ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 7, pp. 2449–2455

Title:

On a Picone's identity for the $\mathcal{A}_{p(x)}$-Laplacian and its applications

Author(s):

S.H. Rasouli

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 7, pp. 2449–2455
Online ISSN: 1735-8515

ON A PICONE'S IDENTITY FOR THE $\mathcal{A}_{p(x)}$-LAPLACIAN AND ITS APPLICATIONS

S.H. RASOULI

(Communicated by Asadollah Aghajani)

Abstract. We present a Picone's identity for the $\mathcal{A}_{p(x)}$-Laplacian, which is an extension of the classic identity for the ordinary Laplace. Also, some applications of our results in Sobolev spaces with variable exponent are suggested.

Keywords: Picone's identity, $\mathcal{A}_{p(x)}$-Laplacian, nonlinear elliptic problems.
MSC(2010): Primary: 35J66; Secondary: 35J92, 35B05, 11Y50.

1. Introduction and preliminaries

Since the pioneering work of M. Picone [15], efforts have been made to establish Picone identities for differential equations of various type (see [7, 13, 19]). Picone identities play an important role in the oscillation theory for ordinary or partial differential equations or systems. Let us recall the classical Picone's identity:

For differentiable functions $v > 0$ and $u \geq 0$, we have

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla(\frac{u^2}{v})\nabla v \geq 0. \tag{1.1}$$

Later Allegreto-Huang [1] presented a Picone's identity for the p-Laplacian, which is an extension of (1.1). As an immediate consequence, they obtained a wide array of applications including the simplicity of the eigenvalues, Sturmian comparison principles, oscillation theorems and Hardy inequalities to name a few.

Recently in [18] the author established a generalization to Picone's identity in the nonlinear framework. They showed, as an application of their results, that the Morse index of the zero solution to a semilinear elliptic boundary value

Article electronically published on December 30, 2017.
Received: 21 September 2015, Accepted: 29 July 2017.

problem is 0 and also established a linear relationship between the components of the solution of a nonlinear elliptic system.

They proved that for differentiable functions $v > 0$ and $u \geq 0$ we have

$$(1.2)\quad |\nabla u|^2 + \frac{|\nabla u|^2}{f'(v)} + (\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}})^2 = |\nabla u|^2 - \nabla(\frac{u^2}{f(v)}) \cdot \nabla v \geq 0$$

where $f(y) \neq 0$ and $f'(y) \geq 1$ for all $y \neq 0$; $f(0) = 0$.

Moreover, $|\nabla u|^2 - \nabla(u^2/f(v)) \cdot \nabla v = 0$ holds if and only if $u = cv$ for an arbitrary constant c. In a recent paper [3] Bal generalized the main result of [18] for the p-Laplacian operator. The aim of this paper is to prove a generalized analogue of Picone's identity for the $\mathcal{A}_{p(x)}$-Laplacian and, using this, we deal with the problems of the type

$$(1.3)\quad \begin{cases} -\mathcal{A}_{p(x)}u = m(x)|u|^{p(x)-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $p \in C(\bar{\Omega})$ such that $2 \leq p(x) < \infty$, Ω is a bounded smooth domain of $\mathbb{R}^N$, $m \in L^\infty(\Omega)$ is a non-negative weight function and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Here $\mathcal{A}_{p(x)}u := div\Big(\alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} A\nabla u\Big)$ and will be called $\mathcal{A}_{p(x)}$-Laplace operator where A: $\overline{\Omega} \to R^{N^2}$ is a symmetric matrix function with $a_{ij}(x) \in L^\infty(\Omega) \bigcap C^1(\overline{\Omega})$ and satisfy:

$$(1.4)\quad \langle A\xi, \xi\rangle = \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \forall x \in \overline{\Omega}, , \xi \in R^N,$$

where $\langle ., . \rangle$ denotes the scalar product on R^N, and α: $\Omega \times R \to (0, \infty)$ is a *Carathéodory* function and for almost every $x \in \Omega$ and every $t \in R$ we have $\lambda \leq \alpha(x,t) \leq \Lambda$ for some positive λ and Λ. The $\mathcal{A}_{p(x)}$-Laplace operator represents a generalization of the $p(x)$-Laplace operator, i.e. $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}.\nabla u)$, which is obtained in the case when $A = Id$ and $\alpha \equiv 1$.

In the last few decades, special attention has been paid to $p(x)$-Laplace type operators since they can model with sufficient accuracy the phenomena arising from the study of electrorheological fluids [16, 17], image restoration [6], mathematical biology [11], dielectric breakdown, electrical resistivity and polycrystal plasticity [4, 5]. In a similar context, we note that a collection of results obtained in the field of partial differential equations involving $p(x)$-Laplace type operators can be found in the survey paper by Harjulehto et al. [12]. Finally, we recall that in the case when $p(x)$ is a constant function, problems involving $\mathcal{A}_p$-Laplace type operators have been widely studied. In this regard we point out the papers by Alvino et al. [2] and El Khalil et al. [8]

and the references therein. In a recent paper [14], the authors established the existence of solutions for a partial differential equation involving the $\mathcal{A}_{p(x)}$-Laplace operator by using the Schauder's fixed point theorem combined with adequate variational arguments.

To discuss problem (1.3), we need some theory on $W_0^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [9, 10]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \leq 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Theorem 1.1 ([10]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

2. Main results

Define $\varphi(s) = |s|^{p(x)-1}s$, $s \in \mathbb{R}$ and $\Phi(\xi) = |\xi|^{p(x)-1}\xi$, $\xi \in \mathbb{R}^n$, for $p(x) > 0$. We begin with the following lemma.

Lemma 2.1 ([13]). *For $X, Y \in \mathbb{R}^n$, we have*

$$F(X, Y) := \langle X, \Phi(X)\rangle + \alpha\langle Y, \Phi(Y)\rangle - (\alpha + 1)\langle X, \Phi(Y)\rangle \geq 0,$$

where the equality holds if and only if $X = Y$.

Next we present our main result.

Theorem 2.2. *Let $v > 0$ and $u \geq 0$ be two non-constant differentiable functions in Ω. For all $x \in \Omega$ define*

$$\begin{aligned} L(u,v) &= \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}} \\ &\quad + (p(x)-1)\alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \frac{u^{p(x)}}{v^{p(x)}}\nabla v\rangle \\ &\quad - p(x)\,\alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \frac{u^{p(x)-1}}{v^{p(x)-1}}\nabla u\rangle, \end{aligned}$$

$$R(u,v) = \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}} - \alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \nabla\Big(\frac{u^p}{v^{p-1}}\Big)\rangle.$$

Then $L(u,v) = R(u,v) \geq 0$. Moreover $L(u,v) = 0$ a.e. in Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. in Ω.

Proof. Expanding $R(u,v)$ by direct calculation one easily sees that $L(u,v) = R(u,v)$. To show $L(u,v) \geq 0$ we proceed as follow:

For simplicity, we let $\alpha(x,u) = \alpha(x,v) = 1$. Since A is positive definite, there exists a square root $\sqrt{A}$ of A which is positive definite. We easily obtain

$$\langle A\nabla u, \nabla u\rangle = |\sqrt{A}\nabla u|^2.$$

Hence we get

$$\begin{aligned} L(u,v) &= |\sqrt{A}\nabla u|^{p(x)} + (p(x)-1)|\sqrt{A}\frac{u}{v}\nabla v|^{p(x)} \\ &\quad - p(x)|\sqrt{A}\frac{u}{v}\nabla v|^{p(x)-2}\langle\sqrt{A}\frac{u}{v}\nabla v, \sqrt{A}\nabla u\rangle. \end{aligned}$$

which is nonnegative (from Lemma 2.1 and cf. [20, p. 85]). Equality holds if $\sqrt{A}\frac{u}{v}\nabla v = \sqrt{A}\nabla u$. Since $\sqrt{A}$ is positive definite, it follows from $\sqrt{A}\frac{u}{v}\nabla v = \sqrt{A}\nabla u$ that $\nabla(\frac{u}{v}) = 0$. □

3. Applications

3.1. **Sturmian Theory.** First we give a nonlinear version of Sturm-type comparison result for $\mathcal{A}_{p(x)}$-Laplacian operator. We start by definition of weak solutions of (1.3).

Definition 3.1. By the weak solution of (1.3) we mean any $u \in W_0^{1,p(x)}(\Omega)$ which satisfies

$$\int_\Omega \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u, \nabla\phi\rangle dx = \int_\Omega m(x)\,|u|^{p(x)-2}\,u\,\phi dx,$$

for any $\phi \in W_0^{1,p(x)}(\Omega)$.

Theorem 3.2. *Let $m_1 < m_2$ be two weight functions of (1.3) and u be a positive solution of (1.3) for $m = m_1$. Then any nontrivial solution v of (1.3) for $m = m_2$, must change sign.*

Proof. Suppose the contrary. We may assume that there exists a solution $v > 0$ of (1.3) in Ω. Then by Picone's identity we have

$$\begin{aligned}0 &\leq \int_\Omega L(u,v) = \int_\Omega R(u,v)\\ &= \int_\Omega \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}} - \alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \nabla\Big(\frac{u^p}{v^{p-1}}\Big)\rangle\\ &= \int_\Omega m_1(x)u^{p(x)} - m_2(x)u^{p(x)}\\ &= \int_\Omega (m_1 - m_2)u^{p(x)} < 0,\end{aligned}$$

which is a contradiction. Hence, v changes sign in Ω. □

3.2. **Nonlinear elliptic system involving the $\mathcal{A}_{p(x)}$-Laplacian.** Let us consider the following coupled nonlinear elliptic system:

$$\begin{aligned}-div\Big(\alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}}A\nabla u\Big) &= v^{p(x)-1} \quad \text{in } \Omega\\ -div\Big(\alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}A\nabla v\Big) &= \frac{v^{2(p(x)-1)}}{u^{p(x)-1}} \quad \text{in } \Omega\\ u > 0, \quad v > 0 \quad &\text{in } \Omega\\ u = 0, \quad v = 0 \quad &\text{on } \partial\Omega.\end{aligned} \tag{3.1}$$

We will show that Picone's Identity yields a linear relationship between u and v.

Theorem 3.3. *Let (u, v) be a weak solution of (3.1). Then $u = c_1 v$ where c_1 is a constant.*

Proof. Let (u, v) be the weak solution of (3.1). Now for any ϕ_1 and ϕ_2 in $W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u, \nabla\phi_1\rangle dx = \int_\Omega v^{p(x)-1}\phi_1 dx, \tag{3.2}$$

$$\int_\Omega \alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \nabla\phi_2\rangle dx = \int_\Omega \frac{v^{2(p(x)-1)}}{u^{p(x)-1}}\phi_2 dx. \tag{3.3}$$

Choosing $\phi_1 = u$ and $\phi_2 = \frac{u^{p(x)}}{v^{p(x)-1}}$ in (3.2) and (3.3) we obtain

$$\begin{aligned}&\int_\Omega \alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}} dx\\&= \int_\Omega uv^{p(x)-1}dx\\&= \int_\Omega \alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \nabla(\frac{u^{p(x)}}{v^{p(x)-1}})\rangle dx.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_\Omega R(u,v)dx = \int_\Omega \big(&\alpha(x,u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\\&- \alpha(x,v)\langle A\nabla v, \nabla v\rangle^{\frac{p(x)-2}{2}}\langle A\nabla v, \nabla\Big(\frac{u^{p(x)}}{v^{p(x)-1}}\Big)\rangle\big)dx\\=&\ 0.\end{aligned}$$

By the positivity of $R(u,v)$ we have that $R(u,v) = 0$ and hence

$$\nabla(\tfrac{u}{v}) = 0$$

which gives $u = c_1 v$ where c_1 is a constant. □

Acknowledgements

The author is extremely grateful to the referees for their helpful suggestions for the improvement of the paper.

References

[1] W. Allegretto and Y. X. Huang, A Picone's identity for the p-Laplacian and applications, *Nonlinear Anal.* **32** (1998), no.7, 819–830.

[2] A. Alvino, V. Ferone and G. Trombetti, On the properties of some nonlinear eigenvalues, *SIAM J. Math. Anal.* **29** (1998) 437–451.

[3] K. Bal, Picone's identity and its applications, *Electron. J. Differential Equations*, **2012** (2012), no. 1, 9 pages.

[4] M. Bocea and M. Mihăilescu, Γ-convergence of power-law functionals with variable exponents, *Nonlinear Anal.* **73** (2010) 110–121.

[5] M. Bocea, M. Mihăilescu and C. Popovici, On the asymptotic behavior of variable exponent power-law functionals and applications, *Ric. Mat.* **59** (2010) 207–238.

[6] Y. Chen, S. Levine and M. Rao, Variable exponent linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006) 1383–1406.

[7] O. Došslý, The Picone identity for a class of partial differential equations, *Math. Bohem.* **127** (2002), no. 4, 581–589.

[8] A. El Khalil, P. Lindqvist and A. Touzani, On the stability of the first eigenvalue of the problem: $A_p u + \lambda g(x)|u|^{p-2}u = 0$ with varying p, *Rend. Mat.* **24** (2004) 321–336.

[9] X.L. Fan, J.S. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001) 749–760.

[10] X.L. Fan and D. Zhao, On the spaces $L^{p(x)}$ and $W^{k,p(x)}$, *J. Math. Anal. Appl.* **263** (2001) 424–446.

[11] G. Fragnelli, Positive periodic solutions for a system of anisotropic parabolic equations, *J. Math. Anal. Appl.* **367** (2010) 204–228.

[12] P. Harjulehto, P. Hasto, U.V. Le and M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.* **72** (2010) 4551–4574.

[13] T. Kusano, J. Jaroš and N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, *Nonlinear Anal.* **40** (2000), no. 1-8, 381–395.

[14] M. Mihailescu and D. Repovs, On a PDE involving the $A_{p(.)}$-Laplace Operator, *Nonlinear Anal.* **75** (2012) 975–981.

[15] M. Picone, Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine, *Ann. Scuola Norm. Sup. Pisa* **11**(1909) 1–141.

[16] K.R. Rajagopal and M. Ružička, Mathematical modeling of electrorheological materials, *Contin. Mech. Thermodyn.* **13** (2001) 59–78.

[17] M. Ružička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math. 1748, Springer-Verlag, Berlin, 2000.

[18] J. Tyagi, A nonlinear picone's identity and its applications, *Appl. Math. Lett.* **26** (2013) 624–626.

[19] N. Yoshida, Picone identities for half-linear elliptic operators with $p(x)$-Laplacians and applications to Sturmian comparison theory, *Nonlinear Anal.* **74** (2011), no. 16, 5631–5642.

[20] N. Yoshida, Picone identity for quasilinear elliptic equations with $p(x)$-Laplacians and Sturmian comparison theory, *Appl. Math. Comput.* **225** (2013) 79–91.

(Sayyed Hashem Rasouli) Department of Mathematics, Faculty of Basic Sciences, Babol Noshirvani University of Technology, Babol, Iran.

E-mail address: s.h.rasouli@nit.ac.ir