u \in L^{p(x)}(Ω) : |\nabla u| \in L^{p(x)}(Ω)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(Ω)} = |u|_{L^{p(x)}(Ω)} + |\nabla u|_{L^{p(x)}(Ω)}.$$

Denote by $W^{1,p(x)}_{0}(Ω)$ the closure of $C^{∞}_{0}(Ω)$ in $W^{1,p(x)}(Ω)$.

**Theorem 1.1** ([10]). The spaces $L^{p(x)}(Ω)$, $W^{1,p(x)}(Ω)$ and $W^{1,p(x)}_{0}(Ω)$ are separable and reflexive Banach spaces.

2. Main results

Define $\varphi(s) = |s|^{p(x)-1} s$, $s \in \mathbb{R}$ and $\Phi(ξ) = |ξ|^{p(x)-1} ξ$, $ξ \in \mathbb{R}^n$, for $p(x) > 0$. We begin with the following lemma.

**Lemma 2.1** ([13]). For $X, Y \in \mathbb{R}^n$, we have

$$F(X,Y) := ⟨X, Φ(X)⟩ + α⟨Y, Φ(Y)⟩ - (α + 1)⟨X, Φ(Y)⟩ ≥ 0,$$

where the equality holds if and only if $X = Y$.

Next we present our main result.
Theorem 2.2. Let \( v > 0 \) and \( u \geq 0 \) be two non-constant differentiable functions in \( \Omega \). For all \( x \in \Omega \) define
\[
L(u, v) = \alpha(x, u) (A\nabla u, \nabla u)^{p(x)} \frac{1}{2} + (p(x) - 1) \alpha(x, v) (A\nabla v, \nabla v)^{p(x) - 2} \langle A\nabla v, \frac{u^{p(x)}}{\nabla v^{p(x)}} \nabla v \rangle
\]
\[
- p(x) \alpha(x, v) (A\nabla v, \nabla v)^{p(x) - 2} \langle A\nabla v, \frac{u^{p(x) - 1}}{\nabla v^{p(x) - 1}} \nabla u \rangle,
\]
\[
R(u, v) = \alpha(x, u) (A\nabla u, \nabla u)^{p(x)} - \alpha(x, v) (A\nabla v, \nabla v)^{p(x) - 2} \langle A\nabla v, \nabla \left( \frac{u^{p(x)}}{\nabla v^{p(x)}} \right) \rangle.
\]
Then \( L(u, v) = R(u, v) \geq 0 \). Moreover \( L(u, v) = 0 \) a.e. in \( \Omega \) if and only if \( \nabla \left( \frac{u}{v} \right) = 0 \) a.e. in \( \Omega \).

Proof. Expanding \( R(u, v) \) by direct calculation one easily sees that \( L(u, v) = R(u, v) \). To show \( L(u, v) = 0 \) we proceed as follow:

For simplicity, we let \( \alpha(x, u) = \alpha(x, v) = 1 \). Since \( A \) is positive definite, there exists a square root \( \sqrt{A} \) of \( A \) which is positive definite. We easily obtain
\[
\langle A\nabla u, \nabla u \rangle = |\sqrt{A}\nabla u|^2.
\]
Hence we get
\[
L(u, v) = |\sqrt{A}\nabla u|^{p(x)} + (p(x) - 1)|\sqrt{A}\frac{u}{v}\nabla v|^{p(x)}
\]
\[
- p(x)|\sqrt{A}\frac{u}{v}\nabla v|^{p(x) - 2} \langle \sqrt{A} u, \sqrt{A} \nabla u \rangle,
\]
which is nonnegative (from Lemma 2.1 and cf. [20, p. 85]). Equality holds if \( \sqrt{A}\frac{u}{v}\nabla v = \sqrt{A}\nabla u \). Since \( \sqrt{A} \) is positive definite, it follows from \( \sqrt{A}\frac{u}{v}\nabla v = \sqrt{A}\nabla u \) that \( \nabla \left( \frac{u}{v} \right) = 0 \).

3. Applications

3.1. Sturmian Theory. First we give a nonlinear version of Sturm-type comparison result for \( A_{p(x)} \)-Laplacian operator. We start by definition of weak solutions of (1.3).

Definition 3.1. By the weak solution of (1.3) we mean any \( u \in W^{1, p(x)}_0(\Omega) \) which satisfies
\[
\int_\Omega \alpha(x, u) (A\nabla u, \nabla u)^{p(x) - 2} \langle A\nabla u, \nabla \phi \rangle dx = \int_\Omega m(x) |u|^{p(x) - 2} u \phi dx,
\]
for any \( \phi \in W^{1, p(x)}_0(\Omega) \).
Theorem 3.2. Let $m_1 < m_2$ be two weight functions of (1.3) and $u$ be a positive solution of (1.3) for $m = m_1$. Then any nontrivial solution $v$ of (1.3) for $m = m_2$, must change sign.

Proof. Suppose the contrary. We may assume that there exists a solution $v > 0$ of (1.3) in $\Omega$. Then by Picone’s identity we have

$$0 \leq \int_{\Omega} L(u,v) = \int_{\Omega} R(u,v)$$

$$= \int_{\Omega} \alpha(x,u)\langle A\nabla u, \nabla u \rangle^{\frac{p(x)-2}{2}} - \alpha(x,v)\langle A\nabla v, \nabla v \rangle^{\frac{p(x)-2}{2}} \langle A\nabla v, \nabla \left( \frac{v^{p(x)}-1}{v^{p(x)}-1} \right) \rangle$$

$$= \int_{\Omega} m_1(x)u^{p(x)} - m_2(x)v^{p(x)}$$

$$= \int_{\Omega} (m_1 - m_2)v^{p(x)} < 0,$$

which is a contradiction. Hence, $v$ changes sign in $\Omega$. \hfill \Box

3.2. Nonlinear elliptic system involving the $A_{p(x)}$-Laplacian. Let us consider the following coupled nonlinear elliptic system:

$$-\text{div} \left( \alpha(x,u)\langle A\nabla u, \nabla u \rangle^{\frac{p(x)-2}{2}} A\nabla u \right) = v^{p(x)-1} \quad \text{in } \Omega \quad (3.1)$$

$$-\text{div} \left( \alpha(x,v)\langle A\nabla v, \nabla v \rangle^{\frac{p(x)-2}{2}} A\nabla v \right) = \frac{\nu^{2(p(x)-1)}}{u^{p(x)-1}} \quad \text{in } \Omega$$

$$u > 0, \quad v > 0 \quad \text{in } \Omega$$

$$u = 0, \quad v = 0 \quad \text{on } \partial \Omega.$$

We will show that Picone’s Identity yields a linear relationship between $u$ and $v$.

Theorem 3.3. Let $(u,v)$ be a weak solution of (3.1). Then $u = c_1 v$ where $c_1$ is a constant.

Proof. Let $(u,v)$ be the weak solution of (3.1). Now for any $\phi_1$ and $\phi_2$ in $W^{1,p(x)}_0(\Omega)$, we have

$$\int_{\Omega} \alpha(x,u)\langle A\nabla u, \nabla u \rangle^{\frac{p(x)-2}{2}} \langle A\nabla u, \nabla \phi_1 \rangle dx = \int_{\Omega} v^{p(x)-1}\phi_1 dx,$$  \hfill (3.2)

$$\int_{\Omega} \alpha(x,v)\langle A\nabla v, \nabla v \rangle^{\frac{p(x)-2}{2}} \langle A\nabla v, \nabla \phi_2 \rangle dx = \int_{\Omega} \frac{\nu^{2(p(x)-1)}}{u^{p(x)-1}} \phi_2 dx.$$  \hfill (3.3)
Choosing $\phi_1 = u$ and $\phi_2 = \frac{u^{p(x)}}{v^{p(x)-1}}$ in (3.2) and (3.3) we obtain

$$
\int_\Omega \alpha(x, u) \langle A\nabla u, \nabla u \rangle^{\frac{p(x)}{2}} dx
= \int_\Omega uv^{p(x)-1} dx
= \int_\Omega \alpha(x, v) \langle A\nabla v, \nabla v \rangle^{\frac{p(x)-2}{2}} \langle A\nabla v, \nabla \left( \frac{u^{p(x)}}{v^{p(x)-1}} \right) \rangle dx.
$$

Hence we have

$$
\int_\Omega R(u, v) dx = \int_\Omega \left( \alpha(x, u) \langle A\nabla u, \nabla u \rangle^{\frac{p(x)}{2}} - \alpha(x, v) \langle A\nabla v, \nabla v \rangle^{\frac{p(x)-2}{2}} \langle A\nabla v, \nabla \left( \frac{u^{p(x)}}{v^{p(x)-1}} \right) \rangle \right) dx = 0.
$$

By the positivity of $R(u, v)$ we have that $R(u, v) = 0$ and hence

$$
\nabla \left( \frac{u}{v} \right) = 0
$$

which gives $u = c_1 v$ where $c_1$ is a constant. \hfill \qed

Acknowledgements

The author is extremely grateful to the referees for their helpful suggestions for the improvement of the paper.

REFERENCES


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