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ON THE NOETHERIAN DIMENSION OF ARTINIAN MODULES WITH HOMOGENEOUS UNISERIAL DIMENSION

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Abstract. In this article, we first show that non-Noetherian Artinian uniserial modules over commutative rings, duo rings, finite R-algebras and right Noetherian rings are 1-atomic exactly like $\mathbb{Z}_{p^\infty}$. Consequently, we show that if R is a right duo (or, a right Noetherian) ring, then the Noetherian dimension of an Artinian module with homogeneous uniserial dimension is less than or equal to 1. In particular, if A is a quotient finite dimensional R-module with homogeneous uniserial dimension, where R is a locally Noetherian (or, a Noetherian duo) ring, then $n\text{-}\dim A \leq 1$. We also show that the Krull dimension of Noetherian modules is bounded by the uniserial dimension of these modules. Moreover, we introduce the concept of qu-uniserial modules and by using this concept, we observe that if A is an Artinian R-module, such that any of its submodules is qu-uniserial, where R is a right duo (or, a right Noetherian) ring, then $n\text{-}\dim A \leq 1$.
Keywords: Noetherian dimension, homogeneous uniserial dimension, atomic modules.
MSC(2010): Primary: 16P60; Secondary: 16P20, 16P40.

1. Introduction

Recently, M. Davoudian and O.A.S. Karamzadeh in [9], have introduced and studied the concept of the dual perfect dimension of an R-module A. This dimension is defined to be the codeviation of the poset of the finitely generated submodules of A. It is well-known and trivial to see that an R-module A satisfies the ascending chain condition on finitely generated submodules if and only if A is Noetherian. This motivated, Davoudian and Karamzadeh to see if this can be extended to the case of modules with Noetherian dimension. They succeeded to show that for Artinian serial module A, the Noetherian dimension and the dual perfect dimension coincide, i.e., $n\text{-}\dim A = dp\text{-}\dim A$, see [9,

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Theorem 4.3]. Consequently, they obtained several results. We know that the Noetherian dimension of any Artinian module over a commutative ring is finite, see [26, 21]. But, even Artinian uniserial modules over noncommutative rings may have any ordinal number as their Noetherian dimension, see for example [19, Example 1]. In this article, we first notice that one can offer a more direct proof of the fact in the title of [9]. Then we try to trade off the concept of uniserial module by a more general concept which is recently introduced in [25], namely, uniserial dimension, and again obtain the above fact with a simple proof. Finally, we should remind the reader that although from the [9, Note added in the proof], we can infer that non-Noetherian Artinian uniserial modules over a commutative (or, a right Noetherian) rings have a similar Loewy Series of length ω to that of $\mathbb{Z}_{p^\infty}$. We should emphasize here that these modules are 1-atomic. In particular, we prove that the Noetherian dimension of an Artinian module with homogeneous uniserial dimension over the previous rings, is less than or equal to 1. It is also shown that for Artinian module A with homogeneous uniserial dimension, over these rings, the Noetherian dimension and the dual perfect dimension of A coincide, i.e., $n\text{-dim}\, A = dp\text{-dim}\, A$. We also show that if A is a quotient finite dimensional(briefly, qfd, i.e., for any submodule B of A, $\frac{A}{B}$ has finite Goldie dimension) R-module with homogeneous uniserial dimension, where R is either locally Noetherian or Noetherian duo, then $n\text{-dim}\, A \leq 1$. We shall call a proper submodule B of A, qu-uniserial if $\frac{A}{B}$ has uniserial dimension, and an R-module A to be qu-uniserial, if any of its proper submodules is contained in a qu-uniserial submodule, equivalently, for each proper submodule B of A, there exists a proper submodule C of A, such that $B \subseteq C$ and $\frac{A}{C}$ is a uniserial module. Using this concept, we observe that if A is an Artinian R-module, such that any of its submodules is qu-uniserial, where R is a right duo (or, a right Noetherian) ring, then $n\text{-dim}\, A \leq 1$. Finally, we show that if A is a Noetherian module, then $k\text{-dim}\, A \leq u.s.\text{dim}\, A$.

If A is an R-module, by $n\text{-dim}\, A$, $k\text{-dim}\, A$ and $dp\text{-dim}\, A$ we mean the Noetherian dimension, the Krull dimension and the dual perfect dimension of A over R, respectively. It is convenient that, when we are dealing with the latter dimensions, to begin our list of ordinals with -1. If an R-module A has Noetherian dimension and α is an ordinal number, then A is called α-atomic if $n\text{-dim}\, A = \alpha$ and $n\text{-dim}\, B < \alpha$ for all proper submodules B of A. An R-module A is called atomic if A is α-atomic for some ordinal α, see [18] (note, atomic modules are also called conotable, dual critical and N-critical in some other articles, see for example [23, 21, 5] and [7]). We also say that an R-module A is called α-critical if $k\text{-dim}\, A = \alpha$ and $k\text{-dim}\, \frac{A}{B} < \alpha$, for all nonzero submodules B of A. We note that if A is an Artinian module with $n\text{-dim}\, A = \alpha$, then for any ordinal $\beta \leq \alpha$, there exists a β-atomic submodule B of A, for it suffices to take B to be minimal with respect to the property

that n-dim $B \geq \alpha$, see [18] and the comment which follows [20, Proposition 1.11]. An R-module A is called uniserial if its submodules are linearly ordered by inclusion and a serial module is a direct sum of uniserial modules.
Throughout this article, all rings are associative with $1 \neq 0$, and all modules are unital right modules. $B \subseteq A$(resp, $B \subset A$) will mean B is a submodule (resp, proper submodule) of A.

2. **Preliminaries**

We need the following result on modules with Noetherian dimension which are just easy dualization of the corresponding result for modules with Krull dimension.

Proposition 2.1 ([18, Proposition 1.1])**.** *An R-module has Noetherian dimension if and only if has Krull dimension.*

Proposition 2.2 ([18, Lemma 1.2])**.** *If B is a submodule of an R-module A, then n-dim $A = \sup\{n\text{-dim}\, B, n\text{-dim}\, \frac{A}{B}\}$ if either side exists.*

Lemma 2.3 ([18, Proposition 1.4])**.** *If A is an R-module and for each submodule B of A, either B or $\frac{A}{B}$ has Noetherian dimension. Then so does M. Moreover, if each proper submodule B of A has Noetherian dimension, then n-dim $A \leq \sup\{n\text{-dim}\, B : B \subset A\} + 1$.*

We also cite the following facts.

Theorem 2.4 ([9, Theorem 4.3])**.** *Let A be an Artinian uniserial R-module. Then n-dim $A = dp$-dim A.*

Lemma 2.5 ([9, Crollary 4.4])**.** *Let A be an Artinian serial R-module. Then n-dim $A = dp$-dim A.*

We have the following definition, see also [25].

Definition 2.6. In order to define uniserial dimension for modules over a ring R , we first define, by transfinite induction, classes ζ_α of R-modules for all ordinals $\alpha \geq 1$. To start with let ζ_1 be the class of non zero uniserial modules. Next consider an ordinal $\alpha > 1$, if ζ_β has been defined for all ordinal $\beta < \alpha$, let ζ_α be the class of those R-modules A such that, for every submodule $B \subset A$, where $\frac{A}{B} \ncong A$, we have $\frac{A}{B} \in \bigcup_{\beta<\alpha} \zeta_\beta$. If an R-module A belongs to some ζ_α, then the least ordinal such α is the uniserial dimension of A, denoted $u.s.$dim A. For $A = 0$ we define $u.s.$dim $A = 0$. If A is non zero and A does not belong to any ζ_α, then we say that $u.s.$dim A is not defined or that A has no uniserial dimension.

Remark 2.7. By the previous definition, if an R-module A has uniserial dimension, then every factor module of A has uniserial dimension. Thus every summand of A has uniserial dimension and if for any submodule $B \subseteq A$, such that $A \ncong \frac{A}{B}$, $u.s.\dim \frac{A}{B}$ is defined, then A has uniserial dimension, in which case if $\alpha = \sup\{u.s.\dim \frac{A}{B} | A \subseteq B, \frac{A}{B} \ncong A\}$, we have $u.s.\dim A \leq \alpha + 1$.

Remark 2.8. Recall that a non-simple module A is called Hopfian if it is not isomorphic to any of its proper factor modules and A is a fully Hopfian if every factor module of A is Hopfian. Also a non-simple module A is said to be anti-Hopfian if A is isomorphic to its proper factor modules.

In the following, we recall some basic properties of modules with uniserial dimension, see also [25].

Lemma 2.9 ([25, Lemma 1.8]). *If A is an R-module and $u.s.\dim A = \alpha$, then for every $0 \leq \beta \leq \alpha$, there exists a factor module $\frac{A}{B}$ of A such that $u.s.\dim \frac{A}{B} = \beta$.*

Proposition 2.10 ([25, Proposition 1.10]). *If A is a fully Hopfian R-module with uniserial dimension and B is a submodule of A, then B has uniserial dimension and $u.s.\dim B \leq u.s.\dim A$.*

Proposition 2.11 ([25, Proposition 1.15]). *If A is an R-module of finite length, then $u.s.\dim A \leq length A$.*

Proposition 2.12 ([25, Proposition 1.3]). *An R-module A has uniserial dimension if and only if for every ascending chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of submodules of A, there exists $n \geq 1$ such that $\frac{A}{A_n}$ is uniserial or $\frac{A}{A_n} \cong \frac{A}{A_k}$ for all $k > n$.*

By the previous proposition, it is clear that every Noetherin module has a uniserial dimension.

Remark 2.13. Since every R-module A with Krulll dimension over commutative Max-ring is Noetherian, see [17, Proposition 4.1]. It implies that over a large class of rings every Artinian module has a uniserial dimension. Moreover over these rings uniserial dimension of Artinian modules are finite, see [13, Corollary 2.5] and Proposition 2.11.

3. Noetherian dimension of Artinian uniserial modules

One of the interesting properties of $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is that: It is Artinian 1-atomic. In [9], it is shown that, the Noetherian dimension of Artinian uniserial modules over commutative rings, right duo rings (i.e., each right ideal is two-sided), right Noetherian rings and finite R-algebras, are less than or equal to 1. In this section, we give a straightforward proof for this fact, which shows that non-Noetherian Artinian uniserial modules over commutative rings, right duo rings and right Noetherian rings are 1-atomic. Also, we conclude that non-Noetherian Artinian serial modules over right duo or right Noetherian rings has Noetherian dimension is equal to 1.

It is well-known that if $A = \sum_{i\in I} A_i$ has a Krull dimension such that $k\text{-dim}\, A_i \leq \alpha$ for all i, then $k\text{-dim}\, A \leq \alpha$. The following is its counterpart for Noetherian dimension of uniserial modules which, is stronger than [9, Proposition 4.6].

Proposition 3.1. *Let $A = \sum_{i\in I} A_i$ be a uniserial R-module such that $n\text{-dim}\, A_i \leq \alpha_i$ and $\alpha = \sup\{\alpha_i : i \in I\}$. Then $n\text{-dim}\, A \leq \alpha + 1$ and if $n\text{-dim}\, A = \alpha + 1$, then A is $\alpha + 1$-atomic.*

Proof. Let B be a proper submodule of A, thus there exists $i \in I$ such that $A_i \nsubseteq B$. Since A is uniserial, so $B \subseteq A_i$. We infer that $n\text{-dim}\, B \leq \alpha_i$ for any proper submodule B of A. Hence $n\text{-dim}\, A \leq \alpha + 1$, by Lemma 2.3. Now, assume that $n\text{-dim}\, A = \alpha + 1$ and B is a proper submodule of A. So there exists $i \in I$ such that $B \subseteq A_i$ and then $n\text{-dim}\, B \leq \alpha_i \leq \alpha < \alpha + 1$. Therefore A is $\alpha + 1$-atomic. □

In view of the previous proposition, the next results are also immediate.

Corollary 3.2. *Let R be a ring with $n\text{-dim}\, R = \alpha$. If A is a uniserial R-module, then $n\text{-dim}\, A \leq \alpha + 1$ and if $n\text{-dim}\, A = \alpha + 1$, then A is $\alpha + 1$-atomic.*

Proof. Let $A = \sum_{i\in I} a_i R$ such that $a_i \in A$, for all $i \in I$. So $n\text{-dim}\, a_i R = n\text{-dim}\, \frac{R}{Ann_r(a_i)} \leq \alpha$ and by in view of the proof previous proposition every proper submodule of A has Noetherian dimension, hence A has Noetherian dimension and $n\text{-dim}\, A \leq \alpha + 1$. Moreover, if $n\text{-dim}\, A = \alpha + 1$, by the proof of previous proposition, A is $\alpha + 1$-atomic. □

In [20], it is shown that every submodule of modules with countable Noetherian dimension is countably generated. Thus, the following result is immediate.

Corollary 3.3. *Let R be a ring whose Noetherian dimension is countable. Then every submodule of a uniserial R-module is countably generated.*

Corollary 3.4. *Let R be a right Noetherian ring. If A is a uniserial R-module, then $n\text{-dim}\, A \leq 1$ and if $n\text{-dim}\, A = 1$, then A is 1-atomic.*

The following is also in [9, Corollary 4.7]. We give a short proof for completeness.

Corollary 3.5. *Let A be an Artinian uniserial R-module over a right duo ring. Then $n\text{-dim}\, A \leq 1$.*

Proof. Put $A = \sum_{i\in I} a_i R$ such that $a_i \in A$, for all $i \in I$. But $\frac{R}{Ann_r(a_i)}$ is an Artinian ring and therefore it is Noetherian. This implies that $a_i R$ as an $\frac{R}{Ann_r(a_i)}$-module is a Noetherian submodule of A. Consequently, $n\text{-dim}\, A \leq 1$, by Proposition 3.1. □

The following result is now immediate. It shows that non-Noetherian Artinian uniserial modules are 1-atomic, it also follows from statement in the " Note added in the proof", in [9].

Theorem 3.6. *Let A be a non-Noetherian Artinian uniserial R-module over a right duo (or, a right Noetherian) ring. Then A is 1-atomic.*

Proof. Let $A = \sum_{i \in I} a_i R$ such that $a_i \in A$, for all $i \in I$. But $a_i R \cong \frac{R}{Ann_r(a_i)}$ is a Noetherian submodule of A. Moreover, the proof of Proposition 3.1, shows that each proper submodule of A is Noetherian, and by Proposition 3.1, A is 1-atomic. □

The following corollary is now immediate.

Corollary 3.7. *Let A be a non-Noetherian Artinian serial module over a right duo (or, a right Noetherian) ring R. Then $n\text{-dim}\, A = 1$.*

Proof. Since $A = \bigoplus_{i=1}^{n} A_i$ (note, A has finite Goldie dimension), where each A_i is an Artinian uniserial R-module. By Theorem 3.6, $n\text{-dim}\, A_i = 1$ for each $i = 1, \ldots, n$. Thus $n\text{-dim}\, A = \sup\{n\text{-dim}\, A_i : i = 1, \ldots, n\} = 1$ and we are done. □

Remark 3.8. We recall that an R-module A is called α-short if for every submodule B of A, either $n\text{-dim}\, B \leq \alpha$ or $n\text{-dim}\, \frac{A}{B} \leq \alpha$ and α is the least ordinal number with this property. We know that if A is α-short module, then either $n\text{-dim}\, A = \alpha$ or $n\text{-dim}\, A = \alpha + 1$, see [10, Proposition 1.12].

From Corollary 3.7 and the previous remark, we obtain the following result, see also [9, Corollary 4.10].

Corollary 3.9. *Let A be an Artinian R-module over a right duo (or, a right Noetherian) ring R, such that for each proper submodule B of A, either B or $\frac{A}{B}$ is a serial module, then $n\text{-dim}\, A \leq 2$.*

Let S be a finite R-algebra (i.e., $S = R + Re_1 + Re_2 + \cdots + Re_n$, $Re_i = e_i R, 1 \leq i \leq n$). It is well known that if A is a S-module with Noetherian dimension, $n\text{-dim}\, A_S = n\text{-dim}\, A_R$, see [23].

Corollary 3.10. *Let R be a commutative ring, and S be a finite R-algebra. If A is an Artinian uniserial R-module, then $n\text{-dim}\, A_S = n\text{-dim}\, A_R \leq 1$ and if A_S is a non-Noetherian module, then A_S is 1-atomic.*

Thus we have the following result, which is also in [9, Corollary 4.14].

Corollary 3.11. *Let R be a commutative ring, and S be a finite R-algebra. If A is an Artinian serial R-module, then $n\text{-dim}\, A_S = n\text{-dim}\, A_R \leq 1$.*

As a consequence of the above result we can easily see that if A is a S-module such that as an R-module is Artinian serial, where S is an $n \times n$

matrices over a right duo ring R, then n-dim $A_S = n$-dim $A_R \leq 1$.

Since every anti-Hopfian module is an Artinian uniserial module, see [14, Proposition 1], thus the following results are now immediate.

Corollary 3.12. *If R is a right duo (or, a right Noetherian) ring and A is a non-Noetherian anti-Hopfian R-module (e.g., $\mathbb{Z}_{p^\infty}$), then A is 1-atomic.*

Corollary 3.13. *Let R be either right duo or right Noetherian. If A is an R-module such that every essential submodule of A is anti-Hopfian, then n-dim $A \leq 2$.*

Proof. It is evident that for each essential submodule E of A, we have n-dim $E \leq 1$, by Corollary 3.12. Moreover each submodule is contained in an essential submodule. In what follows, we observe that n-dim $A \leq 2$, by Lemma 2.3. □

4. Noetherian dimension of Artinian modules with homogeneous uniserial dimension

In [19, Example 3], it is shown that there is a uniserial module A with any Noetherian dimension, i.e., $1 = u.s.\text{dim}\, A < n\text{-dim}\, A$, also $u.s.\text{dim}\, \mathbb{Z} = \omega$, note that $u.s.\text{dim}\, \mathbb{Z}_n < n$, by Proposition 2.11, for all n, let P_n is the first prime number greater than n and $A = \frac{\mathbb{Z}}{(2\times 3\times\cdots\times P_n)\mathbb{Z}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_{P_n}$, which is $u.s.\text{dim}\, A \geq n$. That is there exists quotient module of $\mathbb{Z}$ with uniserail dimension greater than or equal to n, for each positive integer number n. In what follows, we observe that $u.s.\text{dim}\, \mathbb{Z} = \omega$. Therefore $0 = n\text{-dim}\, \mathbb{Z} < u.s.\text{dim}\, \mathbb{Z} = \omega$. Thus, one cannot decide which one is smaller, in general. But we shall briefly, show that the Noetherian dimension of every Artinian module, with homogeneous uniserial dimension, over a right duo (or, a right Noetherian) ring, is less than or equal to 1.

We introduce the next definition.

Definition 4.1. A non-simple module A is called weakly Hopfian (briefly, wh), if for any simple submodule S of A, $A \ncong \frac{A}{S}$, and A is a weakly fully Hopfian (briefly, wfh), if every factor module of A is weakly Hopfian.

Lemma 4.2. *Let A be an Artinian wfh R-module with uniserial dimension. Then for all submodules B of A, there exists submodule D of B, which $\frac{B}{D}$ has uniserial dimension.*

Proof. The proof is by transfinite induction on $u.s.\text{dim}\, A = \alpha$. The case $\alpha = 1$ is clear. Now assume that $\alpha > 1$ and the result is true for all ordinals $\gamma < \alpha$. For any submodule B of A, if B has uniserial dimension, we are through. If B has no uniserial dimension, then B is not simple and contains a simple submoule,

C say. Since $\frac{A}{C} \ncong A$, we infer that $u.s.\dim \frac{A}{C} = \gamma < \alpha$. Finally, by induction hypothesis $\frac{B}{C}$ has a submodule, $\frac{D}{C}$ say, such that $\frac{B}{D}$ has uniserial dimension and we are done. □

Lemma 4.3. *Let R be a right duo (or, a right Noetherian) ring and let A be an Artinian wfh R-module with uniserial dimension. Then n-$\dim A \leq 1$.*

Proof. Let n-$\dim A = \alpha$. By what we have already mentioned in the introduction, there is an α-atomic submodule B of A. By the previous proposition, there exists submodule D of B, such that $\frac{B}{D}$ has uniserial dimension. Note that, by Lemma 2.9, $\frac{B}{D}$ has a submodule $\frac{C}{D}$ such that $\frac{B}{C}$ is a uniserial module. Thus $\alpha = n$-$\dim B = n$-$\dim \frac{B}{C} \leq 1$, by Theorem 3.6 and hence n-$\dim A \leq 1$. □

Since every fully Hopfian module is wfh, we immediately obtain the following result.

Corollary 4.4. *Let A be an Artinian fully Hopfian R-module with uniserial dimension, where R is either right duo or right Noetherian, then n-$\dim A \leq 1$.*

The following is more generalized than Theorem 3.6.

Proposition 4.5. *Let R be a right duo (or, a right Noetherian) ring and let A be a non-Noetherian Artinian module. Then the following statements hold;*

(1) *If A is a uniserial module, then A is 1-atomic.*
(2) *If A is an α-atomic module whose uniserial dimension is greater than 1, then $\alpha < u.s.\dim A$.*
(3) *If A is an α-atomic module with uniserial dimension, then A is 1-atomic.*

Proof. (1) It is clear, by Theorem 3.6.
(2) The proof is by transfinite induction on β, where $u.s.\dim A = \beta$. If $\beta = 2$, then there exists a proper submodule B of A such that $\frac{A}{B} \ncong A$ (for otherwise A is uniserial), so $u.s.\dim \frac{A}{B} < 2$ implies that $\frac{A}{B}$ is an Artinian uniserial module. Thus n-$\dim A = n$-$\dim \frac{A}{B} \leq 1 < 2 = \beta$, by [18]. Inductively we assume that the claim is valid for $\gamma < \beta$. Since A is not uniserial module, thus there exists a proper submodule B of A such that $\frac{A}{B} \ncong A$, therefore $u.s.\dim \frac{A}{B} < u.s.\dim A = \beta$. So By induction hypothesis n-$\dim \frac{A}{B} < u.s.\dim \frac{A}{B} < \beta$. Therefore n-$\dim A = n$-$\dim \frac{A}{B} < \beta$. That is $\alpha < \beta$.
(3) let A be an Artinian α-atomic module and $u.s.\dim A = \beta$. If $\beta = 1$, we are through. Now we suppose that $u.s.\dim A = \beta > 1$. So there exists a submodule B of A such that $u.s.\dim \frac{A}{B} = 1$, by Lemma 2.9. This shows that $\frac{A}{B}$ is an Artinian uniserial module and hence $\frac{A}{B}$ is 1-atomic, by Theorem 3.6. Thus A is 1-atomic. □

We know that every submodule of a uniserial module is uniserial. So it is natural to give the following definition.

Definition 4.6. An R-module A has homogeneous uniserial dimension, if every submodule of A has uniserial dimension.

Example 4.7. Every uniserial module has homogeneous uniserial dimension. Moreover, by Proposition 2.10, every fully Hopfian module with uniserial dimension, has homogeneous uniserial dimension.

The next lemma is also needed, see [23, 18, 7, 3].

Lemma 4.8. *Let A be an R-module with Noetherian dimension. Then* $n\text{-}\dim A = \sup\{n\text{-}\dim \frac{B}{C} : C \subseteq B$ *are submodules of A and* $\frac{B}{C}$ *is atomic*$\}$

We may now present the following fact, whose proof, in view of Proposition 4.5, and the previous lemma, is straightforward and so it is omitted. We should emphasize that this is a generalization of Theorem 3.6.

Proposition 4.9. *Let A be an Artinian R-module, with homogeneous uniserial dimension, where R is either right duo or right Noetherian , then* $n\text{-}\dim A \leq 1$.

Remark 4.10. Let $R = F[x_1, x_2, \dots, x_n]$, where F is a field. So R is a commutative Noetherian ring and every maximal ideal M of R, has rank exactly n (i.e., there exists a chain of prime ideals of length n desending from M, but no longer chain). Let $A = E(S)$ be the injective envelope of a simple R-module S. By [27, Theorem 2], A is an Artinian module and by [7, Poroposion 5], $n\text{-}\dim A = Rank(M)$, where M is a maximal ideal of R such that $S \cong \frac{R}{M}$. Thus there exists Artinian modules with Noetherian dimension of any natural number, see also [9]. Therefore, by taking A to be an Artinian module over R with $n\text{-}\dim A > 1$, we infer that A has no uniserial dimension.

If in Theorem 2.4, we trade off general non-commutative rings, by either right duo or right Noetherian rings, then below, we may extend this theorem to modules with homogeneous uniserial dimension.

Corollary 4.11. *Let A be an Artinian R-module with homogeneous uniserial dimension, where R is either right duo or right Noetherian. Then* $n\text{-}\dim A = dp\text{-}\dim A$.

Proof. Let $dp\text{-}\dim A = \alpha$. If $\alpha = 0$, then clearly $n\text{-}\dim A = 0$. We assume that $\alpha > 0$, i.e., A is not Noetherian. Since $0 < \alpha = dp\text{-}\dim A \leq n\text{-}\dim A \leq 1$, therefore $\alpha = dp\text{-}\dim A = 1$. This completes the proof. □

Corollary 4.12. *Let A be an Artinian R-module, which is a direct sum of modules with homogeneous uniserial dimension, where R is either right duo or right Noetherian. Then* $n\text{-}\dim A = dp\text{-}\dim A$.

We conclude the following result, which is also [9, Theorem 4.3].

Corollary 4.13. *Let A be an Artinian serial R-module, where R is either right duo or right Noetherian. Then* $n\text{-}\dim A = dp\text{-}\dim A$.

The following proposition, which is in [23, Theorem 2.4], [1, Proposition 2.2] and [9, Proposition 1.1], is needed.

Proposition 4.14. *The following statement are equivalent for any R-module A and any ordinal $\alpha \geq 0$:*

(1) *A is a qfd R-module and for any $N \subset P \subseteq A$, there exists X with $N \subseteq X \subset P$ with $n\text{-dim}\, \frac{P}{X} \leq \alpha$.*
(2) *$n\text{-dim}\, A \leq \alpha$.*

A module A is called finitely embedded (f.e.), if A has a finitely generated essential socle. It is easy to see that every module has a f.e. factor module, see [20, Comments, preceding Lemma 1.1]. It is also well-known that for a commutative ring R, every finitely embedded module is Artinian if and only if R is a locally Noetherian ring (i.e., R_M is Noetherian for every maximal ideal M of R), see [27, Theorem 2]. Moreover if R is a Noetherian duo ring, then an R-module A is f.e., if and only if is Artinian, see [13, Theorem 2.4]. In view of this comment and Proposition 4.14, we have the following result, which is a generalization of Proposition 4.9.

Theorem 4.15. *Let A be a qfd R-module with homogeneous uniserial dimension, where R is a locally Noetherian (or, a Noetherian right duo) ring. Then $n\text{-dim}\, A \leq 1$.*

Proof. It suffices to prove that A satisfies part (1) of the previous proposition. For each $N \subset P \subseteq A$, $B = \frac{P}{N}$ has a nonzero quotient module which is finitely embedded, $\frac{B}{C}$ say. But, $\frac{B}{C}$ is an Artinian module with homogeneous uniserial dimension, by the previous comment. In view of Proposition 4.9, $n\text{-dim}\, \frac{B}{C} \leq 1$. Consequently, $n\text{-dim}\, A \leq 1$, by Proposition 4.14 and Remark 2.7. □

Proiposition 2.10, and the previous theorem, immediately yield the following result.

Corollary 4.16. *Let A be fully Hopfian and qfd with uniserial dimension over a locally Noetherian (or, a Noetherian duo) ring R. Then $n\text{-dim}\, A \leq 1$.*

We know that every Noetherian module has both uniserial dimension and Krull dimension. In what follows, we show that the latter dimension is bounded by the former dimension for Noetherian modules.

Theorem 4.17. *Let A be a Noetherian R-module, then $k\text{-dim}\, A \leq u.s.\text{dim}\, A$. Moreover, if $k\text{-dim}\, A = u.s.\text{dim}\, A > 1$, then A is α-critical, where $\alpha = u.s.\text{dim}\, A$.*

Proof. The proof is by transfinite induction on $k\text{-dim}\, A = \gamma$. The case $\gamma = 0$ is clear. Assume that $\gamma > 0$ and the result is true for all ordinals $\beta < \gamma$. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a descending chain of submodules of A, there exists $n \geq 1$, such that $k\text{-dim}\, \frac{A_k}{A_{k+1}} < \gamma$, for all $k \geq n$. By induction hypothesis

k-dim $\frac{A_k}{A_{k+1}} \leq u.s.$dim $\frac{A_k}{A_{k+1}} < u.s.$dim A. Therefore $\gamma \leq \alpha$ and we are done. Now if k-dim $A = u.s.$dim $A = \alpha > 1$, then for each submodule B of A, $\frac{A}{B} \ncong A$ and so $u.s.$dim $\frac{A}{B} < u.s.$dim $A = k$-dim A (note, every Noetherian module is fully Hopfian). Consequently, k-dim $\frac{A}{B} \leq u.s.$dim $\frac{A}{B} < k$-dim A, that is A is α-critical. □

Corollary 4.18. *Let A be a Noetherian uniserial module, then k-dim $A \leq 1$.*

5. qu-uniserial modules

In this section we introduce the concept qu-uniserial modules and extend some of basic results of Artinian modules with uniserial dimension to Artinian qu-uniserial modules.

Next, we give our definition of qu-uniserial modules.

Definition 5.1. Let A be an R-module. A proper submodule B of A is called a qu-uniserial and denoted by $B \overset{qu}{\subsetneq} A$ if $\frac{A}{B}$ has uniserial dimension.

Also, an R-module A is defined to be qu-uniserial if any of its submodules is contained in a qu-uniserial submodule. Equivalently, for each proper submodule B of A, there exists a proper submodule C of A, such that $B \subseteq C$ and $\frac{A}{C}$ is a uniserial module, by Lemma 2.9. In other words, every nonzero quotient module of A has a proper qu-uniserial submodule.

Remark 5.2. It is clear that if A is a module with uniserial dimension, then A is a qu-uniserial module, by Remark 2.7. We recall that a ring R is a Max-ring if and only if , every nonzero R-module has a Maximal submodule or equivalently, every R-submodule of a nonzero module A over a Max-ring R is contained in a maximal submodule of A.

Hence, every module over a Max-ring R, is qu-uniserial. In particular every ring R with identity as an R-module, is qu-uniserial.

Lemma 5.3. *Let A be a qu-uniserial module. Then every factor module of A is a qu-uniserial module.*

Proof. Let $\frac{A}{B}$ be a nonzero quotient module of A and $\frac{C}{B}$ be a proper submodule of $\frac{A}{B}$, so C is a proper submodule of A. Thus there exists a qu-uniserial submodule D of A such that $C \subseteq D$ and $\frac{A}{D} \cong \frac{A/B}{D/B}$ has uniserial dimension. Hence $\frac{D}{B}$ is a qu-uniserial submodule of $\frac{A}{B}$ and $\frac{C}{B} \subseteq \frac{D}{B}$, so $\frac{A}{B}$ is qu-uniserial. □

Lemma 5.4. *Let B be a proper submodule of a module A. If B and $\frac{A}{B}$ are qu-uniserial modules, then A is a qu-uniserial module.*

Proof. Let C be a proper submodule of A. If $C + B = A$, then $C \cap B \subsetneq B$. So there exists a qu-uniserial submodule D of B such that $C \cap B \subseteq D \overset{qu}{\subsetneq} B$. But $\frac{A}{C+D} = \frac{C+B}{C+D} \cong \frac{B}{D}$, so $C + D \overset{qu}{\subsetneq} A$ and $C \subseteq C + D$. Now if $C + B \subsetneq A$, $\frac{C+B}{B}$

may be zero. Then $\frac{C+B}{B} \subsetneq \frac{A}{B}$, so there exists a qu-uniserial submodule $\frac{D}{B}$ of $\frac{A}{B}$ such that $\frac{C+B}{B} \subseteq \frac{D}{B} \overset{qu}{\subsetneq} \frac{A}{B}$. So $C \subseteq C+B \subseteq D \overset{qu}{\subsetneq} A$. Thus A is a qu-uniserial module. □

In view of Lemma 5.3 and Lemma 5.4, we conclude the following results.

Corollary 5.5. *Let $0 \to C \to A \to B \to 0$ be an exact sequence of R-modules. Then:*

(1) *Let A be a qu-uniserial module. Then B is a qu-uniserialmodule.*
(2) *If C and B are qu-uniserial modules, then A is a qu-uniserial module.*

Corollary 5.6. *Let $A = \sum_{i=1}^{n} \oplus A_i$. If A_i is a qu-uniserial module, for each i, then A is a qu-uniserial module.*

Theorem 5.7. *Let B be a small submodule of A. Then $\frac{A}{B}$ is a qu-uniserial module if and only if A is so.*

Proof. We suppose that $\frac{A}{B}$ is a qu-uniserial module. Let K be a proper submodule of A. Since B is a small submodule of A, we infer that $K+B$ is a proper submodule of A. Hence there exists a qu-uniserial submodule $\frac{X}{B}$ of $\frac{A}{B}$ such that $\frac{K+B}{B} \subseteq \frac{X}{B}$. Thus X is a qu-uniserial submodule of A such that $K \subseteq X$. This shows that A is qu-uniserial and we are done. Conversely, let A be a qu-uniserial module. By Lemma 5.3, we infer that $\frac{A}{B}$ is a qu-uniserial module. □

Theorem 5.8. *Let A be an Artinian R-module, such that any of its submodule is qu-uniserial, where R is a right duo (or, a right Noetherian) ring. Then n-$\dim A \leq 1$.*

Proof. It suffices to prove that A satisfies part (1) of the proposition 4.14. For each $N \subset P \subseteq A$, $B = \frac{P}{N}$ has a nonzero quotient module, say $\frac{B}{C}$, which is an Artinian with uniserial dimension, by Remark 2.7. In view of Proposition 4.9, n-$\dim \frac{B}{C} \leq 1$. Consequently, n-$\dim A \leq 1$, by Proposition 4.14. □

Corollary 5.9. *Let R be either right duo ring or right Noetherian ring and A be an Artinian R-module. If there exists a small submodule B of A such that every submodule of $\frac{A}{B}$ is a qu-uniserial module. Then n-$\dim A \leq 1$.*

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References

[1] T. Albu and S. Rizvi, Chain condition on quotient finite dimensional modules, *Comm. Algebra* **29** (2001), no. 5, 1909–1928.

[2] T. Albu and P.F. Smith, Localization of modular lattices, Krull dimension, and the Hopkins-Levitzki theorem (I), *Math. Proc. Cambridge Philos. Soc.* **120** (1996), no. 1, 87–101.

[3] T. Albu and P.F. Smith, Dual Krull dimension and duality, *Rocky Mountain J. Math.* **29** (1999), no. 4, 1153–1165.

[4] T. Albu and L. Teply, Generalized deviation of posets and modular lattices, *Discrete Math.* **214** (2000), no. 1-3, 1–19.

[5] T. Albu and P. Vamos, Global Krull dimension and global dual Krull dimension of valuation rings, in: Abelian Groups, Module Theory, and Topology (Padua, 1997), pp. 37–54, Lecture Notes in Pure and Appl. Math. 201, Dekker, New York, 1998.

[6] F.W. Anderson and K.R. Fuller, Rings and categories of modules, Grad. Texts Math., 13 , Springer, Berlin, 1992.

[7] L. Chambless, N-Dimension and N-critical modules. Application to Artinian modules, *Comm. Algebra.* **8** (1980), no. 16, 1561–1592.

[8] M. Davoudian, A. Halali and N. Shirali, On α-almost Artinian modules, *Open Math.* **14** (2016) 404–413.

[9] M. Davoudian and O.A.S. Karamzadeh, Artinian serial modules over commutative (or, left Noetherian) rings are at most one step away from being Noetherian, *Comm. Algebra* **44** (2016), no. 9, 3907–3917.

[10] M. Davoudian, O.A.S. Karamzadeh and N. Shirali, On α-short modules, *Math. Scand.* **114** (2014), no. 1, 26–37.

[11] A. Facchini, Loewy and Artinian modules over Commutative rings, *Ann. Mat. Pura Appl. (4)* **128** (1980), 359–374.

[12] R. Gordon and J.C. Robson, Krull dimension, Mem. Amer. Math. Soc. 133, Amer. Math. Soc. Providence, RI, 1973.

[13] J. Hashemi, O.A.S. Karamzadeh and N. Shirali, Rings over which the Krull dimension and the Noetherian dimension of all modules coincide, *Comm. Algebra* **37** (2009), no. 2, 650–662.

[14] Y. Hirano and I. Mogami, Modules whose proper submodules are non-hopf kernels, *Comm. Algebra* **15** (1987), no. 8, 1549–1567.

[15] O.A.S. Karamzadeh, Noetherian-dimension, PhD. Thesis, Exeter, 1974.

[16] O.A.S. Karamzadeh and M. Motamedi, On α-$DICC$ modules, *Comm. Algebra* **22** (1994), no. 6, 1933–1944.

[17] O.A.S. Karamzadeh and M. Motamedi, a-Noetherian and Artinian modules, *Comm. Algebra* **23** (1995), no. 10, 3685–3703.

[18] O.A.S. Karamzadeh and A.R. Sajedinejad, Atomic modules, *Comm. Algebra* **29** (2001), no. 7, 2757–2773.

[19] O.A.S. Karamzadeh and A.R. Sajedi Nejad, On the Loewy length and the Noetherian dimension of Artinian modules, *Comm. Algebra.* **30** (2002), no. 3, 1077–1084.

[20] O.A.S. Karamzadeh and N. Shirali, On the countability of Noetherian dimension of modules, *Comm. Algebra* **32** (2004), no. 10, 4073–4083.

[21] D. Kirby, Dimension and length for Artinian modules, *Quart. J. Math. Oxford Ser. (2)* **41** (1990), no. 164, 419–429.

[22] G. Krause, On the Krull-dimension of left Noetherian rings, *J. Algebra* **23** (1972) 88-99.

[23] B. Lemonnier, Déviation des ensembles et groupes abéliens totalement ordonnés, *Bull. Sci. Math. (2)* **96** (1972) 289–303.

[24] J.C. McConell and J.C. Robson, Noncommutative Noetherian Rings, With the cooperation of L.W. Small, Wiley Ser. Pure Appl. Math. Wiley, Chichester, New York, 1987.

[25] Z. Nazemian, A. Ghorbani and M. Behboodi, Uniserial dimension of modules, *J. Algebra* **399** (2014) 894–903.

[26] R.N. Roberts, Krull-dimension for Artinian modules over quasi local commutative Rings, *Quart. J. Math. Oxford Ser. (2)* **26** (1975), no. 103, 269–273.

[27] P. Vamos, The dual of the notion of finitely generated, *J. Lond. Math. Soc.* **43** (1968) 643–646.

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