COMPACT ENDOМОРPHISMS OF CERTAIN SUBАLГEBRAS OF THE DISC ALGEBRA

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Abstract. In this paper we study endomorphisms of the following subalgebras of the disc algebra $A(D)$: The natural uniform subalgebras of $A(D)$, the analytic Lipschitz algebras $Lip_A(D, \alpha)$ of order $\alpha \in (0, 1]$ and the $n$-times differentiable Lipschitz algebras $Lip^n(D, \alpha)$ of order $\alpha \in (0, 1)$. Every nonzero endomorphism $T$ of many commutative semisimple Banach algebras including these subalgebras of $A(D)$ has the form $Tf = f \circ \varphi$ for some $\varphi : \mathbb{D} \to \mathbb{D}$ in them. We show that a sufficient condition for $\varphi$ to induce a compact endomorphism of these algebras is that either $\varphi$ is constant or $\|\varphi\|_D < 1$. We then show that these conditions are also necessary when $\alpha = 1$.

1. Introduction

In this note we consider endomorphisms of three types of Banach algebras of analytic functions on the open unit disc $\mathbb{D}$. We recall that a compact endomorphism of a Banach algebra $B$ is a compact linear map from $B$ into $B$ which preserves multiplication. Let $B$ be a Banach function algebra on a compact Hausdorff space $X$. 

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i.e., an algebra of complex-valued continuous functions on $X$ which separates the points of $X$, contains the constants and is complete under an algebra norm. A Banach function algebra $B$ on $X$ is called natural, if its maximal ideal space is $X$, i.e., each complex homomorphism on $B$ has the form $f \mapsto f(x)$ for some $x \in X$. It is well known that, if $B$ is a natural Banach function algebra on $X$ and $T$ is a nonzero bounded endomorphism of $B$, then there exists a self-map $\varphi$ on $X$ such that $Tf = f \circ \varphi$ for all $f \in B$. Conversely, if $\varphi$ is a self-map on $X$ such that for every $f \in B$, $f \circ \varphi \in B$, then $T : f \mapsto f \circ \varphi$ is an endomorphism of $B$. In each case, we say that $T$ is induced by $\varphi$. If $X$ is a compact plane set and $B$ contains the coordinate map $z$, then obviously $\varphi \in B$. It is interesting to see that under what conditions such $\varphi$ induces compact endomorphisms. For the disc algebra $A(\mathbb{D})$, the uniform algebra of complex-valued functions analytic on the open unit disc $\mathbb{D}$ and continuous on its closure $\overline{\mathbb{D}}$, H. Kamowitz [5] showed that if $T$ is a nonzero endomorphism of the disc algebra $A(\mathbb{D})$ induced by a map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, then $T$ is compact if and only if $\varphi$ is either constant or $\|\varphi\|_{\mathbb{D}} = \sup\{|\varphi(z)| : |z| \leq 1\} < 1$. H. Kamowitz and S. Scheinberg [6] also showed that an endomorphism $T$ of the Lipschitz algebra $\text{Lip}(X, d)$ induced by a map $\varphi : X \to X$ is compact if and only if $\varphi$ is supercontraction, that is $\frac{d(\varphi(x), \varphi(y))}{d(x, y)} \to 0$ as $d(x, y) \to 0$ where $(X, d)$ is a metric space. In [1] it was shown that an endomorphism of $D^n$, the algebra of functions on the closed unit disc $\overline{\mathbb{D}}$ with continuous $n$th derivatives, is compact if and only if $\varphi$ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$. Now we consider the following three types of subalgebras of $A(\mathbb{D})$: The natural uniform subalgebras of $A(\mathbb{D})$, the analytic Lipschitz algebras $\text{Lip}_A(\mathbb{D}, \alpha)$ of order $\alpha$ ($0 < \alpha \leq 1$), and the $n$-times differentiable Lipschitz algebras $\text{Lip}^n(\mathbb{D}, \alpha)$ of order $\alpha$ ($0 < \alpha \leq 1$). We show that a self-map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ in any of these subalgebras induces a compact endomorphism of the subalgebra, if $\varphi$ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$, and in the case that $\alpha = 1$ these conditions are also necessary. In general we have the following:
Proposition 1.1. Let $B$ be a subalgebra of $A(\mathbb{D})$ which is a natural Banach function algebra on $\overline{\mathbb{D}}$ under a norm. If $\varphi \in B$ and $\|\varphi\|_B < 1$, then $\varphi$ induces an endomorphism of $B$.

Proof. By the naturality of $B$, the spectrum $\sigma(\varphi) = \varphi(\mathbb{D})$. So $\sigma(\varphi) \subset \mathbb{D}$, since $\|\varphi\|_B < 1$. Therefore, every $f \in B$ is analytic on a neighborhood of $\sigma(\varphi)$, so $f \circ \varphi \in B$, by the Functional Calculus Theorem. That is, $\varphi$ induces an endomorphism of $B$. □

2. Natural uniform subalgebras of $A(\mathbb{D})$

Let $B$ be a natural uniform subalgebra of $A(\mathbb{D})$ with the uniform norm $\|f\|_B = \sup\{|f(z)| : |z| \leq 1\}$, $(f \in B)$. Then we have

Theorem 2.1. A nonzero endomorphism $T$ of $B$ induced by $\varphi$ is compact if and only if, $\varphi$ is either constant or $\|\varphi\|_B < 1$.

Proof. Obviously, the constant map $\varphi$ induces a compact endomorphism of $B$. Let $\varphi$ be non-constant and $\|\varphi\|_B < 1$. For compactness of $T$, suppose $\{f_n\}$ is a bounded sequence in $B$ with $\|f_n\|_B \leq 1$. Then by Montel Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is uniformly convergent on every compact subset of $\mathbb{D}$, in particular on $\varphi(\mathbb{D})$. So $\|Tf_{n_k} - Tf_{n_j}\|_B = \|f_{n_k} \circ \varphi - f_{n_j} \circ \varphi\|_B = \|f_{n_k} - f_{n_j}\|_{\varphi(\mathbb{D})} \to 0$ as $k, j \to \infty$. By completeness of $B$, $Tf_{n_k} = f_{n_k} \circ \varphi$ is convergent in $B$. That is, $T$ is compact.

Conversely, let $T$ be a nonzero compact endomorphism of $B$ induced by $\varphi$. Suppose for some $c$, $|c| = 1$ and $|\varphi(c)| = 1$. Define $f_n(z) = (\frac{\varphi(c)}{\varphi(z)})^n$ on $\overline{\mathbb{D}}$. By the compactness of $T$, the bounded sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $Tf_{n_k} = f_{n_k} \circ \varphi = (\frac{\varphi(c)}{\varphi(z)})^{n_k}$ converges to a function $f$ in $B$. Since $f_n(\varphi(c)) = 1$ for each $n$, $f(c) = 1$. If $\varphi$ is not constant, the maximum modulus principle implies that $|\varphi(z)| < 1$ whenever $|z| < 1$. So $(\varphi(z))^{n_k} \to 0$ when $|z| < 1$. Therefore, $f = 0$ on $\mathbb{D}$ and hence on $\overline{\mathbb{D}}$, by the continuity of $f$, which is a contradiction. □
3. The analytic Lipschitz algebras of order $\alpha$ ($0 < \alpha \leq 1$)

Let the analytic Lipschitz algebra $\text{Lip}_A(\mathbb{D}, \alpha)$ of order $\alpha$ ($0 < \alpha \leq 1$), be the algebra of all complex-valued functions $f$ on $\mathbb{D}$ which are analytic on $\mathbb{D}$ and satisfy the Lipschitz condition $p_\alpha(f) = \sup\{\frac{|f(z) - f(w)|}{|z-w|^{\alpha}} : z, w \in \mathbb{D}, z \neq w\} < \infty$. The algebra $\text{Lip}_A(\mathbb{D}, \alpha)$ is a Banach function algebra on $\mathbb{D}$, if it is equipped with the norm $\|f\| = \|f\|_D + p_\alpha(f)$. As it is shown in [1], the maximal ideal space of $\text{Lip}_A(\mathbb{D}, \alpha)$ is $\partial \mathbb{D}$, i.e., $\text{Lip}_A(\mathbb{D}, \alpha)$ is natural.

To give necessary conditions under which $\varphi$ induces a compact endomorphism of $\text{Lip}_A(\mathbb{D}) = \text{Lip}_A(\mathbb{D}, 1)$ we need the following lemma [2; Chapter I of Part Six, p 32].

**Lemma 3.1.** Let $f(z)$ be a non-constant analytic function of bounded one in the disc $|z| < 1$, and consider any triangle in this disk that has one of its vertices at $z = 1$. Then for any sequence $\{z_n\}$ of points from the interior of the triangle that converges to $z = 1$, the limit

$$\lim_{n \to \infty} \frac{1 - f(z_n)}{1 - z_n}$$

exists and either always (that is, for every such sequence) equals infinity or always equals a positive number $\alpha_0$.

In the second case, we refer to the number $\alpha_0$ as “the angular derivative” of the function $f$ at $z = 1$.

**Corollary 3.2.** If $\varphi \in \text{Lip}_A(\mathbb{D})$ is non-constant and $\|\varphi\|_D = |\varphi(c)| = 1$ for some $c$ with $|c| = 1$, then $\varphi$ has a nonzero angular derivative at $c$.

**Theorem 3.3.** Let $T$ be a nonzero endomorphism of $\text{Lip}_A(\mathbb{D}, \alpha)$, $0 < \alpha \leq 1$, induced by a map $\varphi : \mathbb{D} \to \mathbb{D}$. Then $T$ is compact, if $\varphi$ is either constant or $\|\varphi\|_D < 1$. For $\alpha = 1$, these conditions are necessary.

**Proof.** If $\varphi$ is constant, clearly $T$ is compact. Let $\|\varphi\|_D < 1$. For the compactness of $T$, we assume that $\{f_n\}$ is a bounded sequence
in $\text{Lip}_A(\mathbb{D}, \alpha)$ with $\|f_n\| = \|f_n\|_{\mathbb{D}} + p_{\alpha}(f_n) \leq 1$. Then $\{f_n\}$ is a bounded sequence of analytic functions on $\mathbb{D}$. By Montel Theorem $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ and its derivative $\{f'_{n_k}\}$ are uniformly convergent on every compact subset of $\mathbb{D}$. We claim that $\{f_{n_k} \circ \varphi\}$ is convergent in $\text{Lip}_A(\mathbb{D}, \alpha)$. Let $\Delta_r = \{z : |z| \leq r\}$ where $\|\varphi\|_{\mathbb{D}} < r < 1$. Then $\Delta_r$ is a compact subset of $\mathbb{D}$ containing the compact set $\varphi(\overline{\mathbb{D}})$. Hence

$$\|f_{n_k} \circ \varphi - f_{n_j} \circ \varphi\|_{\mathbb{D}} = \|f_{n_k} - f_{n_j}\|_{\varphi(\overline{\mathbb{D}})} \to 0 \text{ as } k, j \to \infty,$$

and for all $z, w \in \overline{\mathbb{D}}$, with $z \neq w$ and $k, j \in \mathbb{Z}_+$ we have

$$\left|\frac{|(f_{n_k} \circ \varphi \circ f_{n_j} \circ \varphi)(w) - (f_{n_k} \circ \varphi \circ f_{n_j} \circ \varphi)(z)|}{|w - z|^\alpha}\right|
\leq \frac{|(f_{n_k} - f_{n_j})(\varphi(w)) - (f_{n_k} - f_{n_j})(\varphi(z))|}{|w - z|^\alpha}
\leq \frac{p_{\alpha}(\varphi)}{|w - z|^\alpha} \leq p_{\alpha}(\varphi) \|f'_{n_k} - f'_{n_j}\|_{\Delta_r}.$$

Hence

$$p_{\alpha}(f_{n_k} \circ \varphi - f_{n_j} \circ \varphi) \leq p_{\alpha}(\varphi) \|f'_{n_k} - f'_{n_j}\|_{\Delta_r} \to 0 \text{ as } k, j \to \infty.$$

Therefore, $\{f_{n_k} \circ \varphi\}$ is a Cauchy sequence in $\text{Lip}_A(\mathbb{D}, \alpha)$ and hence $T$ is compact.

Conversely, let $\alpha = 1$ and let $0 \neq T$ be a compact endomorphism of $\text{Lip}_A(\mathbb{D})$ induced by $\varphi$. Suppose for some $c$, $|c| = 1$ and $|\varphi(c)| = 1$. So $\|\varphi\|_{\mathbb{D}} = 1$. Define $f_n(z) = \frac{z^n}{n}$. Then $\|f_n\|_{\mathbb{D}} = \frac{1}{n}$ and $p_1(f_n) \leq 1$. Therefore, $\{f_n\}$ is a bounded sequence in $\text{Lip}_A(\mathbb{D})$. By the compactness of $T$, there exists a subsequence $\{f_{n_k}\}$ such that $T f_{n_k} = f_{n_k} \circ \varphi$ converges in $\text{Lip}_A(\mathbb{D})$. Since $f_{n_k} \to 0$ uniformly on $\mathbb{D}$, $f_{n_k} \circ \varphi \to 0$ in $\text{Lip}_A(\mathbb{D})$. Thus

$$p_1(f_{n_k} \circ \varphi) = \sup_{z, w \in \mathbb{D}} \frac{|\varphi(z) - \varphi(w)|}{n_k(w - z)} \to 0, \text{ as } k \to \infty.$$
Fix $\epsilon > 0$. Then
\[
\sup_{z,w \in \mathbb{D}, z \neq w} \left| \varphi^{n_k}(w) - \varphi^{n_k}(z) \right| n_k(w - z) < \epsilon,
\]
for some $n_k$. In particular,
\[
\frac{1}{n_k} \sup_{z \in \mathbb{D}, z \neq c} \left| \varphi^{n_k}(z) - \varphi^{n_k}(c) \right| < \epsilon,
\]
for some $n_k$. We note that, the self-map $\varphi$ is in $\text{Lip}_A(\mathbb{D})$, since $\text{Lip}_A(\mathbb{D})$ contains the identity map $z$ on $\mathbb{D}$. If $\varphi$ is non-constant, then $\varphi \in \text{Lip}_A(\mathbb{D})$ satisfies the hypotheses of Corollary 3.2, so $\varphi$ has a nonzero angular derivative at $c$. However, by (1)
\[
\frac{1}{n_k} \lim_{z \to c} \left| \varphi^{n_k}(z) - \varphi^{n_k}(c) \right| < \epsilon,
\]
where $\Gamma$ is a triangle in $\mathbb{D}$ that has one of its vertices at $c$. This means that the angular derivative of $\varphi$ at $c$ is zero and this is a contradiction. $\square$

We conjecture that the same conditions are necessary for the compactness of an endomorphism $T$ of $\text{Lip}_A(\mathbb{D}, \alpha)$, $0 < \alpha < 1$.

4. The differentiable Lipschitz algebras of order $\alpha$ ($0 < \alpha \leq 1$)

A complex-valued function $f$ on $\mathbb{D}$ is differentiable on $\mathbb{D}$ if at each point $a \in \mathbb{D},$
\[
f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}
\]
exists. Note that, every differentiable function on $\mathbb{D}$ is analytic on $\mathbb{D}$.

Let the $n$-times differentiable Lipschitz algebra $\text{Lip}^n(\mathbb{D}, \alpha)$ of order $\alpha$ ($0 < \alpha \leq 1$), be the algebra of all complex-valued functions $f$ on $\mathbb{D}$ whose derivatives up to order $n$ exist and for each $k$ ($0 \leq k \leq n$), $f^{(k)}$ satisfy the Lipschitz condition $p_\alpha(f^{(k)}) = \ldots$
compact endomorphisms of certain subalgebras of \( A(D) \)

\[
\sup \left\{ \frac{|f^{(k)}(z)-f^{(k)}(w)|}{|z-w|^{\alpha}} \right\} : z, w \in \mathbb{D}, z \neq w \} < \infty. \quad \text{These algebras are Banach function algebras on } \mathbb{D}, \text{ if they are equipped with the norm}
\]

\[
\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_{\mathbb{D}} + p_{\alpha}(f^{(k)})}{k!}, \quad (f \in Lip^n(\mathbb{D}, \alpha)).
\]

These algebras were introduced in [3], and it was shown that they are natural Banach function algebras. By Proposition 1.1, every self-map \( \varphi : \mathbb{D} \to \mathbb{D} \) in \( Lip^n(\mathbb{D}, \alpha) \) with \( \|\varphi\|_{\mathbb{D}} < 1 \) induces an endomorphism of \( Lip^n(\mathbb{D}, \alpha) \). However, for these algebras this is true without extra condition on \( \varphi \), that is

**Theorem 4.1.** Every map \( \varphi : \mathbb{D} \to \mathbb{D} \) in \( Lip^n(\mathbb{D}, \alpha) \) induces an endomorphism of \( Lip^n(\mathbb{D}, \alpha) \).

**Proof.** We carry out the proof for the case \( n = 1 \); the case \( n > 1 \) is similar. It is enough to show that \( f \circ \varphi \in Lip^1(\mathbb{D}, \alpha) \) for every \( f \in Lip^1(\mathbb{D}, \alpha) \). For every \( z, w \in \mathbb{D} \) with \( \varphi(z) \neq \varphi(w) \) \((\text{so } z \neq w)\) and for every \( f \in Lip^1(\mathbb{D}, \alpha) \) we have

\[
\frac{|f \circ \varphi(z) - f \circ \varphi(w)|}{|z-w|^\alpha} \leq \|f'\|_{\mathbb{D}} \frac{|\varphi(z) - \varphi(w)|}{|z-w|^\alpha} \leq \|f'\|_{\mathbb{D}} p_{\alpha}(\varphi)
\]

and

\[
\frac{|(f \circ \varphi)'(z) - (f \circ \varphi)'(w)|}{|z-w|^\alpha} \leq \frac{|f'(\varphi(z))\varphi'(z) - f'(\varphi(w))\varphi'(w)|}{|\varphi(z) - \varphi(w)|^\alpha} \frac{|\varphi(z) - \varphi(w)|}{z-w} \]

\[
+ \frac{|\varphi'(z) - \varphi'(w)|}{|z-w|^\alpha}|f'(\varphi(w))| \leq p_{\alpha}(f')|\varphi'|^{1+\alpha} + p_{\alpha}(\varphi')\|f'\|_{\mathbb{D}}.
\]

This implies that \( f \circ \varphi \in Lip^1(\mathbb{D}, \alpha) \). \( \square \)

We now consider sufficient conditions that such \( \varphi \) induces compact endomorphisms of \( Lip^n(X, \alpha) \), and we show that in the case \( \alpha = 1 \), these conditions are also necessary. For this we need the following lemma due to Julia [2, Part Six].
Lemma 4.2. Let $f$ be a continuously differentiable complex-valued function on the closed unit disc $\overline{D}$. If $f$ is non-constant and $f(1) = 1 = \|f\|_{D}$, then $f'(1)$ is a strictly positive number.

Theorem 4.3. Let $T$ be a nonzero endomorphism of Lip$^1(D, \alpha)$, $0 < \alpha \leq 1$, induced by a map $\varphi : \overline{D} \to \overline{D}$. Then $T$ is compact, if $\varphi$ is either constant or $\|\varphi\|_{D} < 1$. In the case that $\alpha = 1$, these conditions are necessary.

Proof. When $\varphi$ is constant, it is clear. Let $\|\varphi\|_{D} < 1$. For the compactness of $T$, we assume that $\{f_{n}\}$ is a bounded sequence in Lip$^1(D, \alpha)$ with $\|f_{n}\| = \|f_{n}\|_{D} + \|f'_{n}\|_{D} + p_{\alpha}(f_{n}) + p_{\alpha}(f'_{n}) \leq 1$. Then $\{f_{n}\}$ is a bounded sequence of analytic functions on $\overline{D}$. So by Montel Theorem $\{f_{n}\}$ has a subsequence $\{f_{n_{k}}\}$ such that $\{f_{n_{k}}\}$ and its derivatives $\{f'_{n_{k}}\}$ of each order are uniformly convergent in every compact subset of $\overline{D}$. We denote this subsequence by $\{f_{n}\}$ again. We want to show that $\{f_{n} \circ \varphi\}$ is convergent in Lip$^1(D, \alpha)$. Set $\Delta_{r} = \{z : |z| \leq r\}$ where $\|\varphi\|_{D} < r < 1$. So $\Delta_{r}$ is a compact subset of $\overline{D}$ and $\varphi(\Delta_{r}) \subseteq \Delta_{r}$. Clearly, $\|f_{n} \circ \varphi\|_{D} = \|f_{n}\|_{\varphi(\Delta_{r})}, \|f_{n} \circ \varphi\|_{D} \leq \|f'_{n}\|_{\varphi(\Delta_{r})}$ and for all $z, w \in \overline{D}$ with $\varphi(z) \neq \varphi(w)$ (so $z \neq w$) we have

$$\frac{|f_{n} \circ \varphi(z) - f \circ \varphi(w)|}{|z - w|^\alpha} \leq \frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} \|f'_{n}\|_{\Delta_{r}} \leq p_{\alpha}(\varphi) \|f'_{n}\|_{\Delta_{r}},$$

so that $p_{\alpha}(f_{n} \circ \varphi) \leq p_{\alpha}(\varphi)\|f'_{n}\|_{\Delta_{r}}$, and

$$\frac{|f_{n} \circ \varphi'(z) - f_{n} \circ \varphi'(w)|}{|z - w|^\alpha} = \frac{|f'_{n}(\varphi(z))\varphi'(z) - f'_{n}(\varphi(w))\varphi'(w)|}{|z - w|^\alpha} \leq \frac{|f'_{n}(\varphi(z)) - f'_{n}(\varphi(w))|}{|z - w|^\alpha} \|\varphi'(z)\|_{\Delta_{r}} + \frac{|\varphi'(z) - \varphi'(w)|}{|z - w|^\alpha} \|f'_{n}(\varphi(w))\|_{\Delta_{r}} \leq \|f'_{n}\|_{\Delta_{r}} \|\varphi'(z)\|_{D} + \|f'_{n}\|_{\Delta_{r}} p_{\alpha}(\varphi'),$$

so that $p_{\alpha}(f_{n} \circ \varphi') \leq \|f'_{n}\|_{\Delta_{r}} p_{\alpha}(\varphi') \|\varphi'\|_{D} + \|f'_{n}\|_{\Delta_{r}} p_{\alpha}(\varphi')$.

Hence all sequences $\{\|f_{n} \circ \varphi\|_{D}\}, \{\|f_{n} \circ \varphi\|'_{D}\}, \{p_{\alpha}(f_{n} \circ \varphi)\}$ and $\{p_{\alpha}(f_{n} \circ \varphi')\}$ are Cauchy sequences. Thus $\{f_{n} \circ \varphi\}$ is a Cauchy
sequence in \( \text{Lip}^1(\mathbb{D}, \alpha) \), and hence it is convergent, by the completeness of \( \text{Lip}^1(\mathbb{D}, \alpha) \).

Conversely, let \( \alpha = 1, 0 \neq T \) be compact and \( |c| = 1, |\varphi(c)| = 1 \) for some \( c \). Define \( f_n(z) = \frac{z^n}{n(n-1)} \). Then

\[
\|f_n\| = \|f_n\|_\mathbb{D} + \|f_n\|_\mathbb{D} + p_1(f_n) + p_1(f'_n)
\leq \frac{1}{n(n-1)} + \frac{2}{n-1} + 1.
\]

Therefore, \( \{f_n\} \) is a bounded sequence in \( \text{Lip}^1(\mathbb{D}) = \text{Lip}^1(\mathbb{D}, 1) \). By the compactness of \( T \), there exists a subsequence \( \{f_{n_k}\} \) such that \( Tf_{n_k} = f_{n_k} \circ \varphi \) is convergent in \( \text{Lip}^1(\mathbb{D}) \). Since \( f_{n_k} \to 0 \) uniformly on \( \overline{\mathbb{D}} \), \( f_{n_k} \circ \varphi \to 0 \) in \( \text{Lip}^1(\mathbb{D}) \). Thus

\[
p_1((f_{n_k} \circ \varphi)'(z)) = \sup_{z, w \in \overline{\mathbb{D}}, z \neq w} \frac{|\varphi^{n_k-1}(z)\varphi'(z) - \varphi^{n_k-1}(w)\varphi'(w)|}{(n_k-1)|z-w|}
\leq \frac{1}{n_k-1} \sup_{z, w \in \overline{\mathbb{D}}, z \neq w} \frac{|\varphi^{n_k-1}(z) - \varphi^{n_k-1}(w)|}{z-w} \varphi'(z)
\leq \frac{\varphi'(z) - \varphi'(w)}{z-w} \varphi^{n_k-1}(w) \to 0 \quad \text{as} \quad k \to \infty.
\]

Considering

\[
\frac{1}{n_k-1} \sup_{z, w \in \overline{\mathbb{D}}, z \neq w} \frac{|\varphi'(z) - \varphi'(w)|}{z-w} \varphi^{n_k-1}(w) \leq \frac{1}{n_k-1} p_1(\varphi') \to 0
\]

as \( k \to \infty \),

we have

\[
\frac{1}{n_k-1} \sup_{z, w \in \overline{\mathbb{D}}, z \neq w} \frac{|\varphi^{n_k-1}(z) - \varphi^{n_k-1}(w)|}{z-w} \varphi'(z) \to 0 \quad \text{as} \quad k \to \infty.
\]

In particular,

\[
\frac{1}{n_k-1} \sup_{z \in \overline{\mathbb{D}}, z \neq c} \frac{|\varphi^{n_k-1}(z) - \varphi^{n_k-1}(c)|}{z-c} \varphi'(z) \to 0 \quad \text{as} \quad k \to \infty,
\]
and hence
\[
\frac{1}{n_k-1} \sup_{z \in \mathbb{D}, z \neq c} \left| \frac{\varphi^{n_k-1}(z) - \varphi^{n_k-1}(c)}{z - c} \varphi'(z) \right| < \epsilon,
\]
for arbitrary \( \epsilon > 0 \) and some \( n_k \). Then
\[
\frac{1}{n_k-1} \lim_{z \to c} \left| \frac{\varphi^{n_k-1}(z) - \varphi^{n_k-1}(c)}{z - c} \varphi'(z) \right| \leq \epsilon,
\]
so \(|\varphi'(c)|^2 = |(\varphi'(c))^2\varphi^{n_k-2}(c)| \leq \epsilon\), for any \( \epsilon > 0 \). Hence \( \varphi'(c) = 0 \).

On the other hand, \( g = \frac{\varphi}{\varphi(c)} \in Lip^1(\mathbb{D}) \), has continuous complex derivative on \( \mathbb{D} \), and \( g(c) = 1 = \|g\|_B \). Then by Lemma 4.2, the function \( g \) is constant on \( \mathbb{D} \), so \( \varphi \) must be constant. \( \square \)

**Remark 4.4.** Using a similar method, one can conclude Theorem 4.3 for \( Lip^n(\mathbb{D}, \alpha) \) when \( n \geq 1, 0 < \alpha \leq 1 \).

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