Title:
Filtrations of smooth principal series and Iwasawa modules

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FILTRATIONS OF SMOOTH PRINCIPAL SERIES AND IWASAWA MODULES

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Dedicated to Prof. Freydoon Shahidi on the occasion of his 70th birthday

Abstract. Let $G$ be a reductive $p$-adic group. We consider the general question of whether the reducibility of an induced representation can be detected in a “co-rank one” situation. For smooth complex representations induced from supercuspidal representations, we show that a sufficient condition is the existence of a subquotient that does not appear as a subrepresentation. An important example is the Langlands’ quotient. In addition, we study the same general question for continuous principal series on $p$-adic Banach spaces. Although we do not give an answer in this case, we describe a related filtration on the corresponding Iwasawa modules.

Keywords: Parabolically induced representations, Iwasawa modules, $p$-adic groups.


1. Introduction

In this paper, we study two types of principal series representations of a $p$-adic group $G$: smooth principal series on complex vector spaces and continuous principal series on $p$-adic Banach spaces. Although the properties of continuous and smooth principal series are fundamentally different, we are able to treat both types of principal series in a uniform manner, using a filtration of the group $G$.

Our main motivation is the understanding of continuous principal series, which are important in the $p$-adic Langlands program [3, 7]. Smooth principal series are well-understood. Still, we obtain a new result about reducibility; namely, we show how the question of reducibility can be reduced to determining rank one reducibility (Theorem 3.4). Furthermore, we extend this result to
a more general case of representations induced from supercuspidal representations (Theorem 4.1).

In Section 2, we study the filtration of $G$ coming from the partial order on the Weyl group. We apply this to the smooth principal series representations in Section 3. In Section 4, we study representations induced from supercuspidals and prove a criterion for reducibility (Theorem 4.1). Our approach to continuous principal series is based on the duality theory developed by Schneider and Teitelbaum in [16], which relates Banach space representations to Iwasawa modules. In Section 5, we prove a technical lemma on Iwasawa algebras. Section 6 is a brief review of the duality of [16] applied to continuous principal series. In Section 7, we describe filtrations of Iwasawa modules. We expect that these filtrations could be used in determining the reducibility of principal series. A conjecture about this difficult problem was formulated by Schneider in [14].

2. Partial orders on $W$ and $[W/W_{\Omega}]$

Let $F$ be a nonarchimedean local field and $G$ the group of $F$-rational points of a connected reductive group defined over $F$. Fix a maximal split torus $T$ in $G$ and a minimal parabolic subgroup $P$ containing $T$. Let $\Delta$ be the corresponding set of simple roots. For $\Omega \subseteq \Delta$, we denote by $P_{\Omega} = M_\Omega U_\Omega$ the standard parabolic subgroup corresponding to $\Omega$. The minimal parabolic corresponds to $\emptyset \subseteq \Delta, P = P_\emptyset$.

Let $W$ be the Weyl group of $G$. For $w \in W$, let $C(w)$ denote the double coset $PwP$. The closure of $C(w)$ with respect to the locally compact topology is equal to its relative closure in the Zarisky topology ([5, Proposition 21.27]) and is described as follows.

**Theorem 2.1 ([5, Theorem 21.26]).** Let $w \in W$ and $w = s_1 \cdots s_n$ be a reduced decomposition of $w$. Then the set

$$A_w = \{s_{i_1} \cdots s_{i_m} \mid m \in \mathbb{N}, 1 \leq i_1 < \cdots < i_m \leq n\}$$

depends only on $w$, not on the reduced decomposition, and we have

$$C(w) = \bigcup_{v \in A_w} C(v).$$

We mention here that if $x \in A_w$, then the decomposition $x = s_{i_1} \cdots s_{i_m}$ from the previous theorem is not necessarily a reduced decomposition. However, we can reduce it further to obtain a reduced decomposition of $x$.

Let $\Omega \subseteq \Delta$ and $[W/W_{\Omega}] = \{w \in W \mid w\Omega > 0\}$. Then $G$ has the disjoint union decomposition $G = \bigcup_{w \in [W/W_{\Omega}]} PwP_{\Omega}$. Define a partial order on $[W/W_{\Omega}]$ as follows: $x \leq_{\Omega} y$ if $PxP_{\Omega}$ is contained in the closure of $PyP_{\Omega}$. In the special case when $\Omega = \emptyset$, we obtain a partial order on $W$ and we denote it simply by
Lemma 2.2. Let $\Omega = \{\alpha\}$. If $x, y \in [W/W_\Omega]$ then

\[ x \leq y \iff x \in A_y. \]

Proof. Let $s = s_\alpha$ and $Q = P_\Omega = P \cup PsP$. We have $PyQ = C(ys) = \bigcup_{v \in A_y} C(v)$. If $x \leq y$, then $x \in A_y$, so $xs \in A_ys$. It follows $PxQ \subseteq PyQ$, so $x \leq y$.

Conversely, assume $x \leq \Omega y$. Then $xs \in A_ys$. Let $y = s_1 \cdots s_n s_{n+1}$, where $s_{n+1} = s$, is a reduced decomposition of $ys$. Write $x = s_1 \cdots s_m$ as in Theorem 2.1, $1 \leq i_1 < \cdots < i_m \leq n + 1$, and assume that the decomposition is reduced. Then $s_{i_m} \neq s$, so $i_m < n + 1$ and $x \leq y$. \(\square\)

For each $w \in [W/W_\Omega]$, define

\[
G^\Omega_w = \bigcup_{x \in [W/W_\Omega]} \text{PxP}, \quad G_w = G^\emptyset_w, \\
G^{\Omega,+}_w = \bigcup_{x \in [W/W_\Omega]} \text{PxP}, \quad G^+_w = G^{\emptyset,+}_w.
\]

Lemma 2.3. Let $\Omega = \{\alpha\}$ and $s = s_\alpha$. If $w \in [W/W_\Omega]$, then

1. $G^\Omega_w = G_w$,
2. $G_w^+ = G_w^+ \cup G^{\Omega,+}_w$,
3. $G_{ws} \cap G^{\Omega,+}_w = G^+_w$.

Proof. (1) Clearly, $G^\Omega_w \subseteq G_w$. For converse inclusion, let $y \geq w$, so $w \in A_y$. If $ys > y$, then $y \in [W/W_\Omega]$. It follows $y \geq \Omega w$ and $PyQ \subseteq G^\Omega_w$.

If $ys < y$ then there exists a reduced decomposition $y = s_1 \cdots s_n$ such that $s_n = s$. Let $z = s_1 \cdots s_{n-1}$. Then $z \in [W/W_\Omega]$. Write $w = s_1 \cdots s_m$ as in Theorem 2.1, and assume that the decomposition is reduced. Then $i_m < n$, so $w \in A_z$ and $z \geq w$. It follows that $PzQ \subseteq G^\Omega_w$, so $PyQ = PzsP \subseteq G^\Omega_w$.

(2) Clearly, $G_{ws} \cup G^{\Omega,+}_w \subseteq G^+_w$. For converse inclusion, the proof goes along the same lines as in (1), with inequalities replaced by strict inequalities. The only difference is that in the case $ys < y$, we have to include $z = w$ since $y = ws > w$.

(3) Note that

\[ PyQ \subseteq G^{\Omega,+}_w \iff \begin{cases} y \in [W/W_\Omega], \quad y > w, \quad \text{or,} \\
y = zs, \quad z \in [W/W_\Omega], \quad z > w. \end{cases} \]

In both cases, $y \neq ws$. It follows $G_{ws} \cap G^{\Omega,+}_w \subseteq G^+_w$. 
For the converse inclusion, let \( y > ws \). Then \( y > w \). If \( y \in [W/W_{\Omega}] \), then \( y >_\Omega w \), so \( PyP \subseteq G^\Omega_w \). Otherwise, \( y = zs, z \in [W/W_{\Omega}] \). Similarly as in (1), we get \( z > w \), and consequently \( PyP \subseteq PzQ \subseteq G^\Omega_w \). This proves \( G^+_ws \subseteq G^\Omega_w \). Now, \( G^+_ws \subseteq G_ws \) implies \( G^+_ws \subseteq G_ws \cap G^\Omega_w \), finishing the proof. \qed

3. Filtrations on smooth principal series

Let \( \mathcal{R}(G) \) denote the category of smooth representations of \( G \) on complex vector spaces. We denote by \( \text{Ind}^G_{P_0} : \mathcal{R}(M_\Omega) \to \mathcal{R}(1G) \) the functor of normalized parabolic induction ([4, 6]).

Let \( \chi \) be a smooth character of \( P = P_0 = MU \). Let \( I = I(\chi) \) denote the space of the representation \( \text{Ind}^G_{P_0}(\chi) \). It is the space of all functions \( f : G \to \mathbb{C} \) such that

(i) \( f(muq) = \chi(m)\delta_P^{1/2}(m)f(g) \) for all \( m \in M, u \in U, g \in G \), and

(ii) there exists a compact open subgroup \( K_f \) of \( G \) such that \( f(gk) = f(g) \)

for all \( g \in G, k \in K_f \).

The group \( G \) acts on \( I \) by the right regular action. Define

\[
I_w = \{ f \in I \mid \text{supp} f \subseteq G_w \}, \quad I^\Omega_w = \{ f \in I \mid \text{supp} f \subseteq G^\Omega_w \}, \\
I^+_w = \{ f \in I \mid \text{supp} f \subseteq G^+_w \}, \quad I^{\Omega,+}_w = \{ f \in I \mid \text{supp} f \subseteq G^{\Omega,+}_w \}, \\
J_w = I_w/I^+_w, \quad J^\Omega_w = I^\Omega_w/I^{\Omega,+}_w.
\]

Lemma 3.1. Let \( \Omega = \{ \alpha \} \). If \( w \in [W/W_{\Omega}] \), then

(1) \( I^\Omega_w = I_w \),
(2) \( I^+_w = I_{ws} + I^{\Omega,+}_w \),
(3) \( I_{ws} \cap I^{\Omega,+}_w = I^{\Omega}_w \).

Proof. (1) and (3) follow immediately from Lemma 2.3. For (2), let

\( X = \{ x \in W \mid C(x) \subseteq G^\Omega_w, C(ws) \not\subseteq C(x) \} \).

The second condition is equivalent to \( ws \not\subseteq x \). Since the double cosets are disjoint, it is also equivalent to \( C(ws) \cap \overline{C(x)} = \emptyset \). Let

\( \mathcal{X} = \bigcup_{x \in \mathcal{X}} C(x) \).

Let \( K \) be a maximal compact subgroup such that \( G = PK \). Set \( H = \overline{C(ws)} \). Since \( PH = H \), it follows \( H = P(H \cap K) \).

Let \( f \in I^+_w \). In order to write \( f \) as \( f = f' + f'' \), where \( f' \in I_{ws} \) and \( f'' \in I^{\Omega,+}_w \), we will construct a function \( f' \) from the induced space which coincides with \( f \) on \( C(ws) \) and whose support is disjoint from \( \mathcal{X} \) and from \( H \setminus C(ws) \). Note that such a function has to be invariant under the action of a compact open subgroup of \( G \). We start our construction with a compact open subgroup \( K_f \) such that \( f(gk) = f(g) \) for all \( g \in G, k \in K_f \). The definition of \( \mathcal{X} \) implies that for every \( y \in C(ws) \) there exists a compact open subgroup \( K'_y \) such that
Let \( K_y = K'_y \cap K_f \). Then \( yK_y \cap X = \emptyset \). For \( y \in H \setminus C(ws) \), set \( K_y = K_f \). Then \( f(yk) = 0 \) for all \( y \in H \setminus C(ws) \), \( k \in K_y \). The set \( \{yK_y \mid y \in H \cap K\} \) is a cover of the compact subset \( H \cap K \), so it has a finite subcover \( \{yK_y \mid y \in Y\} \). Let

\[
Y = \bigcup_{y \in Y} yK_y.
\]

Denote by \( f' \) the product of \( f \) with the characteristic function of \( P \). Then \( f' \) is invariant under the action of the compact open subgroup \( \bigcap_{y \in Y} K_y \). We have \( f' \in I \), \( \text{supp} f' \subseteq G^+_w \) and \( \text{supp} f' \cap X = \emptyset \). It follows \( f' \in J_{ws} \). Furthermore, \( f'' = f - f' \in I_{ws}^{\Omega,+} \). This implies \( I_{ws}^{\Omega,+} \subseteq I_{ws} + I_{ws}^{\Omega,+} \). The converse inclusion is obvious. \( \square \)

**Lemma 3.2.** Let \( \Omega = \{\alpha\} \). If \( w \in [W/W_\Omega] \), then \( J_{ws}^{\Omega}/J_{ws} \cong J_w \).

**Proof.** Using Lemma 3.1, we get

\[
J_{ws} = I_{ws}/I_{ws}^+ = I_{ws}/(I_{ws} \cap I_{ws}^{\Omega,+}) \cong (I_{ws} + I_{ws}^{\Omega,+})/I_{ws}^{\Omega,+} = I_{ws}^{\Omega,+}/I_{ws}^{\Omega,+} = I_{ws}/I_{ws}^{\Omega,+}.
\]

Therefore, \( J_{ws} \) embeds into \( J_w^{\Omega} \). Moreover,

\[
J_{ws}^{\Omega}/J_{ws} = (I_{ws}^{\Omega,+}/I_{ws}^{\Omega,+})/(I_{ws}^{\Omega,+}/I_{ws}^{\Omega,+}) \cong I_{ws}^{\Omega}/I_{ws}^{\Omega,+} = I_{ws}/I_{ws}^{\Omega,+} = J_w.
\]

\( \square \)

We obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \downarrow & \downarrow & \downarrow & 0 \\
0 & \rightarrow & I_{ws}^{+} & \rightarrow & I_{ws} & \rightarrow & J_{ws} & \rightarrow & 0 \\
0 & \rightarrow & I_{ws}^{\Omega,+} & \rightarrow & I_{ws}^{\Omega} & \rightarrow & J_{ws}^{\Omega} & \rightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & 0 & & & & 0.
\end{array}
\]

Suppose \( I \) is a proper subrepresentation of \( I \). Define \( I_{ws}^{\Omega} = I \cap I_{ws}^{\Omega} \), \( I_{ws}^{\Omega,+} = I \cap I_{ws}^{\Omega,+} \) and \( J_{ws}^{\Omega} = I_{ws}^{\Omega}/I_{ws}^{\Omega,+} \). We want to select \( w \) and \( \alpha \) so that \( J_{ws}^{\Omega} \) is a proper nonzero submodule of \( J_{ws}^{\Omega} \). Furthermore, we want to pass to representations of \( M_{(\alpha)} \), and obtain from \( J_{ws}^{\Omega} \) a proper subrepresentation of a principal series of \( M_{(\alpha)} \). For this, we use Jacquet modules of parabolically induced representations \( ([4, 6]) \). Since the Jacquet module of \( J_{ws}^{\Omega} \) is precisely a principal series of \( M_{(\alpha)} \), we select \( w \) and \( \alpha \) by considering directly Jacquet modules (see the proof of Theorem 3.4).
For each \( w \in [W/W_\Omega] \), let \( d_\Omega(w) \) be the dimension of the algebraic variety \( P \setminus PwP_\Omega \) over \( F \). Define

\[
G^\Omega_n = \bigcup_{d_\Omega(w) \geq n} PwP_\Omega \quad \text{and} \quad I^\Omega_n = \{ f \in I \mid \text{supp } f \subseteq G^\Omega_n \}.
\]

Let \((\pi, V)\) be a smooth representation of \( G \). For a compact subgroup \( U_0 \subset U_\Omega \), define

\[
V(U_0) = \{ v \in V \mid \int_{U_0} \pi(u)vd\mu = 0 \}.
\]

Define \( V(U_\Omega) \) to be \( \bigcup V(U_0) \), the union over all compact open subgroups \( U_0 \) of \( U_\Omega \). Then \( V(U_\Omega) \) is a subspace of \( V \) [6]. Define \( r^G_{M_\Omega}(V) = V/V(U_\Omega) \). The normalized Jacquet module \( r^G_{M_\Omega}(\pi) \) is the representation of \( M_\Omega \) on the space \( r^G_{M_\Omega}(V) \) given by

\[
r^G_{M_\Omega}(\pi)(m)(v + V(U_\Omega)) = \delta_{P_0}^{-1/2}(m)\pi(m)v + V(U_\Omega),
\]

\( m \in M_\Omega, \ v \in V \) [4]. If \( W \subset V \) is a \( P_\Omega \)-invariant subspace, then \( r^G_{M_\Omega}(V) \) is \( M_\Omega \)-invariant.

**Theorem 3.3** ([6, Theorem 6.3.5]). Let \( P = P_\emptyset = MU \) and \( \Omega \subseteq \Delta \). Let \( \chi \) be a smooth character of \( M, I = \text{Ind}_P^G \chi \). There exists a filtration

\[
0 \subset I^\Omega_{\alpha_1} \subset \cdots \subset I^\Omega_0 = I
\]

by \( P_\Omega \)-stable subspaces such that \( r^G_{M_\Omega}(I^\Omega_{\alpha}/I^\Omega_{\alpha+1}) \cong r^G_{M_\Omega}(I^\Omega_{\alpha})/r^G_{M_\Omega}(I^\Omega_{\alpha+1}) \) is isomorphic to the direct sum \( \bigoplus r^G_{M_\Omega}(J^\Omega_{w}) \), the sum ranging over \( w \in [W/W_\Omega] \) with \( d_\Omega(w) = n \). Furthermore, the normalized Jacquet module \( r^G_{M_\Omega}(J^\Omega_{w}) \) is given by

\[
r^G_{M_\Omega}(J^\Omega_w) \cong \text{Ind}^{M_\Omega}_{w^{-1}PwP_\Omega \cap M_\Omega} \chi.
\]

The proof of the next theorem, as well as the proof of Theorem 4.1, is improved and shortened, as suggested by the referee.

**Theorem 3.4.** Let \( P = P_\emptyset = MU \). Let \( \chi \) be a character of \( M \) and \( I(\chi) = \text{Ind}_P^G \chi \). Suppose that \( I(\chi) \) has an irreducible subquotient which does not appear as a subrepresentation in \( I(\chi) \). Then there exist \( \alpha \in \Delta \) and \( w \in W \) such that \( \text{Ind}^{M(\alpha)}_{P\cap M(\alpha)} w^{-1} \chi \) is reducible.

*Proof.* Denote by \( I \) an irreducible subquotient of \( I(\chi) \) which does not appear as a subrepresentation in \( I(\chi) \). Then there exists \( w' \in W \) such that \( I \) is a subrepresentation of \( I(w'\chi) \). Let \( \chi' = w'\chi, I = I(\chi') \). Define

\[
W_\chi = \{ v \in W \mid \text{Hom}_M(r^G_M(I), v^{-1}\chi') = 0 \}.
\]

Since \( I \) does not appear as a subrepresentation of \( I(\chi) \), \( \text{Hom}_M(r^G_M(I), \chi) = 0 \) so \( w' \in W_\chi \). Moreover, \( 1 \notin W_\chi \) because \( I \) is a subrepresentation of \( I \). Let

\[
W_{\chi'} = \{ v \in W \mid \text{Hom}_M(r^G_M(I), v^{-1}\chi') = 0 \}.
\]

The proof is then completed as before,
$w_0$ be a minimal element in $W_T$. Here, we consider the partial order on $W$ as defined in section 2. Since $w_0 \neq 1$, there exists $\alpha \in \Delta$ such that $w_0 \alpha < 0$. Let

$$s = s_\alpha, \quad w = w_0 s, \quad \Omega = \{\alpha\}, \quad P_\Omega = P \cup PsP.$$  

Then $w \in [W/W_\Omega]$. In addition, $w < w_0$, so $w \notin W_T$. The Frobenius reciprocity gives us

$$0 \neq \text{Hom}_M(r_M^G(I), w^{-1} \chi') = \text{Hom}_M(r_M^{M_\Omega} \circ r_M^G(I), w^{-1} \chi')$$
$$\cong \text{Hom}_{M_\Omega}(r_M^{M_\Omega}(I), \text{Ind}_{P \cap M_\Omega}^{M_\Omega}(w^{-1} \chi')).$$

In the same way, $0 = \text{Hom}_{M_\Omega}(r_M^{M_\Omega}(I), \text{Ind}_{P \cap M_\Omega}^{M_\Omega}(sw^{-1} \chi'))$. Hence,

$$\text{Ind}_{P \cap M_\Omega}^{M_\Omega}(w^{-1} \chi') \nsimeq \text{Ind}_{P \cap M_\Omega}^{M_\Omega}(sw^{-1} \chi').$$

Since these representations have the same composition factors ([6, Theorem 6.3.11]), if irreducible, they would be isomorphic. It follows that $\text{Ind}_{P \cap M_\Omega}^{M_\Omega}(w^{-1} \chi') = \text{Ind}_{P \cap M_\Omega}^{M_\Omega}(w^{-1} w' \chi)$ is reducible.

**Remark 3.5.** Theorem 3.4 does not remain true if $I(\chi)$ is reducible, but we remove the hypothesis that $I(\chi)$ has a subquotient which does not appear as a subrepresentation in $I(\chi)$. The following example was communicated to us by Marko Tadić. Let $\chi_1, \chi_2$ be two different characters of $F^\times$ of order 2. The unitary principal series representation of $GSp(6, F)$ induced from $\chi_1 \otimes \chi_2 \otimes \chi_1 \chi_2$ has length 2, but we have irreducibility for all rank one Levi subgroups, and even for all rank two Levi subgroups (because unitary principal series for $GL(3, F)$ and $GSp(4, F)$ are irreducible [13]).

**Example 3.6 (Langlands’ Quotients).** Let $\chi_0$ be a character of $M = M_\emptyset$ which is square-integrable modulo center. For $\Omega \subset \Delta$, the induced representation $\text{Ind}_{M_\Omega}^{M_\emptyset}(\chi_0)$ decomposes as a direct sum of tempered representations, $\text{Ind}_{M_\Omega}^{M_\emptyset}(\chi_0) = \bigoplus \tau_i$. Let $\nu$ be an unramified character of $M_\Omega$ which takes positive real values and is strictly positive (in Weyl chamber). Then $\text{Ind}_{M_\Omega}^{G_\emptyset}(\nu \tau_i)$ has a unique irreducible quotient $\pi$ (Langlands’ quotient). It appears with multiplicity one in $\text{Ind}_{M_\Omega}^{G_\emptyset}(\nu \tau_i)$. Moreover, it can be shown using central exponents and uniqueness of Langlands’ data that $\pi$ does not appear as a subquotient in $\text{Ind}_{M_\Omega}^{G_\emptyset}(\nu \tau_j)$ for any $\tau_j \neq \tau_i$.

Assume that $\text{Ind}_{M_\Omega}^{G_\emptyset}(\nu \tau_i)$ is reducible. Corresponding to $\nu$ is a character of $M$, which we will denote by the same letter $\nu$, such that $\text{Ind}_{M_\emptyset}^{G_\emptyset}(\nu \tau_i)$ is a subrepresentation of $\text{Ind}_{M_\emptyset}^{G_\emptyset}(\nu \chi_0)$. Then $\pi$ does not appear as a subrepresentation in $\text{Ind}_{M_\emptyset}^{G_\emptyset}(\nu \chi_0)$ and we can apply Theorem 3.4. It follows that there exist $\alpha \in \Delta$ and $w \in W$ such that $\text{Ind}_{M_\emptyset}^{M_\emptyset}(w^{-1} \chi_0)$ is reducible.
4. Generalization

In this section, we generalize Theorem 3.4.

**Theorem 4.1.** Let $\Theta \subseteq \Delta$. Let $P_\Theta = M_\Theta U_\Theta$. Let $\sigma$ be an irreducible supercuspidal representation of $M_\Theta$ and $I(\sigma) = \text{Ind}^G_{P_\Theta} \sigma$. Suppose that $I(\sigma)$ has an irreducible subquotient which does not appear as a subrepresentation in $I(\sigma)$. Then there exists $w_0 \in W$ and $\Omega \subseteq \Delta$ such that $\Omega = w_0(\Theta) \cup \{\alpha\}$ and $\text{Ind}^{\text{M}_\Omega}_{w_0(\text{M}_\Theta)} w_0 \sigma$ is reducible.

We give a brief proof. For more detail, see [1].

**Proof.** Set $W(\Theta, \Theta) = \{w \in W \mid w(\Theta) = \Theta\}$. Denote by $I$ an irreducible subquotient of $I(\sigma)$ which does not appear as a subrepresentation in $I(\sigma)$. Then there exists $w' \in W(\Theta, \Theta)$ such that $I$ is a subrepresentation of $I(w' \sigma)$. Let $\sigma' = w' \sigma$, $I = I(\sigma')$. Define

$$W_{\Theta} = \{v \in W \mid v(\Theta) \subseteq \Delta \text{ and } \text{Hom}_{v(\text{M}_\Theta)}(r^G_{v(\text{M}_\Theta)}(I), \upsilon \sigma') = 0\}.$$  

Similarly as in the proof of Theorem 3.4, we see that $(w')^{-1} \in W_{\Theta}$ and $1 \notin W_{\Theta}$. Let $w_{\min}$ be an element of $W_{\Theta}$ of minimum length and $\Theta' = w_{\min}(\Theta)$. Now, we apply [17, Lemma 2.1.2]. In particular, let $\Theta = \Theta_1, \ldots, \Theta_n = \Theta'$ be a sequence of associate subsets of $\Delta$ as in [17, Lemma 2.1.2]. Then for any $1 \leq i \leq n - 1$ there exists a simple root $\alpha_i$ such that $\Theta_{i+1}$ is the conjugate of $\Theta_i$ in $\Theta_i \cup \{\alpha_i\}$. We have $w_{\min} = w_{n-1} \ldots w_1$, where $w_i \in W(\Theta_i, \Theta_{i+1})$. Set $y = w_{n-1}$ and $w = w_{n-2} \ldots w_1$. By minimality, $w \notin W_{\Theta}$ and we have

$$\text{Hom}_{w(\text{M}_\Theta)}(r^G_{w(\text{M}_\Theta)}(I), \upsilon \sigma') \neq 0,$$

and

$$\text{Hom}_{y w(\text{M}_\Theta)}(r^G_{y w(\text{M}_\Theta)}(I), y w \sigma') = 0.$$  

Let $\alpha_1, \ldots, \alpha_{n-1}$ be as in [17, Lemma 2.1.2]. Set $\alpha = \alpha_{n-1}$ and $\Omega = w(\Theta) \cup \{\alpha\}$. Then $y w(\Theta)$ is the conjugate of $w(\Theta)$ in $\Omega$. The Frobenius reciprocity gives us

$$\text{Hom}_{w(\text{M}_\Theta)}(r^G_{w(\text{M}_\Theta)}(I), \upsilon \sigma') = \text{Hom}_{w(\text{M}_\Theta)}(r^G_{w(\text{M}_\Theta)}(I), \upsilon \sigma') \cong \text{Hom}_{w(\text{M}_\Theta)}(r^G_{M_\Omega}(I), \text{Ind}^{\text{M}_\Omega}_{w(\text{M}_\Theta)}(w \sigma')) \neq 0.$$  

In the same way, $\text{Hom}_{M_\Omega}(r^G_{M_\Omega}(I), \text{Ind}^{\text{M}_\Omega}_{y w(\text{M}_\Theta)}(y w \sigma')) = 0$. Hence,

$$\text{Ind}^{\text{M}_\Omega}_{w(\text{M}_\Theta)}(w \sigma') \neq \text{Ind}^{\text{M}_\Omega}_{y w(\text{M}_\Theta)}(y w \sigma').$$  

As in the proof of Theorem 3.4, it follows that these representations are reducible. Hence, $\text{Ind}^{\text{M}_\Omega}_{w(\text{M}_\Theta)}(w w' \sigma) = \text{Ind}^{\text{M}_\Omega}_{w(\text{M}_\Theta)}(w \sigma')$ is reducible. Note that reducibility implies that $y w(\Theta) = w(\Theta)$. To complete the proof, we may select $w_0$ to be either $w w'$ or $y w w'$.

Define $W(\sigma) = \{w \in W(\Theta, \Theta) \mid w \sigma \cong \sigma\}$ and call $\sigma$ regular if $W(\sigma) = \{1\}$.

**Corollary 4.2.** Let $P_\Theta = M_\Theta U_\Theta$. Let $\sigma$ be an irreducible supercuspidal representation of $M_\Theta$. Suppose that $\sigma$ is regular. Then $I(\sigma) = \text{Ind}^G_{P_\Theta} \sigma$ is reducible.
if and only if there exist $w_0 \in W$ and $\Omega \subseteq \Delta$ such that $\Omega = w_0(\Theta) \cup \{\alpha\}$ and $\text{Ind}_{w_0(M_\Theta)} w_0 \sigma$ is reducible.

Proof. Suppose that $\sigma$ is regular. Then $r^G_{M_\Theta}(I(\sigma)) = \bigoplus_{w \in W(\Theta, \Theta)} w \sigma$ is a direct sum of mutually inequivalent components ([6, Proposition 6.4.1]). Since the Jacquet functor is exact, it follows that every component of $I(\sigma)$ appears with multiplicity one. Furthermore, if $V$ is an irreducible subrepresentation of $I(\sigma)$, then the Frobenius reciprocity gives us $\text{Hom}_G(V, I(\sigma)) \cong \text{Hom}_{M_\Theta}(r^G_{M_\Theta}(V), \sigma)$. Since the multiplicity of $\sigma$ in $r^G_{M_\Theta}(I(\sigma))$ is one, it follows that $I(\sigma)$ has a unique irreducible subrepresentation. Therefore, if $I(\sigma)$ is reducible, it satisfies the conditions of Theorem 4.1. □

In the case of principal series, the previous corollary also follows from Rodier’s work on the principal series induced from regular characters [12].

5. Iwasawa algebras

We start by reviewing some results on projective limits. We refer to [11] for definitions of a projective system and a projective limit. The following two propositions follow from Proposition 1.1.3, Proposition 1.1.4 and Corollary 1.1.8 of [11].

Proposition 5.1. Let $(X_i)$ be a projective system of compact Hausdorff topological spaces over the directed set $I$, and let $X = \text{proj lim } X_i$.

(a) If $X_i$ is totally disconnected, for all $i \in I$, then $X$ is also a compact Hausdorff totally disconnected topological space.

(b) If $X_i$ is nonempty, for all $i \in I$, then $X$ is also nonempty.

Proposition 5.2. Let $(X_i)$ be a projective system of compact Hausdorff spaces, $X = \text{proj lim } X_i$, and let $\varphi_i : X \to X_i$ be the projections.

(a) If $Y$ is a closed subspace of $X$, then $Y = \text{proj lim } \varphi_i(Y)$.

(b) If $Y$ is a subspace of $X$, then $\overline{Y} = \text{proj lim } \varphi_i(Y)$, where $\overline{Y}$ is the closure of $Y$ in $X$.

(c) If $Y$ and $Y'$ are subspaces of $X$ and $\varphi_i(Y) = \varphi_i(Y')$ for each $i$, then their closures in $X$ coincide: $\overline{Y} = \overline{Y'}$.

Next, we review the definition and basic properties of Iwasawa algebras ([10], [15]). Let $H$ be a profinite group. Let $N(H)$ denote the family of all open normal subgroups of $H$. Then $H = \text{proj lim }_{N \in N(H)} H/N$ is a projective limit, as a topological group, of the finite groups $H/N$. Let $K$ be a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_K$ its ring of integers. The group rings $\mathcal{O}_K[H/N]$, $N \in N(H)$, form a projective system of rings. The Iwasawa algebra of $H$ over $\mathcal{O}_K$ is defined as $\mathcal{O}_K[[H]] = \text{proj lim }_{N \in N(H)} \mathcal{O}_K[H/N]$. 
We equip $o_K[[H]]$ with the projective limit topology. Then $o_K[[H]]$ is a torsion free and compact linear-topological $o_K$-module. It has a structure of a topological ring; the ring multiplication is continuous. The inclusion map $H \hookrightarrow o_K[[H]]$ is a homeomorphism onto its image.

Define $K[[H]] = K \otimes_{o_K} o_K[[H]]$, endowed with the finest locally convex topology such that the inclusion of $o_K[[H]]$ is continuous. Then the multiplication on $K[[H]]$ is separately continuous.

Let $A$ be a closed subset of $H$. For $N \in \mathcal{N}(H)$, define $A_N = \{aN \mid a \in A\} \subseteq H/N$. Let $(A_N)$ denote the $o_K$-submodule of $o_K[H/N]$ generated by $A_N$. Then $(A_N)$, $N \in \mathcal{N}(H)$, is a projective system of topological $o_K$-modules. Define

$$
\Lambda^o(A) = \operatorname{projlim}_{N \in \mathcal{N}(H)} (A_N) \quad \text{and} \quad \Lambda(A) = K \otimes_{o_K} \Lambda^o(A).
$$

Then $\Lambda^o(A)$ is a closed $o_K$-submodule of $o_K[[H]]$ and $\Lambda(A)$ is a closed $K$-subspace of $K[[H]]$.

**Lemma 5.3.** Let $A, B$ be closed subsets of $H$.

(a) $\Lambda^o(A)$ is a compact Hausdorff totally disconnected topological space.

(b) $\Lambda^o(A \cup B) = \Lambda^o(A) + \Lambda^o(B)$, and $\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)$.

(c) $\Lambda^o(A \cap B) = \Lambda^o(A) \cap \Lambda^o(B)$, and $\Lambda(A \cap B) = \Lambda(A) \cap \Lambda(B)$.

**Proof.** (a) follows from Proposition 5.1. (b) follows from Proposition 5.2, because

$$(A_N \cup B_N) = (A_N) + (B_N) \quad \text{for all } N \in \mathcal{N}(H).$$

(c) Set $C = A \cap B$. Since

$$(C_N) \subseteq (A_N) \cap (B_N) \quad \text{for all } N \in \mathcal{N}(H),$$

we immediately get $\Lambda^o(C) \subseteq \Lambda^o(A) \cap \Lambda^o(B)$. Assume $\Lambda^o(C) \neq \Lambda^o(A) \cap \Lambda^o(B)$. Then there exists $\mu \in \Lambda^o(A) \cap \Lambda^o(B)$ such that $\mu \notin \Lambda^o(C)$. Write

$$
\mu = (\mu_N)_{N \in \mathcal{N}(H)}, \quad \mu_N \in o_K[H/N].
$$

Then there exists $N_0 \in \mathcal{N}(H)$ such that $\mu_{N_0} \notin (C_{N_0})$. Write

$$
\mu_{N_0} = \alpha_1 a_1 N_0 + \cdots + \alpha_k a_k N_0 = \alpha_1 b_1 N_0 + \cdots + \alpha_k b_k N_0,
$$

where $\alpha_i \in o_K$, $a_i \in A$, $b_i \in B$, and $a_i N_0 = b_i N_0$ for all $i$. We can decompose $\mu$ as

$$
\mu = \mu^1 + \cdots + \mu^k, \quad \mu^i \in \Lambda^o(a_i N_0).
$$

Note that $\mu^i_{N_0} = \alpha_i a_i N_0$. Select $\ell \in \{1, \ldots, k\}$ such that $\alpha_i a_i N_0 \notin (C_{N_0})$. Then $a_i N_0 \neq c N_0$ for any $c \in C$. Let $\lambda = \mu^\ell$. Then $\lambda \in \Lambda^o(a_i N_0) = \Lambda^o(b_i N_0)$ and $\lambda \notin \Lambda^o(C)$.

For any $c \in C$ there exists $N_c \in \mathcal{N}(H)$ such that $a_i N_0 \cap c N_c = \emptyset$. Then $(c N_c \mid c \in C)$ is an open cover of $C$. The set $B_1 = B \setminus \bigcup_{c \in C} c N_c$ is closed, and disjoint from $A$. For any $b \in B_1$ there exists $N_b \in \mathcal{N}(H)$ such that $A \cap b N_b = \emptyset$. Then $(b N_b \mid b \in B_1)$ is an open cover of $B_1$. By compactness, it
has a finite subcover. It follows that we can find \( N_1 \in \mathcal{N}(H) \), \( N_1 \leq N_0 \), such that \( A \cap bN_1 = \emptyset \) for all \( b \in B_1 \). Now, write
\[
\lambda_{N_1} = \alpha'_1N_1 + \cdots + \alpha'_sN_1.
\]
Since \( \alpha'_1 + \cdots + \alpha'_s = \alpha_\ell \neq 0 \), at least one coefficient \( \alpha'_i \) is not zero. Since \( \alpha'_i \notin bN_1 \) for any \( b \in B \), we have \( \alpha'_iN_1 \notin \langle B_{N_1} \rangle \). This contradicts \( \lambda \in A^\ell(bN_0) \).

\[\square\]

6. Continuous principal series

From now on, \( F \) is a finite extension of \( \mathbb{Q}_p \), and \( K \) is a finite extension of \( F \). As before, \( P = P_0 \) is a minimal parabolic subgroup of \( G \). Let \( \chi : P \to K^\times \) be a continuous character. Let
\[\text{Ind}_{P\!}^G(\chi^{-1}) = \{ f : G \to K \text{ continuous} \mid f(gp) = \chi(p)f(g) \forall p \in P, g \in G \},\]
where \( G \) acts on the left by \( g \cdot f(h) = f(g^{-1}h) \). Here, we take the left action because we will use the duality \([16]\).

Let \( G_0 \subset G \) be a maximal compact subgroup which satisfies the Iwasawa decomposition \( G = G_0P \). If \( P_0 = P \cap G_0 \) and \( \chi_0 = \chi|_{P_0} \), then restriction gives an isomorphism \( \text{Ind}_{P_0}^{G_0} (\chi^{-1}) \cong \text{Ind}_{P_0}^{G_0} (\chi_0^{-1}) \).

Let \( K(\chi_0) \) denote the one dimensional representation of \( P_0 \) on \( G \) given by \( \chi_0 \). The continuous dual of \( \text{Ind}_{P_0}^{G_0} (\chi_0^{-1}) \) is isomorphic to
\[\text{M}(\chi_0) = K[[G_0]] \otimes_{K[[P_0]]} K(\chi_0).\]
The isomorphism \( \text{Ind}_{P_0}^{G_0} (\chi^{-1}) \cong \text{Ind}_{P_0}^{G_0} (\chi_0^{-1}) \) induces a \( G \)-module structure on \( \text{M}(\chi_0) \). We denote this \( G \)-module by \( \text{M}(\chi_0) \). Hence, \( \text{M}(\chi_0) \) is a \( G \)-module and \( K[[G_0]] \)-module. It follows from Theorem 3.5 of \([16]\) that there is a bijection between \( G \)-invariant closed subspaces of \( \text{Ind}_{P_0}^{G_0} (\chi^{-1}) \) and \( G \)-invariant \( K[[G_0]] \)-quotient modules of \( \text{M}(\chi) \).

7. Iwasawa modules

Similarly to open subsets of \( G \) defined in Section 2, we define certain closed subsets of \( G \). For \( w \in W \), define
\[
H_w = \overline{PwP} = \bigcup_{x \leq w} PxP, \quad H_w^- = \bigcup_{x < w} PxP.
\]
Let \( \alpha \in \Delta, \Omega = \{ \alpha \} \), and \( s = s_\alpha \). For \( w \in [W_\Omega \setminus W] = \{ w \in W \mid w^{-1}\Omega > 0 \} \), define
\[
H_w^\Omega = \bigcup_{x \in [W_\Omega \setminus W]} P_{1\Omega}xP, \quad H_w^{\Omega,-} = \bigcup_{x \in [W_\Omega \setminus W]} P_{1\Omega}xP.
\]

For \( X \subset K[[G_0]] \), we denote by \( [X] \) the image of \( X \) in \( \text{M}(\chi) \). There exists a compact set \( C \subset G_0 \) such that \( C \) is a set of coset representatives of \( G/P \) (see
Let $M_w = \Lambda(C \cap H_w)$, $M_w^- = \Lambda(C \cap H_w^-)$, and $N_w = M_w/M_w^-$. Similarly, for $w \in [W_\Omega \setminus W]$, we define

$$ M_w^\Omega = \Lambda(C \cap H_w^\Omega), \quad M_w^\Omega^- = \Lambda(C \cap H_w^{\Omega^-}), \quad N_w^\Omega - M_w^\Omega^-.$$

**Lemma 7.1.** Let $\Omega = \{\alpha\}$ and $s = s_\alpha$. If $w \in [W_\Omega \setminus W]$, then

(a) $M_w^\Omega = M_{sw}$,
(b) $M_w \cap M_w^{\Omega^-} = M_w^-,$
(c) $M_w + M_w^{\Omega^-} = M_{sw}^-.$

**Proof.** (a) is clear, because $H_w^\Omega = H_{sw}$. For (b) and (c), note that

$$ H_w \cap H_w^{\Omega^-} = \left( \bigcup_{x \leq w} P_x P \right) \cap \left( \bigcup_{x \in [W_\Omega \setminus W]} P_{\Omega x} P \right) = H_w^- $$

and

$$ H_w \cup H_w^{\Omega^-} = \left( \bigcup_{x \leq w} P_x P \right) \cup \left( \bigcup_{x \in [W_\Omega \setminus W]} P_{\Omega x} P \right) = H_{sw}^-.$$

Lemma 5.3 implies

$$ \Lambda(C \cap H_w) \cap \Lambda(C \cap H_w^{\Omega^-}) = \Lambda((C \cap H_w) \cap (C \cap H_w^{\Omega^-})) = \Lambda(C \cap (H_w \cap H_w^{\Omega^-})) = \Lambda(C \cap H_w^-).$$

It follows $M_w \cap M_w^{\Omega^-} = M_w^-$. Similarly,

$$ \Lambda(C \cap H_w) + \Lambda(C \cap H_w^{\Omega^-}) = \Lambda((C \cap H_w) \cup (C \cap H_w^{\Omega^-})) = \Lambda(C \cap (H_w \cup H_w^{\Omega^-})) = \Lambda(C \cap H_{sw}^-)$$

gives $M_w + M_w^{\Omega^-} = M_{sw}^-$. \hfill \Box

**Lemma 7.2.** Let $\Omega = \{\alpha\}$. If $w \in [W_\Omega \setminus W]$, then $N_w^\Omega / N_w \cong N_{sw}^-.$

**Proof.** Using Lemma 7.1, we get

$$ N_w \cong M_w/M_w^- \cong M_w/(M_w \cap M_w^{\Omega^-}) \cong (M_w + M_w^{\Omega^-})/M_w^{\Omega^-} \cong M_{sw}^- / M_w^{\Omega^-}.$$

Then

$$ N_w^\Omega / N_w \cong (M_{sw} / M_{sw}^{\Omega^-}) / (M_{sw}^- / M_{sw}^{\Omega^-}) \cong M_{sw} / M_{sw}^- \cong N_{sw}^-.$$

\hfill \Box
Hence, we have the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & M_w^- & \to & M_w & \to & N_w & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_w^{\Omega,-} & \to & M_w^{\Omega} & \to & N_w^{\Omega} & \to & 0. \\
\downarrow & & & & \downarrow & & \downarrow & & \\
N_{sw} & \to & 0
\end{array}
\]

Note that \( M^{(\lambda)} \) is a cyclic \( K[[G_0]] \)-module, generated by [1]. Let \( S \) be a \( K[[G_0]] \)-submodule and \( G \)-submodule of \( M^{(\lambda)} \). Assume that \( S \neq 0 \) and \( S \neq M^{(\lambda)} \). Then \([1] \notin S \) and \( S \cap M_1 = 0 \). Moreover, there exists a minimal \( w' \in W \) such that \( S \cap M_{w'} \neq 0 \). Write \( w' = sw \), where \( s = s_\alpha \) is a simple reflection. Hence,

\[ S \cap M_{sw} \neq 0, \quad S \cap M_x = 0, \text{ for all } x < sw. \]

In particular, \( S \cap M_w = 0, S \cap M_w^{\Omega,-} = 0, S \cap M_{sw}^{-} = 0. \)

**Lemma 7.3.** \( S \cap M_w^{\Omega} \) is isomorphic to a proper submodule of \( N_w^{\Omega} \).

**Proof.** Since \( S \cap M_w^{\Omega,-} = 0 \), we have

\[
S \cap M_w^{\Omega} \cong (S \cap M_w^{\Omega})/(S \cap M_w^{\Omega,-}) \\
\cong (S \cap M_w^{\Omega}) + M_w^{\Omega,-}/M_w^{\Omega,-} \subseteq M_w^{\Omega}/M_w^{\Omega,-} = N_w^{\Omega}.
\]

We have to prove \((S \cap M_w^{\Omega}) + M_w^{\Omega,-} \neq M_w^{\Omega} \). Assume, on the contrary, that \((S \cap M_w^{\Omega}) + M_w^{\Omega,-} = M_w^{\Omega} \). Then we can write \([w] \in M_w^{\Omega} \) as

\[ [w] = [\sigma] + [\nu], \quad [\sigma] \in S \cap M_w^{\Omega}, [\nu] \in M_w^{\Omega,-}. \]

Then

\[ [\sigma] = [w] - [\nu] \in M_w + M_w^{\Omega,-} = M_{sw}^{-}. \]

Since \( S \cap M_{sw}^{-} = 0 \), the equation above implies \([w] \in M_w^{\Omega,-} \), a contradiction. \( \square \)

To follow the approach of Section 3, we would need a method for associating to \( N_w^{\Omega} \) a module corresponding to a principal series representation of a rank one group. The method used in Section 3 is the Jacquet functor. In the theory of \( p \)-adic Banach space representations, we still do not have a functor that plays the role of the Jacquet functor. Such a functor is defined for certain locally analytic representations in [8] and for mod \( p \) representations in [9].
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