WEAK COMPACTNESS OF THE SET OF 
$\varepsilon$-EXTENSIONS

SH. REZAPOUR

Abstract. Let $W$ be a subspace of a normed space $X$, $\varepsilon > 0$ be given and let $f \in W^*$. Every extension of $f$ (in the sense of Hahn-Banach Theorem) is a $\varepsilon$-extension of $f$. We determine when the set of all $\varepsilon$-extensions of every $f \in W^*$ is weakly compact. Finally by using of $\varepsilon$-extensions, we characterize weak*-closed $\varepsilon$-weakly Chebyshev subspaces of $X^*$.

1. Introduction

Let $X$ be a (complex or real) normed space, $\varepsilon > 0$ be given and let $W$ be a subspace of $X$. A point $y_0 \in W$ is said to be a $\varepsilon$-approximation for $x \in X$ if $\|x - y_0\| \leq d(x, W) + \varepsilon$. For $x \in X$, put

$$P_{W, \varepsilon}(x) = \{y \in W : \|x - y\| \leq d(x, W) + \varepsilon\}.$$ 

It is clear that $P_{W, \varepsilon}(x)$ is a non-empty, bounded and convex subset of $X$. Also, $P_{W, \varepsilon}(x)$ is closed for all $x \in X$, if $W$ is closed.

Recently, the author has defined $\varepsilon$-quasi Chebyshev and $\varepsilon$-weakly Chebyshev subspaces of a normed space (see [8,9]). A subspace $W$ of a normed space $X$ is called $\varepsilon$-quasi Chebyshev ($\varepsilon$-Weakly Chebyshev) if $P_{W, \varepsilon}(x)$ is compact (weakly compact) for all $x \in X$. Also,
some types of proximinality are investigated (see [2]-[7]).

Let $W$ be a subspace of a normed space $X$, $\varepsilon > 0$ be given and let $f \in W^*$. A linear functional $g \in X^*$ is called $\varepsilon$-extension of $f$ if $g|_W = f$ and $\|g\| - \varepsilon \leq \|f\| \leq \|g\|$. We denote the set of all $\varepsilon$-extensions of $f$ by $E_\varepsilon(f)$. Note that $E_\varepsilon(f)$ is a non-empty, closed and convex subset of $X^*$.

Recall that if $W$ is a subspace of a normed space $X$, then $W^\perp = \{f \in X^* : f(w) = 0 \text{ for all } w \in W\}$. Also if $M$ is a subspace of $X^*$, then $\frac{1}{2} M = \{x \in X : f(x) = 0 \text{ for all } f \in M\}$ and $\|f\|_{M^\perp} = \sup\{\|\varphi(f)\| : \|\varphi\| \leq 1, \varphi \in M^\perp\}$, for all $f \in X^*$. Finally, $\overline{x}$ stands for the canonical image of $x \in X$ in the second dual $X^{**}$.

We conclude this section by a list of known lemmas needed in the proof of the main results.

**Lemma 1.1.** [8; Lemma 2.1]. Let $W$ be a subspace of a normed linear space $X$, $x \in X$, $y_0 \in W$ and $\varepsilon > 0$ be given. Then, $y_0 \in P_{W,\varepsilon}(x)$ if and only if $\|x - y_0\| \leq \|x - y_0\|_{W^\perp} + \varepsilon$, where $\|x - y_0\|_{W^\perp} = \sup\{\|f(x - y_0)\| : \|f\| \leq 1, f \in W^\perp\}$.

**Lemma 1.2.** [8; Lemma 2.2]. Let $W$ be a closed subspace of a normed linear space $X$, $x \in X$ and $\varepsilon > 0$ be given. Then, $M \subseteq P_{W,\varepsilon}(x)$ if and only if there exists $f \in X^*$ such that $\|f\| = 1$, $f|_W = 0$ and $f(x - y) \geq \|x - y\| - \varepsilon$ for all $y \in M$.

**Lemma 1.3.** [8; Corollary 2.12(a)]. All closed subspaces of a normed space $X$ are $\varepsilon$-quasi Chebyshev in $X$ if and only if $X$ is finite dimensional.
Lemma 1.4. [9; Corollary 2.8(a)]. All closed subspaces of a normed space $X$ are $\varepsilon$-weakly Chebyshev in $X$ if and only if $X$ is reflexive.

Lemma 1.5. [3; Lemma 1.4]. Let $X$ be a normed space and let $W$ be weak$^*$-closed subspace of $X^*$. Then, $\|f\|_{W^*} = \|f|_{W^*}\|_W$ for all $f \in X^*$.

2. Weak compactness of the set of $\varepsilon$-extensions

Now, we are ready to state and prove our results on $\varepsilon$-extensions.

Lemma 2.1. Let $W$ be a subspace of a normed space $X$ and $f, g \in X^*$. Then

$$g \in P_{W^*,\varepsilon}(f) \iff \hat{f} - g \in E_\varepsilon(f|_{W^*}).$$

Proof. Let $g \in P_{W^*,\varepsilon}(f)$. Then, by Lemma 1.2, there exists $\Lambda \in X^{**}$ such that $\|\Lambda\| = 1$, $\Lambda|_{W^*} = 0$ and $\Lambda(f - g) \geq \|f - g\| - \varepsilon$. It follows that, $\|f - g\| - \varepsilon \leq \|(\hat{f} - g)|_{W^*}\| \leq \|f - g\|$. Therefore, $\hat{f} - g$ is an $\varepsilon$-extension of $f|_{W^*}$. Now, let $f - g$ be an $\varepsilon$-extension of $\hat{f}|_{W^*}$. Then, $\hat{g}|_{W^*} = 0$ and hence $g \in W^*$. On the other hand, $\|f - g\| = \|\hat{f} - g\| \leq \|\hat{f}|_{W^*}\| + \varepsilon = \|(\hat{f} - g)|_{W^*}\| + \varepsilon = \|f - g\|_{W^*} + \varepsilon$. Therefore, by Lemma 1.1, $g \in P_{W^*,\varepsilon}(f)$. □

Lemma 2.2. Let $W$ be a subspace of a normed space $X$, $f \in W^*$ and let $\hat{f} \in X^*$ be an extension of $f$. Then

$$E_\varepsilon(\hat{f}|_{W^*}) = \{\hat{g} : g \in E_\varepsilon(f)\}.$$

Proof. Let $g \in X^*$ be such that $\hat{g}$ is a $\varepsilon$-extension of $\hat{f}|_{W^*}$. Then, $\hat{g}(\varphi) = (\hat{f})(\varphi)$, for all $\varphi \in W^*$. Thus,
\[ g(x) = \hat{x}(g) = \hat{g}(\hat{x}) = (\hat{f})(\hat{x}) = \tilde{f}(x) = f(x), \text{ for all } x \in W. \]

Since
\[ \|g\| - \varepsilon = \|\hat{g}\| - \varepsilon \leq \|\hat{f}\|_{W^1} = \|\tilde{f}\| = \|f\| \leq \|\hat{g}\| = \|g\|, \]

\( g \) is a \( \varepsilon \)-extension of \( f \). Now, let \( g \in X^* \) be a \( \varepsilon \)-extension of \( f \). Then, \( g(x) = f(x) = \tilde{f}(x) \), for all \( x \in W \). Since \( \hat{g} \) and \( \hat{f} \) are continuous with respect to the weak-topology on \( X^{**} \) and \( W \) is dense in \( W^{\perp\perp} \) for the weak-topology on \( X^{**} \), we have \( \hat{g}(\varphi) = (\hat{f})(\varphi) \), for all \( \varphi \in W^{\perp\perp} \). Hence, \( \hat{g}\|_{W^1} = (\hat{f})\|_{W^1} \) and
\[ \|\hat{g} - \varepsilon = \|g\| - \varepsilon \leq \|f\| = \|\hat{f}\| = \|\hat{f}\|_{W^1} \leq \|g\| = \|\hat{g}\|. \]

Therefore, \( \hat{g} \) is a \( \varepsilon \)-extension of \( (\hat{f})\|_{W^1} \). \( \square \)

**Theorem 2.3.** Let \( W \) be a subspace of a normed space \( X \) and let \( \varepsilon > 0 \) be given. Then the following are equivalent:

(a) For each \( f \in W^* \), \( E_\varepsilon(f) \) is weakly compact.

(b) For each \( f \in X^* \), the set of all \( g \in X^* \) such that \( \hat{g} \) is a \( \varepsilon \)-extension of \( f\|_{W^1} \) is weakly compact.

(c) \( W^\perp \) is a \( \varepsilon \)-weakly Chebyshev subspace of \( X^* \).

**Proof.** (a)⇒(b). For each \( f \in X^* \), \( f_1 \in W^* \), where \( f_1 = f\|_W \). Let \( \{g_n\}_{n \geq 1} \) be an arbitrary sequence in \( X^* \) such that \( g_n \) is a \( \varepsilon \)-extension of \( f\|_{W^1} \) (\( n = 1, 2, \ldots \)). Then, by Lemma 2.2, \( \{g_n\}_{n \geq 1} \) is a sequence in the set of all \( \varepsilon \)-extensions of \( f_1 \). Therefore, \( \{g_n\}_{n \geq 1} \) has a weakly convergent subsequence, and hence the set of all \( g \in X^* \) such that \( \hat{g} \) is a \( \varepsilon \)-extension of \( f\|_{W^1} \) is weakly compact.
(b)⇒(a). Let \( f \in W^* \) be given. Then, \( f \) has an extension \( \tilde{f} \in X^* \), by Hahn-Banach Theorem. Now, (a) is an immediate consequence of Lemma 2.2.

(b)⇒(c). Let \( f \in X^* \) be given and let \( \{g_n\}_{n \geq 1} \) be a sequence in \( P_{W^\perp,\varepsilon}(f) \). Then, by Lemma 2.1, \( \{f - g_n\}_{n \geq 1} \) is a sequence in the set of all \( g \in X^* \) such that \( \tilde{g} \) is a \( \varepsilon \)-extension of \( \tilde{f}|_{W^\perp} \). Thus, there exists a weakly convergent subsequence \( \{f - g_{n_k}\}_{k \geq 1} \) of \( \{f - g_n\}_{n \geq 1} \). Therefore, \( \{g_{n_k}\}_{k \geq 1} \) is a weakly convergent subsequence of \( \{g_n\}_{n \geq 1} \). Hence, \( W^\perp \) is a \( \varepsilon \)-weakly Chebyshev subspace of \( X^* \).

(c)⇒(b). Let \( f \in X^* \) be given and let \( \{g_n\}_{n \geq 1} \) be a sequence in \( X^* \) such that \( \tilde{g}_n \) is an extension of \( \tilde{f}|_{W^\perp} \), for all \( n \geq 1 \). Then, by Lemma 2.1, \( \{f - g_n\}_{n \geq 1} \) is a sequence in \( P_{W^\perp,\varepsilon}(f) \). Thus, there exists a weakly convergent subsequence \( \{f - g_{n_k}\}_{k \geq 1} \) of \( \{f - g_n\}_{n \geq 1} \). Therefore, \( \{g_n\}_{n \geq 1} \) has a weakly convergent subsequence. □

The proof of the following Theorem is similar to that of Theorem 2.3.

**Theorem 2.4.** Let \( W \) be a subspace of a normed space \( X \) and let \( \varepsilon > 0 \) be given. Then the following are equivalent:

(a) For each \( f \in W^* \), the set \( E_{\varepsilon}(f) \) is compact.

(b) For each \( f \in X^* \), the set of all \( g \in X^* \) such that \( \tilde{g} \) is a \( \varepsilon \)-extension of \( \tilde{f}|_{W^\perp} \), is compact.

(c) \( W^\perp \) is a \( \varepsilon \)-quasi Chebyshev subspace of \( X^* \).

The following corollary is a consequence of Theorems 2.3, 2.4, Lemma 1.3 and Lemma 1.4.
Corollary 2.5. Let $X$ be a normed space.

(a) For each subspace $W$ of $X$ and for each $f \in W^*$, the set of all $\varepsilon$-extensions of $f$ is weakly compact if and only if $X$ is reflexive.

(b) For each subspace $W$ of $X$ and $f \in W^*$, the set of all $\varepsilon$-extensions of $f$ is compact if and only if $X$ is finite dimensional.

3. $\varepsilon$-weakly Chebyshev subspaces in dual spaces

In this section by using of $\varepsilon$-extensions, we shall characterize weak*-closed $\varepsilon$-weakly Chebyshev subspaces in dual spaces.

A subspace $W$ of a normed space $X$ is said to have the property $(\varepsilon-W)$ if for every $f \in W^*$ the set $E_\varepsilon(f)$ is weakly compact.

A subspace $M$ of $X^*$ is said to have the property $(\varepsilon-W^*)$ if the set

$$D_{x,\varepsilon}^M = \{y \in X : f(y) = f(x) \text{ for all } f \in M \& \|y\| \leq \|x\|_M + \varepsilon\}$$

is weakly compact for all $x \in X$, where

$$\|x\|_M = \sup\{|f(x)| : \|f\| \leq 1, \ f \in M\}.$$

Note that $D_{x,\varepsilon}^M$ is a non-empty, closed, bounded and convex subset of $X$ for all $x \in X$.

Theorem 3.1. Let $X$ be a normed space, $M$ be a subspace of $X^*$ and let $\varepsilon > 0$ be given. Then the following are equivalent:

(a) $M$ is a $\varepsilon$-weakly Chebyshev subspace of $X^*$.

(b) $M^\perp$ has the property $(\varepsilon-W^*)$.

(c) $\perp M$ has the property $(\varepsilon-W)$. 
Weak compactness of the set of \( \varepsilon \)-extensions

\textbf{Proof.} (a)\( \Rightarrow \) (b). If \( D_{f,\varepsilon}^{M} \) is not weakly compact for some \( f \in X^{*} \), then there exists a sequence \( \left\{ g_{n} \right\}_{n \geq 1} \) in \( D_{f,\varepsilon}^{M} \) without a weakly convergent subsequence. Since \( M \) is weak\(^*\)-closed, \( g_{1} - g_{n} \in M \) for all \( n \geq 1 \) and

\[ \left\| g_{1} - (g_{1} - g_{n}) \right\| = \left\| g_{n} \right\| \leq \left\| f \right\|_{M^{\perp}} + \varepsilon = \left\| g_{1} - (g_{1} - g_{n}) \right\|_{M^{\perp}} + \varepsilon \]

for all \( n \geq 1 \). Therefore, \( g_{1} - g_{n} \in P_{M,\varepsilon}(g_{1}) \) for all \( n \geq 1 \). It follows that \( M \) is not \( \varepsilon \)-weakly Chebyshev subspace of \( X^{*} \). Hence, (a) implies (b).

(b)\( \Rightarrow \) (c). suppose that \( M \) does not have the property \( (\varepsilon - W) \). Then, there exists \( f_{0} \in (\dagger M)^{*} \) and a sequence \( \left\{ g_{n} \right\}_{n \geq 1} \) in \( E_{f_{0},\varepsilon} \) without a convergent subsequence. Since, \( M^{\perp} = (\dagger M)^{\perp} \), \( M \) is dense in \( M^{\perp} \) for the weak topology on \( X^{**} \) ([1; Propositions 1.10.15 and 2.6.6]). Next, define \( F_{0} : X^{**} \rightarrow \Phi \) by \( F_{0}(\varphi) = \varphi(f_{0}) \) and \( F_{n} : X^{**} \rightarrow \Phi \) by \( F_{n}(\varphi) = \varphi(g_{n}) \) for all \( n \geq 1 \), where \( \Phi \) is the real or complex field. Then, \( F_{0}, F_{1}, F_{2}, \ldots \) are continuous with respect to the weak-topology on \( X^{**} \). Therefore, \( \varphi(g_{n}) = \varphi(f_{0}) \) for all \( n \geq 1 \) and \( \varphi \in M^{\perp} \). By Lemma 1.5, \( g_{n} \in D_{g_{1},\varepsilon}^{M} \) for all \( n \geq 1 \), because \( \left\| g_{n} \right\|_{M^{\perp}} = \left\| f_{0} \right\| \geq \left\| g_{n} \right\| + \varepsilon \) for all \( n \geq 1 \). Thus, \( M^{\perp} \) does not have the property \( (\varphi - W^{*}) \). Hence, (b) implies (c).

(c)\( \Rightarrow \) (a). suppose that \( P_{M,\varepsilon}(f_{0}) \) is not weakly compact for some \( f_{0} \in X^{*} \). Then, there exists a sequence \( \left\{ g_{n} \right\}_{n \geq 1} \) in \( P_{M,\varepsilon}(f_{0}) \) without a weakly convergent subsequence. By Lemma 1.1, \( \left\| f_{0} - g_{n} \right\| \leq \left\| f_{0} - g_{n} \right\|_{M^{\perp}} + \varepsilon = \left\| f_{0} \right\| + \varepsilon \) for all \( n \geq 1 \). It follows that \( f_{0} - g_{n} \in D_{f_{0},\varepsilon}^{M} \) for all \( n \geq 1 \). If we define the linear functional \( \Lambda_{0} \in (\dagger M)^{*} \) by \( \Lambda_{0} = f_{0}^{*} \), then by Lemma 1.5, \( \Lambda_{0}(x) = f_{0}(x) - g_{n}(x) \) and \( \left\| \Lambda_{0} \right\| + \varepsilon = \left\| f_{0} \right\| + \varepsilon \geq \left\| f_{0} - g_{n} \right\| \) for all \( n \geq 1 \) and all \( x \in M \). It follows that \( f_{0} - g_{n} \in E_{\Lambda_{0},\varepsilon} \) for all \( n \geq 1 \). Therefore, \( M \) does not have the property \( (\varepsilon - W) \). \( \Box \)

**Corollary 3.2.** Let \( X \) be a normed space and \( \varepsilon > 0 \) be given. Then the following are equivalent:

- \( \varepsilon \)-weak compactness of \( X^{*} \)
- \( \varepsilon \)-weakly Chebyshev subspace of \( X^{*} \)
- \( \varepsilon \)-weakly compact subset of \( X^{*} \)
(a) All weak*-closed subspaces of a $X^*$ are $\varepsilon$-weakly Chebyshev.

(b) All subspaces of $X$ have the property ($\varepsilon - W$).

(c) $X$ is reflexive.

REFERENCES


Sh. Rezapour
Department of Mathematics
Azarbaijan University of Tabrity Moalem
Azarshahr 51745-406
Tabriz, Iran
e-mail: sh.rezapour@azaruniv.edu
shahramrezapour@yahoo.ca