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FILTRATIONS OF SMOOTH PRINCIPAL SERIES AND IWASAWA MODULES

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Dedicated to Prof. Freydoon Shahidi on the occasion of his 70th birthday

ABSTRACT. Let G be a reductive p-adic group. We consider the general question of whether the reducibility of an induced representation can be detected in a "co-rank one" situation. For smooth complex representations induced from supercuspidal representations, we show that a sufficient condition is the existence of a subquotient that does not appear as a subrepresentation. An important example is the Langlands' quotient. In addition, we study the same general question for continuous principal series on p-adic Banach spaces. Although we do not give an answer in this case, we describe a related filtration on the corresponding Iwasawa modules.

Keywords: Parabolically induced representations, Iwasawa modules, p-adic groups.
MSC(2010): Primary: 22E50; Secondary: 20G05.

1. Introduction

In this paper, we study two types of principal series representations of a p-adic group G: smooth principal series on complex vector spaces and continuous principal series on p-adic Banach spaces. Although the properties of continuous and smooth principal series are fundamentally different, we are able to treat both types of principal series in a uniform manner, using a filtration of the group G.

Our main motivation is the understanding of continuous principal series, which are important in the p-adic Langlands program [3, 7]. Smooth principal series are well-understood. Still, we obtain a new result about reducibility; namely, we show how the question of reducibility can be reduced to determining rank one reducibility (Theorem 3.4). Furthermore, we extend this result to

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a more general case of representations induced from supercuspidal representations (Theorem 4.1).

In Section 2, we study the filtration of G coming from the partial order on the Weyl group. We apply this to the smooth principal series representations in Section 3. In Section 4, we study representations induced from supercuspidals and prove a criterion for reducibility (Theorem 4.1). Our approach to continuous principal series is based on the duality theory developed by Schneider and Teitelbaum in [16], which relates Banach space representations to Iwasawa modules. In Section 5, we prove a technical lemma on Iwasawa algebras. Section 6 is a brief review of the duality of [16] applied to continuous principal series. In Section 7, we describe filtrations of Iwasawa modules. We expect that these filtrations could be used in determining the reducibility of principal series. A conjecture about this difficult problem was formulated by Schneider in [14].

2. Partial orders on W and $[W/W_\Omega]$

Let F be a nonarchimedean local field and G the group of F-rational points of a connected reductive group defined over F. Fix a maximal split torus T in G and a minimal parabolic subgroup P containing T. Let Δ be the corresponding set of simple roots. For $\Omega \subseteq \Delta$, we denote by $P_\Omega = M_\Omega U_\Omega$ the standard parabolic subgroup corresponding to Ω. The minimal parabolic corresponds to $\emptyset \subseteq \Delta$, $P = P_\emptyset$.

Let W be the Weyl group of G. For $w \in W$, let $C(w)$ denote the double coset PwP. The closure of $C(w)$ with respect to the locally compact topology is equal to its relative closure in the Zarisky topology ([5, Proposition 21.27]) and is described as follows.

Theorem 2.1 ([5, Theorem 21.26])**.** *Let $w \in W$ and $w = s_1 \cdots s_n$ be a reduced decomposition of w. Then the set*

$$A_w = \{s_{i_1} \cdots s_{i_m} \mid m \in \mathbb{N}, 1 \leq i_1 < \cdots < i_m \leq n\}$$

depends only on w, not on the reduced decomposition, and we have

$$\overline{C(w)} = \bigcup_{v \in A_w} C(v).$$

We mention here that if $x \in A_w$, then the decomposition $x = s_{i_1} \cdots s_{i_m}$ from the previous theorem is not necessarily a reduced decomposition. However, we can reduce it further to obtain a reduced decomposition of x.

Let $\Omega \subseteq \Delta$ and $[W/W_\Omega] = \{w \in W \mid w\Omega > 0\}$. Then G has the disjoint union decomposition $G = \bigcup_{w \in [W/W_\Omega]} PwP_\Omega$. Define a partial order on $[W/W_\Omega]$ as follows: $x \leq_\Omega y$ if PxP_Ω is contained in the closure of PyP_Ω. In the special case when $\Omega = \emptyset$, we obtain a partial order on W and we denote it simply by

$\leq$. Hence, for $x, y \in W$,

$$x \leq y \iff x \in A_y.$$

Lemma 2.2. *Let $\Omega = \{\alpha\}$. If $x, y \in [W/W_\Omega]$ then*

$$x \leq y \iff x \leq_\Omega y.$$

Proof. Let $s = s_\alpha$ and $Q = P_\Omega = P \cup PsP$. We have $\overline{PyQ} = \overline{C(ys)} = \bigcup_{v \in A_{ys}} C(v)$. If $x \leq y$, then $x \in A_y$, so $xs \in A_{ys}$. It follows $PxQ \subseteq \overline{PyQ}$, so $x \leq_\Omega y$.

Conversely, assume $x \leq_\Omega y$. Then $xs \in A_{ys}$. Let $y = s_1 \cdots s_n$ be a reduced decomposition of y. Then $ys = s_1 \cdots s_n s_{n+1}$, where $s_{n+1} = s$, is a reduced decomposition of ys. Write $x = s_{i_1} \cdots s_{i_m}$ as in Theorem 2.1, $1 \leq i_1 < \cdots < i_m \leq n+1$, and assume that the decomposition is reduced. Then $s_{i_m} \neq s$, so $i_m < n+1$ and $x \leq y$. $\square$

For each $w \in [W/W_\Omega]$, define

$$G_w^\Omega = \bigcup_{\substack{x \in [W/W_\Omega] \\ x \geq w}} PxP_\Omega, \qquad G_w = G_w^\emptyset,$$

$$G_w^{\Omega,+} = \bigcup_{\substack{x \in [W/W_\Omega] \\ x > w}} PxP_\Omega, \qquad G_w^+ = G_w^{\emptyset,+}.$$

Lemma 2.3. *Let $\Omega = \{\alpha\}$ and $s = s_\alpha$. If $w \in [W/W_\Omega]$, then*

(1) $G_w^\Omega = G_w$,
(2) $G_w^+ = G_{ws} \cup G_w^{\Omega,+}$,
(3) $G_{ws} \cap G_w^{\Omega,+} = G_{ws}^+$.

Proof. (1) Clearly, $G_w^\Omega \subseteq G_w$. For converse inclusion, let $y \geq w$, so $w \in A_y$. If $ys > y$, then $y \in [W/W_\Omega]$. It follows $y \geq_\Omega w$ and $PyP \subseteq G_w^\Omega$.

If $ys < y$ then there exists a reduced decomposition $y = s_1 \cdots s_n$ such that $s_n = s$. Let $z = s_1 \cdots s_{n-1}$. Then $z \in [W/W_\Omega]$. Write $w = s_{i_1} \cdots s_{i_m}$ as in Theorem 2.1, and assume that the decomposition is reduced. Then $i_m < n$, so $w \in A_z$ and $z \geq w$. It follows that $PzQ \subseteq G_w^\Omega$, so $PyP = PzsP \subseteq G_w^\Omega$.

(2) Clearly, $G_{ws} \cup G_w^{\Omega,+} \subseteq G_w^+$. For converse inclusion, the proof goes along the same lines as in (1), with inequalities replaced by strict inequalities. The only difference is that in the case $ys < y$, we have to include $z = w$ since $y = ws > w$.

(3) Note that

$$PyP \subseteq G_w^{\Omega,+} \iff \begin{cases} y \in [W/W_\Omega], \quad y > w, \quad \text{or,} \\ y = zs,\ z \in [W/W_\Omega], \quad z > w. \end{cases}$$

In both cases, $y \neq ws$. It follows $G_{ws} \cap G_w^{\Omega,+} \subseteq G_{ws}^+$.

For the converse inclusion, let $y > ws$. Then $y > w$. If $y \in [W/W_\Omega]$, then $y >_\Omega w$, so $PyP \subseteq G_w^{\Omega,+}$. Otherwise, $y = zs$, $z \in [W/W_\Omega]$. Similarly as in (1), we get $z > w$, and consequently $PyP \subseteq PzQ \subseteq G_w^{\Omega,+}$. This proves $G_{ws}^+ \subseteq G_w^{\Omega,+}$. Now, $G_{ws}^+ \subseteq G_{ws}$ implies $G_{ws}^+ \subseteq G_{ws} \cap G_w^{\Omega,+}$, finishing the proof. $\square$

3. **Filtrations on smooth principal series**

Let $\mathfrak{R}(G)$ denote the category of smooth representations of G on complex vector spaces. We denote by $\mathrm{Ind}_{P_\Omega}^G : \mathfrak{R}(M_\Omega) \to \mathfrak{R}(1G)$ the functor of normalized parabolic induction ([4, 6]).

Let χ be a smooth character of $P = P_\emptyset = MU$. Let $I = I(\chi)$ denote the space of the representation $\mathrm{Ind}_P^G(\chi)$. It is the space of all functions $f : G \to \mathbb{C}$ such that

(i) $f(mug) = \chi(m)\delta_P^{1/2}(m)f(g)$ for all $m \in M$, $u \in U$, $g \in G$, and
(ii) there exists a compact open subgroup K_f of G such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$.

The group G acts on I by the right regular action. Define

$$\begin{aligned} I_w &= \{f \in I \mid \operatorname{supp} f \subseteq G_w\}, & I_w^\Omega &= \{f \in I \mid \operatorname{supp} f \subseteq G_w^\Omega\},\\ I_w^+ &= \{f \in I \mid \operatorname{supp} f \subseteq G_w^+\}, & I_w^{\Omega,+} &= \{f \in I \mid \operatorname{supp} f \subseteq G_w^{\Omega,+}\},\\ J_w &= I_w/I_w^+, & J_w^\Omega &= I_w^\Omega/I_w^{\Omega,+}. \end{aligned}$$

Lemma 3.1. *Let $\Omega = \{\alpha\}$. If $w \in [W/W_\Omega]$, then*

(1) $I_w^\Omega = I_w$,
(2) $I_w^+ = I_{ws} + I_w^{\Omega,+}$,
(3) $I_{ws} \cap I_w^{\Omega,+} = I_{ws}^+$.

Proof. (1) and (3) follow immediately from Lemma 2.3. For (2), let

$$X = \{x \in W \mid C(x) \subseteq G_w^{\Omega,+},\, C(ws) \not\subseteq \overline{C(x)}\}.$$

The second condition is equivalent to $ws \not\leq x$. Since the double cosets are disjoint, it is also equivalent to $C(ws) \cap \overline{C(x)} = \emptyset$. Let

$$\mathcal{X} = \bigcup_{x \in X} C(x).$$

Let K be a maximal compact subgroup such that $G = PK$. Set $H = \overline{C(ws)}$. Since $PH = H$, it follows $H = P(H \cap K)$.

Let $f \in I_w^+$. In order to write f as $f = f' + f''$, where $f' \in I_{ws}$ and $f'' \in I_w^{\Omega,+}$, we will construct a function f' from the induced space which coincides with f on $C(ws)$ and whose support is disjoint from $\mathcal{X}$ and from $H \setminus C(ws)$. Note that such a function has to be invariant under the action of a compact open subgroup of G. We start our construction with a compact open subgroup K_f such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$. The definition of $\mathcal{X}$ implies that for every $y \in C(ws)$ there exists a compact open subgroup K'_y such that

$yK'_y \cap \mathcal{X} = \emptyset$. Let $K_y = K'_y \cap K_f$. Then $yK_y \cap \mathcal{X} = \emptyset$. For $y \in H \setminus C(ws)$, set $K_y = K_f$. Then $f(yk) = 0$ for all $y \in H \setminus C(ws)$, $k \in K_y$. The set $\{yK_y \mid y \in H \cap K\}$ is a cover of the compact subset $H \cap K$, so it has a finite subcover $\{yK_y \mid y \in Y\}$. Let

$$\mathcal{Y} = \bigcup_{y \in Y} yK_y.$$

Denote by f' the product of f with the characteristic function of $P\mathcal{Y}$. Then f' is invariant under the action of the compact open subgroup $\bigcap_{y \in Y} K_y$. We have $f' \in I$, $\operatorname{supp} f' \subseteq G_w^+$ and $(\operatorname{supp} f') \cap \mathcal{X} = \emptyset$. It follows $f' \in I_{ws}$. Furthermore, $f'' = f - f' \in I_w^{\Omega,+}$. This implies $I_w^+ \subseteq I_{ws} + I_w^{\Omega,+}$. The converse inclusion is obvious. □

Lemma 3.2. *Let $\Omega = \{\alpha\}$. If $w \in [W/W_\Omega]$, then $J_w^\Omega / J_{ws} \cong J_w$.*

Proof. Using Lemma 3.1, we get

$$J_{ws} = I_{ws}/I_{ws}^+ = I_{ws}/(I_{ws} \cap I_w^{\Omega,+}) \cong (I_{ws} + I_w^{\Omega,+})/I_w^{\Omega,+} = I_w^+/I_w^{\Omega,+}.$$

Therefore, J_{ws} embeds into J_w^Ω. Moreover,

$$J_w^\Omega/J_{ws} = (I_w^\Omega/I_w^{\Omega,+})/(I_w^+/I_w^{\Omega,+}) \cong I_w^\Omega/I_w^+ = I_w/I_w^+ = J_w.$$

□

We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & I_{ws}^+ & \to & I_{ws} & \to & J_{ws} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_w^{\Omega,+} & \to & I_w^\Omega & \to & J_w^\Omega & \to & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & J_w & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Suppose $\mathcal{I}$ is a proper subrepresentation of I. Define $\mathcal{I}_w^\Omega = \mathcal{I} \cap I_w^\Omega$, $\mathcal{I}_w^{\Omega,+} = \mathcal{I} \cap I_w^{\Omega,+}$ and $\mathcal{J}_w^\Omega = \mathcal{I}_w^\Omega/\mathcal{I}_w^{\Omega,+}$. We want to select w and α so that $\mathcal{J}_w^\Omega$ is a proper nonzero submodule of J_w^Ω. Furthermore, we want to pass to representations of $M_{\{\alpha\}}$, and obtain from $\mathcal{J}_w^\Omega$ a proper subrepresentation of a principal series of $M_{\{\alpha\}}$. For this, we use Jacquet modules of parabolically induced representations ([4, 6]). Since the Jacquet module of J_w^Ω is precisely a principal series of $M_{\{\alpha\}}$, we select w and α by considering directly Jacquet modules (see the proof of Theorem 3.4).

For each $w \in [W/W_\Omega]$, let $d_\Omega(w)$ be the dimension of the algebraic variety $P \setminus PwP_\Omega$ over F. Define

$$G_n^\Omega = \bigcup_{d_\Omega(w) \geq n} PwP_\Omega \quad \text{and} \quad I_n^\Omega = \{f \in I \mid \operatorname{supp} f \subseteq G_n^\Omega\}.$$

Let (π, V) be a smooth representation of G. For a compact subgroup $U_0 \subset U_\Omega$, define

$$V(U_0) = \{v \in V \mid \int_{U_0} \pi(u) v du = 0\}.$$

Define $V(U_\Omega)$ to be $\bigcup V(U_0)$, the union over all compact open subgroups U_0 of U_Ω. Then $V(U_\Omega)$ is a subspace of V [6]. Define $r^G_{M_\Omega}(V) = V/V(U_\Omega)$. The normalized Jacquet module $r^G_{M_\Omega}(\pi)$ is the representation of M_Ω on the space $r^G_{M_\Omega}(V)$ given by

$$r^G_{M_\Omega}(\pi)(m)(v + V(U_\Omega)) = \delta_{P_\Omega}^{-1/2}(m)\pi(m)v + V(U_\Omega),$$

$m \in M_\Omega$, $v \in V$ [4]. If $W \subset V$ is a P_Ω-invariant subspace, then $r^G_{M_\Omega}(V)$ is M_Ω-invariant.

Theorem 3.3 ([6, Theorem 6.3.5])**.** *Let $P = P_\emptyset = MU$ and $\Omega \subseteq \Delta$. Let χ be a smooth character of M, $I = \operatorname{Ind}_P^G \chi$. There exists a filtration*

$$0 \subseteq I^\Omega_{n_\ell} \subseteq \cdots \subseteq I_0^\Omega = I$$

by P_Ω-stable subspaces such that $r^G_{M_\Omega}(I_n^\Omega / I^\Omega_{n+1}) \cong r^G_{M_\Omega}(I_n^\Omega)/r^G_{M_\Omega}(I^\Omega_{n+1})$ is isomorphic to the direct sum $\bigoplus r^G_{M_\Omega}(J_w^\Omega)$, the sum ranging over $w \in [W/W_\Omega]$ with $d_\Omega(w) = n$. Furthermore, the normalized Jacquet module $r^G_{M_\Omega}(J_w^\Omega)$ is given by

$$r^G_{M_\Omega}(J_w^\Omega) \cong \operatorname{Ind}^{M_\Omega}_{w^{-1}Pw \cap M_\Omega} w^{-1}\chi.$$

The proof of the next theorem, as well as the proof of Theorem 4.1, is improved and shortened, as suggested by the referee.

Theorem 3.4. *Let $P = P_\emptyset = MU$. Let χ be a character of M and $I(\chi) = \operatorname{Ind}_P^G \chi$. Suppose that $I(\chi)$ has an irreducible subquotient which does not appear as a subrepresentation in $I(\chi)$. Then there exist $\alpha \in \Delta$ and $w \in W$ such that $\operatorname{Ind}^{M_{\{\alpha\}}}_{P \cap M_{\{\alpha\}}} w^{-1}\chi$ is reducible.*

Proof. Denote by $\mathcal{I}$ an irreducible subquotient of $I(\chi)$ which does not appear as a subrepresentation in $I(\chi)$. Then there exists $w' \in W$ such that $\mathcal{I}$ is a subrepresentation of $I(w'\chi)$. Let $\chi' = w'\chi$, $I = I(\chi')$. Define

$$W_{\mathcal{I}} = \{v \in W \mid \operatorname{Hom}_M(r^G_M(\mathcal{I}), v^{-1}\chi') = 0\}.$$

Since $\mathcal{I}$ does not appear as a subrepresentation of $I(\chi)$, $\operatorname{Hom}_M(r^G_P(\mathcal{I}), \chi) = 0$ so $w' \in W_{\mathcal{I}}$. Moreover, $1 \notin W_{\mathcal{I}}$ because $\mathcal{I}$ is a subrepresentation of I. Let

w_0 be a minimal element in $W_{\mathcal{I}}$. Here, we consider the partial order on W as defined in section 2. Since $w_0 \neq 1$, there exists $\alpha \in \Delta$ such that $w_0\alpha < 0$. Let

$$s = s_\alpha, \quad w = w_0 s, \quad \Omega = \{\alpha\}, \quad P_\Omega = P \cup PsP.$$

Then $w \in [W/W_\Omega]$. In addition, $w < w_0$, so $w \notin W_{\mathcal{I}}$. The Frobenius reciprocity gives us

$$\begin{aligned} 0 \neq \mathrm{Hom}_M(r_M^G(\mathcal{I}), w^{-1}\chi') &= \mathrm{Hom}_M(r_M^{M_\Omega} \circ r_{M_\Omega}^G(\mathcal{I}), w^{-1}\chi') \\ &\cong \mathrm{Hom}_{M_\Omega}(r_{M_\Omega}^G(\mathcal{I}), \mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(w^{-1}\chi')). \end{aligned}$$

In the same way, $0 = \mathrm{Hom}_{M_\Omega}(r_{M_\Omega}^G(\mathcal{I}), \mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(sw^{-1}\chi'))$. Hence,

$$\mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(w^{-1}\chi') \not\cong \mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(sw^{-1}\chi').$$

Since these representations have the same composition factors ([6, Theorem 6.3.11]), if irreducible, they would be isomorphic. It follows that $\mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(w^{-1}\chi')$ $= \mathrm{Ind}_{P\cap M_\Omega}^{M_\Omega}(w^{-1}w'\chi)$ is reducible. □

Remark 3.5. Theorem 3.4 does not remain true if $I(\chi)$ is reducible, but we remove the hypothesis that $I(\chi)$ has a subquotient which does not appear as a subrepresentation in $I(\chi)$. The following example was communicated to us by Marko Tadić. Let χ_1, χ_2 be two different characters of $F^\times$ of order 2. The unitary principal series representation of $GSp(6, F)$ induced from $\chi_1 \otimes \chi_2 \otimes \chi_1\chi_2$ has length 2, but we have irreducibility for all rank one Levi subgroups, and even for all rank two Levi subgroups (because unitary principal series for $GL(3, F)$ and $GSp(4, F)$ are irreducible [13]).

Example 3.6 (Langlands' Quotients)**.** Let χ_0 be a character of $M = M_\emptyset$ which is square-integrable modulo center. For $\Omega \subset \Delta$, the induced representation $\mathrm{Ind}_M^{M_\Omega} \chi_0$ decomposes as a direct sum of tempered representations, $\mathrm{Ind}_M^{M_\Omega} \chi_0 = \oplus\tau_i$. Let ν be an unramified character of M_Ω which takes positive real values and is strictly positive (in Weyl chamber). Then $\mathrm{Ind}_{M_\Omega}^G(\nu\tau_i)$ has a unique irreducible quotient π (Langlands' quotient). It appears with multiplicity one in $\mathrm{Ind}_{M_\Omega}^G(\nu\tau_i)$. Moreover, it can be shown using central exponents and uniqueness of Langlands' data that π does not appear as a subquotient in $\mathrm{Ind}_{M_\Omega}^G(\nu\tau_j)$ for any $\tau_j \not\cong \tau_i$.

Assume that $\mathrm{Ind}_{M_\Omega}^G(\nu\tau_i)$ is reducible. Corresponding to ν is a character of M, which we will denote by the same letter ν, such that $\mathrm{Ind}_{M_\Omega}^G(\nu\tau_i)$ is a subrepresentation of $\mathrm{Ind}_M^G(\nu\chi_0)$. Then π does not appear as a subrepresentation in $\mathrm{Ind}_M^G(\nu\chi_0)$ and we can apply Theorem 3.4. It follows that there exist $\alpha \in \Delta$ and $w \in W$ such that $\mathrm{Ind}_M^{M_{\{\alpha\}}} w^{-1}(\nu\chi_0)$ is reducible.

4. Generalization

In this section, we generalize Theorem 3.4.

Theorem 4.1. *Let $\Theta \subseteq \Delta$. Let $P_\Theta = M_\Theta U_\Theta$. Let σ be an irreducible supercuspidal representation of M_Θ and $I(\sigma) = \mathrm{Ind}_{P_\Theta}^G \sigma$. Suppose that $I(\sigma)$ has an irreducible subquotient which does not appear as a subrepresentation in $I(\sigma)$. Then there exist $w_0 \in W$ and $\Omega \subseteq \Delta$ such that $\Omega = w_0(\Theta) \cup \{\alpha\}$ and $\mathrm{Ind}_{w_0(M_\Theta)}^{M_\Omega} w_0\sigma$ is reducible.*

We give a brief proof. For more detail, see [1].

Proof. Set $W(\Theta,\Theta) = \{w \in W \mid w(\Theta) = \Theta\}$. Denote by $\mathcal{I}$ an irreducible subquotient of $I(\sigma)$ which does not appear as a subrepresentation in $I(\sigma)$. Then there exists $w' \in W(\Theta,\Theta)$ such that $\mathcal{I}$ is a subrepresentation of $I(w'\sigma)$. Let $\sigma' = w'\sigma$, $I = I(\sigma')$. Define

$$W_{\mathcal{I}} = \{v \in W \mid v(\Theta) \subseteq \Delta \text{ and } \mathrm{Hom}_{v(M_\Theta)}(r^G_{v(M_\Theta)}(\mathcal{I}), v\sigma') = 0\}.$$

Similarly as in the proof of Theorem 3.4, we see that $(w')^{-1} \in W_{\mathcal{I}}$ and $1 \notin W_{\mathcal{I}}$. Let w_{min} be an element of $W_{\mathcal{I}}$ of minimum length and $\Theta' = w_{min}(\Theta)$. Now, we apply [17, Lemma 2.1.2]. In particular, let $\Theta = \Theta_1, \dots, \Theta_n = \Theta'$ be a sequence of associate subsets of Δ as in [17, Lemma 2.1.2]. Then for any $1 \leq i \leq n-1$ there exists a simple root α_i such that Θ_{i+1} is the conjugate of Θ_i in $\Theta_i \cup \{\alpha_i\}$. We have $w_{min} = w_{n-1} \dots w_1$, where $w_i \in W(\Theta_i, \Theta_{i+1})$. Set $y = w_{n-1}$ and $w = w_{n-2} \dots w_1$. By minimality, $w \notin W_{\mathcal{I}}$ and we have

$$\mathrm{Hom}_{w(M_\Theta)}(r^G_{w(M_\Theta)}(\mathcal{I}), w\sigma') \neq 0, \quad \mathrm{Hom}_{yw(M_\Theta)}(r^G_{yw(M_\Theta)}(\mathcal{I}), yw\sigma') = 0.$$

Let $\alpha_1, \dots, \alpha_{n-1}$ be as in [17, Lemma 2.1.2]. Set $\alpha = \alpha_{n-1}$ and $\Omega = w(\Theta) \cup \{\alpha\}$. Then $yw(\Theta)$ is the conjugate of $w(\Theta)$ in Ω. The Frobenius reciprocity gives us

$$\begin{aligned}\mathrm{Hom}_{w(M_\Theta)}(r^G_{w(M_\Theta)}(\mathcal{I}), w\sigma') &= \mathrm{Hom}_{w(M_\Theta)}(r^{M_\Omega}_{w(M_\Theta)} \circ r^G_{M_\Omega}(\mathcal{I}), w\sigma') \\ &\cong \mathrm{Hom}_{M_\Omega}(r^G_{M_\Omega}(\mathcal{I}), \mathrm{Ind}^{M_\Omega}_{w(M_\Theta)}(w\sigma')) \neq 0.\end{aligned}$$

In the same way, $\mathrm{Hom}_{M_\Omega}(r^G_{M_\Omega}(\mathcal{I}), \mathrm{Ind}^{M_\Omega}_{yw(M_\Theta)}(yw\sigma')) = 0$. Hence,

$$\mathrm{Ind}^{M_\Omega}_{w(M_\Theta)}(w\sigma') \ncong \mathrm{Ind}^{M_\Omega}_{yw(M_\Theta)}(yw\sigma').$$

As in the proof of Theorem 3.4, it follows that these representations are reducible. Hence, $\mathrm{Ind}^{M_\Omega}_{w(M_\Theta)}(ww'\sigma) = \mathrm{Ind}^{M_\Omega}_{w(M_\Theta)}(w\sigma')$ is reducible. Note that reducibility implies that $yw(\Theta) = w(\Theta)$. To complete the proof, we may select w_0 to be either ww' or yww'. □

Define $W(\sigma) = \{w \in W(\Theta,\Theta) \mid w\sigma \cong \sigma\}$ and call σ regular if $W(\sigma) = \{1\}$.

Corollary 4.2. *Let $P_\Theta = M_\Theta U_\Theta$. Let σ be an irreducible supercuspidal representation of M_Θ. Suppose that σ is regular. Then $I(\sigma) = \mathrm{Ind}^G_{P_\Theta} \sigma$ is reducible* *if and only if there exist* $w_0 \in W$ *and* $\Omega \subseteq \Delta$ *such that* $\Omega = w_0(\Theta) \cup \{\alpha\}$ *and* $\mathrm{Ind}_{w_0(M_\Theta)}^{M_\Omega} w_0\sigma$ *is reducible.*

Proof. Suppose that σ is regular. Then $r_{M_\Theta}^G(I(\sigma)) = \bigoplus_{w \in W(\Theta,\Theta)} w\sigma$ is a direct sum of mutually inequivalent components ([6, Proposition 6.4.1]). Since the Jacquet functor is exact, it follows that every component of $I(\sigma)$ appears with multiplicity one. Furthermore, if V is an irreducible subrepresentation of $I(\sigma)$, then the Frobenius reciprocity gives us $\mathrm{Hom}_G(V, I(\sigma)) \cong \mathrm{Hom}_{M_\Theta}(r_{M_\Theta}^G(V), \sigma)$. Since the multiplicity of σ in $r_{M_\Theta}^G(I(\sigma))$ is one, it follows that $I(\sigma)$ has a unique irreducible subrepresentation. Therefore, if $I(\sigma)$ is reducible, it satisfies the conditions of Theorem 4.1. $\square$

In the case of principal series, the previous corollary also follows from Rodier's work on the principal series induced from regular characters [12].

5. Iwasawa algebras

We start by reviewing some results on projective limits. We refer to [11] for definitions of a projective system and a projective limit. The following two propositions follow from Proposition 1.1.3, Proposition 1.1.4 and Corollary 1.1.8 of [11].

Proposition 5.1. *Let* (X_i) *be a projective system of compact Hausdorff topological spaces over the directed set* I*, and let* $X = \mathrm{proj\,lim}\, X_i$*.*

- (a) *If* X_i *is totally disconnected, for all* $i \in I$*, then* X *is also a compact Hausdorff totally disconnected topological space.*
- (b) *If* X_i *is nonempty, for all* $i \in I$*, then* X *is also nonempty.*

Proposition 5.2. *Let* (X_i) *be a projective system of compact Hausdorff spaces,* $X = \mathrm{proj\,lim}\, X_i$*, and let* $\varphi_i : X \to X_i$ *be the projections.*

- (a) *If* Y *is a closed subspace of* X*, then* $Y = \mathrm{proj\,lim}\, \varphi_i(Y)$*.*
- (b) *If* Y *is a subspace of* X*, then* $\overline{Y} = \mathrm{proj\,lim}\, \varphi_i(Y)$*, where* $\overline{Y}$ *is the closure of* Y *in* X*.*
- (c) *If* Y *and* Y' *are subspaces of* X *and* $\varphi_i(Y) = \varphi_i(Y')$ *for each* i*, then their closures in* X *coincide:* $\overline{Y} = \overline{Y'}$*.*

Next, we review the definition and basic properties of Iwasawa algebras ([10], [15]). Let H be a profinite group. Let $\mathcal{N}(H)$ denote the family of all open normal subgroups of H. Then $H = \mathrm{proj\,lim}_{N \in \mathcal{N}(H)} H/N$ is a projective limit, as a topological group, of the finite groups H/N. Let K be a finite extension of $\mathbb{Q}_p$, and o_K its ring of integers. The group rings $o_K[H/N]$, $N \in \mathcal{N}(H)$, form a projective system of rings. The Iwasawa algebra of H over o_K is defined as

$$o_K[[H]] = \mathop{\mathrm{proj\,lim}}_{N \in \mathcal{N}(H)} o_K[H/N].$$

We equip $o_K[[H]]$ with the projective limit topology. Then $o_K[[H]]$ is a torsion free and compact linear-topological o_K-module. It has a structure of a topological ring; the ring multiplication is continuous. The inclusion map $H \hookrightarrow o_K[[H]]$ is a homeomorphism onto its image.

Define $K[[H]] = K \otimes_{o_K} o_K[[H]]$, endowed with the finest locally convex topology such that the inclusion of $o_K[[H]]$ is continuous. Then the multiplication on $K[[H]]$ is separately continuous.

Let A be a closed subset of H. For $N \in \mathcal{N}(H)$, define $A_N = \{aN \mid a \in A\} \subseteq H/N$. Let $\langle A_N \rangle$ denote the o_K-submodule of $o_K[H/N]$ generated by A_N. Then $\langle A_N \rangle$, $N \in \mathcal{N}(H)$, is a projective system of topological o_K-modules. Define

$$\Lambda^o(A) = \operatorname*{proj\,lim}_{N \in \mathcal{N}(H)} \langle A_N \rangle \quad \text{and} \quad \Lambda(A) = K \otimes_{o_K} \Lambda^o(A).$$

Then $\Lambda^o(A)$ is a closed o_K-submodule of $o_K[[H]]$ and $\Lambda(A)$ is a closed K-subspace of $K[[H]]$.

Lemma 5.3. *Let A, B be closed subsets of H.*

(a) *$\Lambda^o(A)$ is a compact Hausdorff totally disconnected topological space.*
(b) *$\Lambda^o(A \cup B) = \Lambda^o(A) + \Lambda^o(B)$, and $\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)$.*
(c) *$\Lambda^o(A \cap B) = \Lambda^o(A) \cap \Lambda^o(B)$, and $\Lambda(A \cap B) = \Lambda(A) \cap \Lambda(B)$.*

Proof. (a) follows from Proposition 5.1. (b) follows from Proposition 5.2, because

$$\langle A_N \cup B_N \rangle = \langle A_N \rangle + \langle B_N \rangle \quad \text{for all } N \in \mathcal{N}(H).$$

(c) Set $C = A \cap B$. Since

$$\langle C_N \rangle \subseteq \langle A_N \rangle \cap \langle B_N \rangle \quad \text{for all } N \in \mathcal{N}(H),$$

we immediately get $\Lambda^o(C) \subseteq \Lambda^o(A) \cap \Lambda^o(B)$. Assume $\Lambda^o(C) \neq \Lambda^o(A) \cap \Lambda^o(B)$. Then there exists $\mu \in \Lambda^o(A) \cap \Lambda^o(B)$ such that $\mu \notin \Lambda^o(C)$. Write

$$\mu = (\mu_N)_{N \in \mathcal{N}(H)}, \quad \mu_N \in o_K[H/N].$$

Then there exists $N_0 \in \mathcal{N}(H)$ such that $\mu_{N_0} \notin \langle C_{N_0} \rangle$. Write

$$\mu_{N_0} = \alpha_1 a_1 N_0 + \cdots + \alpha_k a_k N_0 = \alpha_1 b_1 N_0 + \cdots + \alpha_k b_k N_0,$$

where $\alpha_i \in o_K$, $a_i \in A$, $b_i \in B$, and $a_i N_0 = b_i N_0$ for all i. We can decompose μ as

$$\mu = \mu^1 + \cdots + \mu^k, \quad \mu^i \in \Lambda^o(a_i N_0).$$

Note that $(\mu^i)_{N_0} = \alpha_i a_i N_0$. Select $\ell \in \{1, \ldots, k\}$ such that $\alpha_\ell a_\ell N_0 \notin \langle C_{N_0} \rangle$. Then $a_\ell N_0 \neq c N_0$ for any $c \in C$. Let $\lambda = \mu^\ell$. Then $\lambda \in \Lambda^o(a_\ell N_0) = \Lambda^o(b_\ell N_0)$ and $\lambda \notin \Lambda^o(C)$.

For any $c \in C$ there exists $N_c \in \mathcal{N}(H)$ such that $a_\ell N_0 \cap c N_c = \emptyset$. Then $\{c N_c \mid c \in C\}$ is an open cover of C. The set $B_1 = B \setminus \bigcup_{c \in C} c N_c$ is closed, and disjoint from A. For any $b \in B_1$ there exists $N_b \in \mathcal{N}(H)$ such that $A \cap b N_b = \emptyset$. Then $\{b N_b \mid b \in B_1\}$ is an open cover of B_1. By compactness, it has a finite subcover. It follows that we can find $N_1 \in \mathcal{N}(H)$, $N_1 \leq N_0$, such that $A \cap bN_1 = \emptyset$ for all $b \in B_1$. Now, write

$$\lambda_{N_1} = \alpha_1' a_1' N_1 + \cdots + \alpha_s' a_s' N_1.$$

Since $\alpha_1' + \cdots + \alpha_s' = \alpha_\ell \neq 0$, at least one coefficient α_i' is not zero. Since $a_i' \notin bN_1$ for any $b \in B$, we have $\alpha_i' a_i' N_1 \notin \langle B_{N_1} \rangle$. This contradicts $\lambda \in \Lambda^o(b_i N_0)$. □

6. Continuous principal series

From now on, F is a finite extension of $\mathbb{Q}_p$, and K is a finite extension of F. As before, $P = P_\emptyset$ is a minimal parabolic subgroup of G. Let $\chi : P \to K^\times$ be a continuous character. Let

$${}^c\mathrm{Ind}_P^G(\chi^{-1}) = \{f : G \to K \text{ continuous } \mid f(gp) = \chi(p) f(g) \, \forall p \in P, g \in G\},$$

where G acts on the left by $g \cdot f(h) = f(g^{-1}h)$. Here, we take the left action because we will use the duality [16].

Let $G_0 \subset G$ be a maximal compact subgroup which satisfies the Iwasawa decomposition $G = G_0 P$. If $P_0 = P \cap G_0$ and $\chi_0 = \chi|_{P_0}$, then restriction gives an isomorphism ${}^c\mathrm{Ind}_P^G(\chi^{-1}) \cong {}^c\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$.

Let $K^{(\chi_0)}$ denote the one dimensional representation of P_0 on K given by χ_0. The continuous dual of ${}^c\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ is isomorphic to

$$M^{(\chi_0)} = K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}.$$

The isomorphism ${}^c\mathrm{Ind}_P^G(\chi^{-1}) \cong {}^c\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ induces a G-module structure on $M^{(\chi_0)}$. We denote this G-module by $M^{(\chi)}$. Hence, $M^{(\chi)}$ is a G-module and $K[[G_0]]$-module. It follows from Theorem 3.5 of [16] that there is a bijection between G-invariant closed subspaces of ${}^c\mathrm{Ind}_P^G(\chi^{-1})$ and G-invariant $K[[G_0]]$-quotient modules of $M^{(\chi)}$.

7. Iwasawa modules

Similarly to open subsets of G defined in Section 2, we define certain closed subsets of G. For $w \in W$, define

$$H_w = \overline{PwP} = \bigcup_{x \leq w} PxP, \qquad H_w^- = \bigcup_{x < w} PxP.$$

Let $\alpha \in \Delta$, $\Omega = \{\alpha\}$, and $s = s_\alpha$. For $w \in [W_\Omega \setminus W] = \{w \in W \mid w^{-1}\Omega > 0\}$, define

$$H_w^\Omega = \bigcup_{\substack{x \in [W_\Omega \setminus W] \\ x \leq w}} P_\Omega x P, \qquad H_w^{\Omega,-} = \bigcup_{\substack{x \in [W_\Omega \setminus W] \\ x < w}} P_\Omega x P.$$

For $X \subset K[[G_0]]$, we denote by $[X]$ the image of X in $M^{(\chi)}$. There exists a compact set $C \subset G_0$ such that C is a set of coset representatives of G/P (see

[2] for an explicit description of such a set). Then the map $\mu \mapsto [\mu]$ defines an isomorphism between $\Lambda(C)$ and $M^{(\chi)}$ ([2], Corollary 15). Fix such a compact set C. The sets H_w, H_w^-, H_w^Ω and $H_w^{\Omega,-}$ are closed in G, so we can define, for $w \in W$,

$$M_w = [\Lambda(C \cap H_w)], \quad M_w^- = [\Lambda(C \cap H_w^-)], \quad \text{and} \quad N_w = M_w/M_w^-.$$

Similarly, for $w \in [W_\Omega \setminus W]$, we define

$$M_w^\Omega = [\Lambda(C \cap H_w^\Omega)], \quad M_w^{\Omega,-} = [\Lambda(C \cap H_w^{\Omega,-})], \quad \text{and} \quad N_w^\Omega = M_w^\Omega/M_w^{\Omega,-}.$$

Lemma 7.1. *Let $\Omega = \{\alpha\}$ and $s = s_\alpha$. If $w \in [W_\Omega \setminus W]$, then*
 (a) $M_w^\Omega = M_{sw}$,
 (b) $M_w \cap M_w^{\Omega,-} = M_w^-$,
 (c) $M_w + M_w^{\Omega,-} = M_{sw}^-$.

Proof. (a) is clear, because $H_w^\Omega = H_{sw}$. For (b) and (c), note that

$$H_w \cap H_w^{\Omega,-} = \left(\bigcup_{x \le w} PxP\right) \cap \left(\bigcup_{\substack{x \in [W_\Omega \setminus W] \\ x < w}} P_\Omega x P\right) = H_w^-$$

and

$$H_w \cup H_w^{\Omega,-} = \left(\bigcup_{x \le w} PxP\right) \cup \left(\bigcup_{\substack{x \in [W_\Omega \setminus W] \\ x < w}} P_\Omega x P\right) = H_{sw}^-.$$

Lemma 5.3 implies

$$\begin{aligned}\Lambda(C \cap H_w) \cap \Lambda(C \cap H_w^{\Omega,-}) &= \Lambda((C \cap H_w) \cap (C \cap H_w^{\Omega,-})) \\ &= \Lambda(C \cap (H_w \cap H_w^{\Omega,-})) = \Lambda(C \cap H_w^-).\end{aligned}$$

It follows $M_w \cap M_w^{\Omega,-} = M_w^-$. Similarly,

$$\begin{aligned}\Lambda(C \cap H_w) + \Lambda(C \cap H_w^{\Omega,-}) &= \Lambda((C \cap H_w) \cup (C \cap H_w^{\Omega,-})) \\ &= \Lambda(C \cap (H_w \cup H_w^{\Omega,-})) = \Lambda(C \cap H_{sw}^-)\end{aligned}$$

gives $M_w + M_w^{\Omega,-} = M_{sw}^-$. ☐

Lemma 7.2. *Let $\Omega = \{\alpha\}$. If $w \in [W_\Omega \setminus W]$, then $N_w^\Omega / N_w \cong N_{sw}$.*

Proof. Using Lemma 7.1, we get

$$N_w \cong M_w/M_w^- \cong M_w/(M_w \cap M_w^{\Omega,-}) \cong (M_w + M_w^{\Omega,-})/M_w^{\Omega,-} \cong M_{sw}^-/M_w^{\Omega,-}.$$

Then

$$N_w^\Omega/N_w \cong (M_{sw}/M_w^{\Omega,-})/(M_{sw}^-/M_w^{\Omega,-}) \cong M_{sw}/M_{sw}^- \cong N_{sw}.$$

☐

Hence, we have the following commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & M_w^- & \to & M_w & \to & N_w & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_w^{\Omega,-} & \to & M_w^\Omega & \to & N_w^\Omega & \to & 0. \\ & & & & & & \downarrow & & \\ & & & & & & N_{sw} & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Note that $M^{(\chi)}$ is a cyclic $K[[G_0]]$-module, generated by [1]. Let S be a $K[[G_0]]$-submodule and G-submodule of $M^{(\chi)}$. Assume that $S \neq 0$ and $S \neq M^{(\chi)}$. Then $[1] \notin S$ and $S \cap M_1 = 0$. Moreover, there exists a minimal $w' \in W$ such that $S \cap M_{w'} \neq 0$. Write $w' = sw$, where $s = s_\alpha$ is a simple reflection. Hence,

$$S \cap M_{sw} \neq 0, \quad S \cap M_x = 0, \text{ for all } x < sw.$$

In particular, $S \cap M_w = 0$, $S \cap M_w^{\Omega,-} = 0$, $S \cap M_{sw}^- = 0$.

Lemma 7.3. *$S \cap M_w^\Omega$ is isomorphic to a proper submodule of N_w^Ω.*

Proof. Since $S \cap M_w^{\Omega,-} = 0$, we have

$$\begin{aligned} S \cap M_w^\Omega &\cong (S \cap M_w^\Omega)/(S \cap M_w^{\Omega,-}) \\ &\cong (S \cap M_w^\Omega) + M_w^{\Omega,-}/M_w^{\Omega,-} \subseteq M_w^\Omega/M_w^{\Omega,-} = N_w^\Omega. \end{aligned}$$

We have to prove $(S \cap M_w^\Omega) + M_w^{\Omega,-} \neq M_w^\Omega$. Assume, on the contrary, that $(S \cap M_w^\Omega) + M_w^{\Omega,-} = M_w^\Omega$. Then we can write $[w] \in M_w^\Omega$ as

$$[w] = [\sigma] + [\nu], \quad [\sigma] \in S \cap M_w^\Omega,\ [\nu] \in M_w^{\Omega,-}.$$

Then

$$[\sigma] = [w] - [\nu] \in M_w + M_w^{\Omega,-} = M_{sw}^-.$$

Since $S \cap M_{sw}^- = 0$, the equation above implies $[w] \in M_w^{\Omega,-}$, a contradiction. □

To follow the approach of Section 3, we would need a method for associating to N_w^Ω a module corresponding to a principal series representation of a rank one group. The method used in Section 3 is the Jacquet functor. In the theory of p-adic Banach space representations, we still do not have a functor that plays the role of the Jacquet functor. Such a functor is defined for certain locally analytic representations in [8] and for mod p representations in [9].

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