ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the

Iranian Mathematical Society

in Honor of Professor Freydoon Shahidi's 70th birthday

Vol. 43 (2017), No. 4, pp. 17–54

Title:

Symmetric powers and the Satake transform

Author(s):

B. Casselman

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 4, pp. 17–54
Online ISSN: 1735-8515

SYMMETRIC POWERS AND THE SATAKE TRANSFORM

B. CASSELMAN

Abstract. This paper gives several examples of the basic functions introduced in recent years by Ngô. These are mainly conjectures based on computer experiment.

Keywords: Satake transform, basic functions.
MSC(2010): Primary: 11F70; Secondary: 22E50.

1. **Introduction**

At the core of the program laid out by Langlands many years ago are the L functions he associated to automorphic forms. In particular, one would like to know that they have meromorphic continuation and functional equation, and to know the structure of their poles. There have been so far two principal approaches to this problem. One is due to Shahidi and Kim, in which they take advantage of certain low-dimensional accidents. Shahidi has tried extending this method by looking at Kac-Moody groups of infinite dimension, but so far without success. In any case this approach, although attractive in many ways, is necessarily limited, since it relies on Whittaker models of automorphic forms, and is hence restricted to generic representations and forms.

Another approach can be found in [14]. Godement and Jacquet were loosely following [36] (who was in turn following Tate). They proved meromorphic continuation and the functional equation for the function $L(s, \pi, \sigma)$ attached by Langlands to any automorphic representation π occurring on $\mathrm{GL}(F)\backslash\mathrm{GL}(\mathbb{A})$ and the standard representation σ of $\mathrm{GL}_n(\mathbb{C})$. They did this by Fourier analysis on Schwartz spaces of the matrix algebras M_n, both local and global, combined with a global Poisson summation formula.

Up to now, neither approach appeared to be capable of further development. In recent years, however, the following question has become of new interest:

Is an approach similar to that of Godement and Jacquet possible for other groups? What would replace the matrix algebra? What would replace the Schwartz functions?

Article electronically published on August 30, 2017.
Received: 1 February 2016, Accepted: 27 October 2016.

Langlands' program *'Beyond Endoscopy'* suggests that one might answer some of these questions by using in the Trace Formula certain functions that are not of compact support. Ngô, Lafforgue, and others have proposed a particular class of such functions associated to monoids defined by Vinberg. This seems to fit well an idea of Langlands and Arthur about a Trace Formula associated to each finite-dimensional representation of an L-group. For GL_n and the standard representation, these functions would be those in $\mathcal{S}(\mathrm{M}_n)$.

There are now several characterizations of the local functions that would replace Schwartz functions on matrix algebras, at least those which are unramified (these are called **basic functions**) but none is so far very concrete. In this paper I shall describe computations of basic functions that I have made for a small number of cases. The formulas I arrive at are so far only conjectural, but there is much reason to believe that they are correct.

According to the definition of basic functions, computing them takes place in two major steps. One involves computing Satake transforms and their inverses. The other involves decomposing symmetric powers of irreducible representations of Langlands' dual groups. The latter is a classic problem, up to now resolved only in a very limited sense. One intriguing, perhaps surprising, by-product of my computer experiments is a hint that one might in the end come up with a completely explicit formula for the decomposition of symmetric powers of irreducible finite-dimensional representations of complex groups.

In the last part of this paper I'll present explicit formulas for all basic functions associated to GL_2. In this case, evaluating basic functions reduces to finding the irreducible decomposition of all symmetric powers of irreducible representations. There is a classic formula for this, which I'll explain, but I'll also present an algorithm that seems to be new. The nature of the new algorithm leads one to think that similar results might well be within reach for all semi-simple groups.

This article comes in three main parts. The first five sections are a survey of the Satake transform, with emphasis on how to compute it. The next four look at some of the results of computation. The last few look closely at GL_2, for which results are nearly complete.

Throughout, let

$$\begin{aligned} \mathfrak{k} &= \text{a } \mathfrak{p}\text{-adic field} \\ \mathfrak{o} &= \text{the ring of integers in } \mathfrak{k} \\ \mathfrak{p} &= \text{the prime ideal of } \mathfrak{o} \\ \varpi &= \text{ a generator of } \mathfrak{p} \\ \mathbb{F}_q &= \mathfrak{o}/\mathfrak{p} \end{aligned}$$

and

$$\begin{aligned} G &= \text{a split reductive group defined over } \mathfrak{k} \\ K &= G(\mathfrak{o}) \\ B &= \text{a Borel subgroup} \\ &= AN \end{aligned}$$

$$
\begin{aligned}
\Sigma &= \text{corresponding set of roots} \\
\Delta &= \text{simple roots} \\
W &= \text{Weyl group} \; .
\end{aligned}
$$

From time to time G will be a specific group.

Parts of this essay have been written at the Morningside Center in Beijing, at the Tata Institute in Mumbai, and during a conference at AIM in San Jose. I am extremely grateful to all of these institutions. I also wish to thank Wen-wei Li, Ali Altug, and Stavros Kousidis for helpful comments.

2. **Preliminaries**

In this section, let

$$
\begin{aligned}
G &= \mathrm{GL}_2(\mathfrak{k}) \\
B &= \text{subgroup of upper triangular matrices} \\
A &= \text{diagonal matrices in } G \\
\mathfrak{Det}(g) &= -\log_q |\det(g)| \\
\varpi^{m,n} &= \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix} .
\end{aligned}
$$

Thus

$$
\mathfrak{Det}(\varpi^{m,n}) = m + n.
$$

The unramified characters of A, hence the unramified representations of G, are parametrized by α, β in $\mathbb{C}^\times$:

$$
\chi_{[\alpha,\beta]} \colon \; \varpi^{m,n} \longmapsto \alpha^m \beta^n .
$$

To each of these is associated the unramified principal series representation $\pi_{[\alpha,\beta]}$ of G. The subspace of K-fixed vectors in this has dimension one. Any function in the Hecke algebra $\mathcal{H}(G//K)$ acts on this by scalar multiplication. This gives rise to the Satake homomorphism $\mathfrak{S}$ from $\mathcal{H}(G//K)$ to the ring of symmetric polynomials in $\alpha^{\pm 1}$, $\beta^{\pm 1}$. It is an isomorphism.

The Hecke algebra has as basis the characteristic functions

$$
F_{m,n} = \mathfrak{char}(K\varpi^{m,n}K) \quad (m \ge n).
$$

An explicit formula for the Satake transform is well known:

$$
\mathfrak{S} \colon \; F_{m,n} \longmapsto \begin{cases}
\alpha^n\beta^n & m = n \\
q^{1/2} \cdot \alpha^n\beta^n \cdot (\alpha+\beta) & m = n+1 \\
q^{(m-n)/2} \cdot \alpha^n\beta^n \cdot (\alpha^{m-n} + \cdots + \beta^{m-n}) & \\
\quad - q^{((m-1)-(n+1))/2} \cdot \alpha^{n+1}\beta^{n+1} \cdot (\alpha^{m-n-2} + \cdots \beta^{m-n-2}) & \\
 & m \ge n+3.
\end{cases}
$$

There is an intriguing way to express these. The representation $\pi_{\alpha,\beta}$ is generically isomorphic to $\pi_{\beta,\alpha}$, and it is this that is ultimately reponsible for the fact that the expressions on the right are symmetric. This also says that the expressions are affine functions on $(\mathbb{C}^\times)^2$ modulo swaps, which parametrizes the semi-simple conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$. The expressions on the right are characters of certain irreducible representations of $\mathrm{GL}_2(\mathbb{C})$.

Let σ_{std} be the standard representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$, and let σ_m be the corresponding representation on the symmetric power $S^m(\sigma_{\mathrm{std}})$ (so that $\sigma_1 = \sigma_{\mathrm{std}}$). It has dimension $m+1$. The character of $\det^n \cdot \sigma_m$ evaluated on the semi-simple matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

is

$$\tau_{m+n,n} = \alpha^n \beta^n (\alpha^m + \alpha^{m-1}\beta + \cdots + \beta^m).$$

The formulas above can now be abbreviated as

$$\mathfrak{S}\colon\ F_{m,n} \longmapsto \begin{cases} q^{(m-n)/2}\tau_{m,n} & m = n, n+1 \\ q^{(m-n)/2}\tau_{m,n} - q^{((m-1)-(n+1))/2}\tau_{m-1,n+1} & m \geq n+2. \end{cases}$$

There are reasons to choose occasionally a very slightly different basis of the Hecke algebra:

$$f_{m,n} = q^{-(m-n)/2} F_{m,n}.$$

With this new basis:

$$\mathfrak{S}\colon\ f_{m,n} \longmapsto \begin{cases} \tau_{n,n} & m = n \\ \tau_{n+1,n} & m = n+1 \\ \tau_{m,n} - q^{-1}\cdot \tau_{m-1,n+1} & m \geq n+2. \end{cases}$$

Such an equation now expresses one basis for the Hecke algebra in terms of another. The matrix of the Satake isomorphism is now unipotent, with entries in the ring $\mathbb{Z}[q^{-1}]$. Since $(I - xT)^{-1} = I + xT + x^2T^2 + \cdots$, its inverse is easy to calculate:

$$\text{(2.1)} \qquad \mathfrak{S}^{-1}\colon\ \tau_{m,n} \longmapsto f_{m,n} + q^{-1} f_{m-1,n+1} + q^{-2} f_{m-2,n+2} + \cdots .$$

The sum in the last expression is over all i such that $m-i \geq n+i$. Reverting to the $F_{m,n}$ as basis, this implies that

$$q^{m/2}\tau_{m,0} = \sum_{\substack{k \geq \ell \geq 0 \\ k+\ell = m}} F_{k,\ell}\,.$$

The term $\tau_{m,0}$ on the left is the character of $S^m(\sigma_{\mathrm{std}})$. The right hand side is the characteristic function $\mathfrak{M}_m$ of $M_2(\mathfrak{o}) \cap \mathfrak{det}^{-1}(m)$. We therefore obtain

Tamagawa's formula (in this case originating with Hecke):

$$\begin{aligned} L(s-1/2, \pi_{[\alpha,\beta]}, \sigma_{\mathrm{std}}) &= \frac{1}{(1-q^{1/2}\alpha q^{-s})(1-q^{1/2}\beta q^{-s})} \\ &= (1+q^{-(s-1/2)}\alpha+\cdots)(1+q^{-(s-1/2)}\beta+\cdots) \\ &= \textstyle\sum_{m\ge 0} q^{m/2}(\alpha^m+\cdots+\beta^m)q^{-ms} \\ &= \textstyle\sum_{m\ge 0} q^{m/2}\,\tau_{m,0}\,q^{-ms} \\ &= \textstyle\sum_{m\ge 0} \mathfrak{M}_m\,q^{-ms}\,. \end{aligned}$$

In effect, this says that the Satake transform of the characteristic function of $\mathfrak{M}(\mathfrak{o})\cap \mathrm{GL}_2(\mathfrak{k})$ is (at least formally) the symmetric rational function

$$\frac{1}{(1-q^{1/2}\alpha)(1-q^{1/2}\beta)}\,.$$

3. **Basic functions**

Something similar happens for $G=\mathrm{GL}_n(\mathfrak{k})$. In this case, the torus A is isomorphic to a product of n copies of $\mathfrak{k}^\times$. Unramified representations of G are parametrized by elements (α_i) of $(\mathbb{C}^\times)^n$, modulo permutations. This may be identified with the set of semi-simple conjugacy classes in the Langlands dual $\widehat{G}$, which happens to be $\mathrm{GL}_n(\mathbb{C})$.

The map $f\mapsto \pi(f)$ for unramified representations π of G induces an isomorphism of the Hecke algebra $\mathcal{H}(G//K)$ with the ring of symmetric polynomials $\mathbb{C}[\alpha_1^{\pm1},\ldots,\alpha_n^{\pm1}]^{\mathfrak{S}_n}$, which may be identified with the ring of conjugation-invariant affine functions on $\widehat{G}$. It was Tamagawa who first showed that under a natural extension of the Satake transform the characteristic function of $\mathrm{M}_n(\mathfrak{o})\cap \mathrm{GL}_n(\mathfrak{k})$ maps to

$$\frac{1}{\prod_{i=1}^n (1-q^{(n-1)/2}\alpha_i)} = L(-(n-1)/2, \pi_{[\alpha]}, \sigma_{\mathrm{std}}).$$

This extends to an isomorphism of $\mathcal{S}(\mathrm{M}_n(\mathfrak{k}))^{K\times K}$ with the space of meromorphic functions of the form

$$\frac{P(\alpha)}{\prod_{i=1}^n (1-q^{(n-1)/2}\alpha_i)}\ ,$$

in which P is a symmetric polynomial.

This can be generalized to other groups and other representations. The simplest place to start is perhaps with the Langlands dual. Suppose $\widehat{G}_{\mathrm{sc}}$ to be any simply connected semi-simple complex group, σ an irreducible representation of it. Let $\widehat{G}$ be the quotient of $\widehat{G}_{\mathrm{sc}}$ by the kernel of σ, which is then a faithful representation of $\widehat{G}$. Alternatively, one could start with an arbitrary semi-simple $\widehat{G}$ and a faithful representation, but it turns out that that makes some of the discussion a little more awkward. In my computer programs, all representations are parametrized by dominant weights of a group $\widehat{G}_{\text{sc}}$ whose root system is specified.

One can embed $\widehat{G}$ into a reductive group $\widehat{G}_\sigma$ whose centre is isomorphic to $\mathbb{C}^\times$, and extend σ to a representation of $\widehat{G}_\sigma$ such that $\sigma(z) = z \cdot I$. More precisely, define

$$\widehat{G}_\sigma = \frac{\mathbb{C}^\times \times \widehat{G}}{\{(\sigma(z), 1/z) \mid z \in Z(\widehat{G})\}} .$$

The dual $\mathfrak{p}$-adic group G_σ hence possesses a dual homomorphism

$$G_\sigma \longrightarrow \mathfrak{k}^\times ,$$

which is conventionally expressed in the literature as det. As earlier, let $\mathfrak{Det} = -\log_q \det$.

For example:

$$\begin{aligned} \widehat{G} &= \mathrm{SL}_2,\ \sigma = \sigma_{2k+1},\ \widehat{G}_\sigma = \mathrm{GL}_2 \\ \widehat{G} &= \mathrm{PGL}_2,\ \sigma = \sigma_{2k},\ \widehat{G}_\sigma = \mathbb{C}^\times \times \mathrm{PGL}_2 \\ \widehat{G} &= \mathrm{SL}_n,\ \sigma = \sigma_{\text{std}},\ \widehat{G}_\sigma = \mathrm{GL}_n \\ \widehat{G} &= \mathrm{Sp}_{2n},\ \sigma = \sigma_{\text{std}},\ \widehat{G}_\sigma = \mathrm{GSp}_{2n}. \end{aligned}$$

I recall that GSp_{2n} is the group of all $2n \times 2n$ matrices X such that

$$(3.1) \qquad {}^t X\, J\, X = c_X\, J,$$

for some scalar c_X, in which

$$J = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 & & 1 \\ & \dots & \\ 1 & & 0 \end{bmatrix}.$$

In general, the group dual to GSp_{2n} is isomorphic to GSpin_{2n+1}, but since the root systems B_2 and C_2 are isomorphic the dual of GSp_4 is again GSp_4. The homomorphism I have called det then takes X to c_X. (Do not be too bothered by the fact that $\det \neq \det$.)

Unramified representations π of G_σ are parametrized by semi-simple conjugacy classes $\mathfrak{F}_\pi$ of $\widehat{G}_\sigma$. The associated L-function is

$$L(s, \pi, \sigma) = \frac{1}{\det\big(I - \sigma(\mathfrak{F}_\pi) q^{-s}\big)} .$$

Theorem 3.1. *There exists on G_σ a unique function φ_σ whose Satake transform evaluated at π is $L(0, \pi, \sigma)$.*

Proof. This is a direct consequence of Molien's formula

$$\frac{1}{\det(I - Ax)} = \sum_k \operatorname{trace}\,[S^k(A)]\, x^k \quad (A \in \mathrm{GL}_n(\mathbb{C}))$$

which is in turn a consequence of the equation

$$\frac{1}{\prod_{i=1}^{n}(1-\alpha_i)} = \prod (1+\alpha_i+\alpha_i^2+\cdots) = \sum_{k\in\mathbb{N}^n} \alpha^k \quad (\alpha = (\alpha_i)).$$

□

In these circumstances, the Satake transform of $\varphi_\sigma \cdot |\det|^s$ is $L(s,\pi,\sigma)$.

Remark 3.2. It is significant that in all cases in which meromorphic continuation and functional equation of a global L-function is known, there is a group involved in which a map like det exists, and the factor q^{-s} in the L-function is essentially $|\det|^s$. It is often forgotten, or at least not generally acknowledged, that in Langlands' original definition of L-functions this factor comes from the local Frobenius automorphism, which cannot be ignored if G is quasi-split but not split over $\mathfrak{k}$.

I'll say more about this, since it is commonly misunderstood. Suppose G to be any unramified reductive group defined over $\mathfrak{k}$. It may be defined by Galois descent from a split group over an unramified extension $\mathfrak{l}/\mathfrak{k}$, by means of a single outer automorphism Φ preserving B and T. The group $G(\mathfrak{k})$ is then the points g of $G(\mathfrak{l})$ such that $g^{\mathfrak{F}} = g^{\Phi}$. The dual group $\widehat{G}$ is defined as before in terms of the root system dual to that of G, but the full L-group ${}^LG$ is the semi-direct product $\widehat{G} \rtimes W_{\mathfrak{k}}$, in which the Frobenius of the Weil group acts on $\widehat{G}$ by means of the complex automorphism $\widehat{\Phi}$ dual to Φ. The Satake isomorphism now takes its values in the affine ring of conjugate-invariant functions on the coset $\widehat{G} \times \mathfrak{F}$. An unramified representation π gives rise to a homomorphism from this to $\mathbb{C}$, or in other words a semi-simple conjugacy class $\mathfrak{F}_\pi$ in ${}^LG$, called by Langlands its **Frobenius-Hecke** class. The L-function associated to π is then

$$L(s,\pi,\sigma) = \frac{1}{\det(I - \sigma(\mathfrak{F}_\pi)|\mathfrak{F}|^s)}\,.$$

Here $|\mathfrak{F}| = q^{-1}$ is derived from the isomorphism of the maximal abelian quotient of $W_{\mathfrak{k}}$ with $\mathfrak{k}^\times$ given by local class field theory. This formulation makes it apparent that Langlands' definition is a generalization of that of the Artin L-function, which in fact is a special case when G is the trivial group of one element. The paper [6] is a good reference for this material.

In this papre I deal only with split groups, but the Satake transform for arbitrary unramified groups has some new and interesting features. I'll deal with them elsewhere.

The function φ_σ is defined by its Fourier transform. *Can one find a more explicit description?* As I shall explain later, computer programs offer some interesting conjectures.

The example of GL_n suggests that a certain shift of s in the L-function would be a good idea. To see what shift, let's look at GL_n more closely. Let

$\varepsilon_i^\vee$ be the coweight of GL_n taking x to the diagonal matrix with 1 along the diagonal except an x in the i-th place. The coroots of SL_n are

$$\begin{array}{llll} \varepsilon_1^\vee - \varepsilon_2^\vee & & & \\ \varepsilon_1^\vee - \varepsilon_3^\vee, & \varepsilon_2^\vee - \varepsilon_3^\vee & & \\ & \ldots & & \\ \varepsilon_1^\vee - \varepsilon_n^\vee, & \varepsilon_2^\vee - \varepsilon_n^\vee, & \ldots\ , & \varepsilon_{n-1}^\vee - \varepsilon_n^\vee \end{array}$$

so that the sum of positive coroots is

$$2\rho^\vee = (n-1)\varepsilon_1 + (n-3)\varepsilon_2 + \cdots\ - (n-1)\varepsilon_n\,.$$

Hence, we may express $(n-1)/2$ as $\langle \varepsilon_1, \rho^\vee \rangle$.

Keep in mind that ε_1 is the highest weight of the standard representation on $\mathbb{C}^n$. It is therefore suggested that for general $\widehat{G}$ we set $\sigma = \sigma_\lambda$ (with highest weight λ) and look at

$$\begin{aligned} L(s - \langle \lambda, \rho^\vee \rangle, \pi, \sigma) &= \frac{1}{\det(I - q^{\langle \lambda, \rho^\vee \rangle} \sigma(\mathfrak{F}_\pi) q^{-s})} \\ &= \textstyle\sum_{k \ge 0} q^{k \langle \lambda, \rho^\vee \rangle} \operatorname{trace} [S^k(\sigma)](\mathfrak{F}_\pi)\, q^{-ks}\,. \end{aligned}$$

Let $\Phi_{\lambda,k}$ be the unique function on G_σ whose Satake transform at π is $q^{k\langle \lambda, \rho^\vee \rangle}$ trace $[S^k(\sigma)](\mathfrak{F}_\pi)$. Thus

$$\Phi_{\lambda,k} = q^{k\langle \lambda, \rho^\vee \rangle} \varphi_{\lambda,k}\,.$$

There are reasons for working with both Φ and φ.

The function Φ_λ is called the **basic function** associated to G_σ and $\sigma = \sigma_\lambda$.

The support of $\varphi_{\lambda,k}$ or $\Phi_{\lambda,k}$ lies in $\mathfrak{det}^{-1}(k)$. Set $\Phi_\sigma = \sum \Phi_{\sigma,k}$. The Satake transform of $\Phi_\sigma \cdot |\det|^s$, at least formally, is $L(s - \langle \lambda, \rho^\vee \rangle, \pi, \sigma)$.

One hopes to apply the trace formula to global functions whose local component is such a Φ_λ almost everywhere. It is natural to ask, *is there a more explicit way to describe the function Φ_σ (or, equivalently, each $\Phi_{\sigma,k}$)?*

There are at the moment many interesting answers to this question. One is geometrical. To the representation σ and group G_σ is associated a monoid M_σ. This is an algebraic variety defined over $\mathfrak{k}$ in which G_σ embeds as a Zariski-open subvariety, and on which $G_\sigma \times G_\sigma$ acts compatibly with its action on G_σ. The variety M_σ is in general singular, and it is conjectured that the behaviour of Φ_σ is related to this singularity. This has been confirmed to some extent in [7], and conjectures extend their results.

Furthermore, there ought to be some space of functions on G_σ, designated by some the **Schwartz space** of the monoid, characterized by their behaviour near the singularity. One example is GL_n, embedded in M_n. Here the monoid is non-singular, and the correct space is just $\mathcal{S}(\mathrm{M}_n)$. This conjectured Schwartz space has been defined in analytical terms by Lafforgue, in terms of a Plancherel formula extending that of G itself.

Although it is not entirely clear exactly what properties of Φ_λ will be needed in the long run, one thing one probably wants is some understanding of its asymptotic behaviour as $\mathfrak{det} \to \infty$. For G itself, the Plancherel formula is closely related to the asymptotic behaviour of matrix coefficients, and consequently also to Jacquet modules. Does Lafforgue's formula tell us that something similar is going on here? Is there an algebraic characterization of the asymptotic behaviour of Schwartz functions on the monoid?

In the rest of this essay I'll give some idea of how to compute basic functions (by machine). The results will turn out to be fairly interesting. From the very definition of basic functions it is immediate that this computation involves two main processes: (1) computing the decomposition of symmetric powers of irreducible representations; (2) computing the inverse Satake transform. I'll next look at the second of these, which is the more straightforward.

4. **The Satake transform**

The characteristic functions of the double cosets $K\varpi^\lambda K$ form a basis of the Hecke algebra, as λ ranges over $X_*^{++}(A)$. I introduce as well a normalization:

$$\begin{aligned} F_\lambda &= \mathfrak{char}(K\varpi^\lambda K), \\ f_\lambda &= q^{-\langle \lambda, \rho^\vee \rangle} F_\lambda . \end{aligned}$$

Macdonald's formula asserts that the Satake transform of f_λ is

$$\mathfrak{S}(f_\lambda) = \sum_{0 \ll \mu \leq \lambda} S_{\mu,\lambda}(q^{-1}) \tau_\mu .$$

Here $S_{\mu,\lambda}$ is a polynomial with integral coefficients, τ_μ the character of the irreducible representation with dominant weight μ. Also, $0 \ll \mu$ means that μ is dominant, and $\mu \leq \lambda$ means that $\lambda - \mu$ is a sum of positive roots. The point of the normalization of f_λ is that $S_{\lambda,\lambda} = 1$. I shall later say more about the polynomial $S_{\mu,\lambda}$. We have already seen how this goes when $\widehat{G}_{\mathrm{sc}} = \mathrm{SL}_2$.

This formula makes sense because the lattice $X^*(\widehat{A})$ of weights of the dual torus $\widehat{A}$ in the dual group $\widehat{G}$ is the same as $X_*(A)$, and because the Weyl group and the dual Weyl group may be identified. From now on I shall identify f_λ with its Satake transform $\mathfrak{S}(f_\lambda)$. Macdonald's formula therefore relates two different bases of the W-invariant polynomials in the group algebra of the lattice $A/A(\mathfrak{o})$, which may be identified with $X_*(A)$.

Since $S_{\lambda,\lambda} = 1$ and $S_{\mu,\lambda} \neq 0$ only when $\mu \leq \lambda$, the matrix of the linear map $\mathfrak{S}$ is unipotent, and may be solved to give

$$\tau_\lambda = \sum_{0 \ll \mu \leq \lambda} K_{\mu,\lambda}(q^{-1}) f_\mu .$$

The $K_{\mu,\lambda}$ are remarkable polynomials with integral coefficients (see [21]).

- $K_{\mu,\lambda}(q^{-1}) = q^{-\langle \lambda - \mu, \rho^\vee \rangle} P_{\mu,\lambda}(q)$ *where the* $P_{\mu,\lambda}(q)$ *are Kazhdan-Lusztig polynomials for the pair* λ, μ *in the affine Grassmannian. They hence* *have non-negative coefficients and constant term 1. The degree of $P_{\mu,\lambda}$ is less than $\langle \lambda - \mu, \rho^\vee \rangle$ if $\mu \neq \lambda$, and $P_{\lambda,\lambda} = 1$.*

• *The $P_{\mu,\lambda}$ are the q-weights of the weight μ in the representation π^λ, which is defined by a q-version of a vector partition function introduced by Kostant. In particular, $P_{\mu,\lambda}(1)$ is the multiplicity of μ in V^λ.*

• *They are also the Poincaré polynomials of a subtle filtration defined by Ranee Brylinski on the weight spaces V^λ_μ.*

Of these, the first two are clearly important in understanding basic functions. The significance of the last remains something of a mystery.

For example, suppose $G = \mathrm{GL}_2$. The dominant weights may be identified with pairs (m, n), $m \geq n$. The positive root α is $(1, -1)$, $\rho^\vee = \alpha^\vee/2$, and $\langle \alpha, \rho^\vee \rangle = 1$. Equation (2.1) tells us that

$$K_{\lambda - n\alpha,\lambda}(1/q) = q^{-n}, \quad P_{\lambda - n\alpha,\lambda}(q) = 1.$$

What is the relevance to basic functions? The representation of $\widehat{G}_{\mathrm{sc}}$ on the m-th symmetric power $S^m(\sigma_\lambda)$ can be decomposed as $\sum m_\mu \sigma_\mu$. The sum is over dominant $\mu \leq m\lambda$. By Molien's formula the inverse of $\sum m_\mu \tau_\mu$ with respect to the Satake transform is

$$\begin{aligned} \varphi_{\lambda,m} &= \textstyle\sum_{0 \ll \mu \leq m\lambda} m_\mu \tau_\mu \\ &= \textstyle\sum_{0 \ll \mu \leq m\lambda} m_\mu \Big(\sum_{0 \ll \nu \leq \mu} K_{\nu,\mu}(q^{-1}) f_\nu \Big) \\ &= \textstyle\sum_{0 \ll \nu \leq m\lambda} f_\nu \Big(\sum_{\nu \leq \mu \leq m\lambda} m_\mu K_{\nu,\mu}(q^{-1}) \Big). \end{aligned}$$

The coefficient of f_ν is a polynomial in q^{-1}. From this and the first remark about $K_{\mu,\lambda}$ the following is immediate:

Proposition 4.1. *The multiplicity m_ν of σ_ν is the constant term in the coefficient of f_ν.*

I can now offer a new proof of the formula of Hecke and Tamagawa.

Theorem 4.2. *Suppose $\widehat{G}$ to be GL_n, σ is its standard representation (with highest weight ε_1). The polynomial $P_{\mu,\lambda} = 1$ for all $0 \ll \mu \leq \lambda = k\varepsilon_1$.*

This follows easily from what I have just said, together with known facts about $S^k(\sigma)$—it is irreducible, and since it has as explicit basis the monomials $x_1^{k_1} \ldots x_n^{k_n}$ with $\sum k_i = k$, all weight multiplicities are 1.

Already around 1963, Satake and Shimura (independently) looked at the groups GSp_{2n} to see if they could obtain a formula like Tamagawa's. The group GSp_{2n} is that of all $2n \times 2n$ matrices satisfying equation (3.1). The scalar in that equation is what I call $\det(g)$, although it is not in fact the determinant. This group in turn is embedded in the monoid MSp_{2n}, for which, roughly speaking, the scalar is allowed to be 0. The space MSp_{2n} is a singular cone for $n \geq 2$.

What Satake and Shimura did was to copy Tamagawa's calculation, but replacing the matrix algebra by $\mathrm{MSp}_{2n}(\mathfrak{o})$. What they found in the case of GSp_4 was that its Satake transform was (in modern notation)

$$Q \cdot L(s - \langle \varepsilon_1, \rho^\vee \rangle, \pi, \sigma),$$

where Q was a certain non-trivial polynomial in the group algebra of $\widehat{A}$, (see [33, Section 3]). After this, there were one or two brief attempts to understand what was going on, but in essence research on this topic came to a dead halt.

For larger n, Satake expressed the transform of the characteristic function of $\mathrm{MSp}_{2n}(\mathfrak{o})$ as an infinite series, and conjectured that this was the Taylor series of a rational function that we would now write as

$$Q \cdot L(s - \langle \lambda, \rho^\vee \rangle, \pi, \sigma),$$

with Q an invariant polynomial. The Langlands dual of GSp_{2n} is GSpin_{2n+1}, as pointed out in [2, Section 2]. What we see now is that in Satake's conjecture the L-factor is Langlands' L-function associated to the spin representation $\sigma = \sigma_\lambda$ of the Langlands dual, although it is not clear that Satake recognized this. Keep in mind that at that time Langlands' definition of L-functions had not yet been made, and that it was not entirely clear at that time what 'good' L-functions were.

The point is that Satake and Shimura apparently never came to ask the question that now seems the natural one. Instead of asking, what is the transform of $\mathfrak{char}(\mathrm{MSp}_4(\mathfrak{o}))$, they might have asked, *what function has as its transform* $L(s - \langle \varepsilon_1, \rho^\vee \rangle, \pi, \sigma)$? It is not clear that they could have answered this satisfactorily, however, because the answer that we can see now depends on technology developed (by Lusztig et al.) quite a bit later.

I'll give later a conjectural answer to this question, produced by computation. It will be both surprisingly simple and surprisingly interesting.

5. Details of the Satake transform

In this section, let G be an arbitrary split reductive group defined over $\mathfrak{k}$.

As λ ranges over the positive cone in $X_*(A)$, the functions $F_\lambda = \mathfrak{char}(K\varpi^\lambda K)$ make up a basis of the Hecke algebra $\mathcal{H}$. Their Satake transforms make up a basis of W-invariant polynomials in $X_*(A)$. But $X_*(A) = X^*(\widehat{A})$, and another basis of W-invariant polynomials is made up of irreducible characters of irreducible finite-dimensional representations of the dual group $\widehat{G}$.. The Satake homomorphism is given explicitly in terms of these bases according a well known formula due to Ian Macdonald.

Let χ be any unramified character of A. The original form of Macdonald's formula is for the spherical function Γ_χ defined by the corresponding principal series representation of G. which is completely determined by its values on

$$A^{--} = \big\{ a \in A \,\big|\, |\alpha(a)| \le 1 \text{ for all } \alpha \in \Delta \big\}.$$

(I use here Macdonald's notation. The cone $X_*^{--}(A)$ is acute, whereas $X_*^{-}(A)$ is obtuse.)

There are a few variants of the formula, describing the asymptotic behaviour of very general matrix coefficients in terms of Jacquet modules. The only version needed here is an explicit formula for unramified spherical functions, in which case the asymptotic behaviour is valid everywhere.

Let I be the Iwahori subgroup of K, the inverse image of the Borel subgroup in $G(\mathbb{F}_q)$ corresponding to P. Define

$$\mu_G = \frac{\mathrm{meas}(I w_\ell I)}{\mathrm{meas}(K)} = \frac{1}{\sum_{w \in W} q^{-\ell(w)}} \, .$$

This is the proportion of points in $G(\mathbb{F}_q)$ contained in the largest open Bruhat cell. It is also the ratio of two natural Haar measures on G—one of them is such that $\mathrm{meas}(K) = 1$, while the other is determined by a natural G-invariant differential form on G, one used in defining a Tamagawa measure.

Let $\gamma/_\chi$ be the unique function in the induced representation I_χ fixed by K and such that $\gamma/_\chi(1) = 1$. The associated spherical function is defined by the formula

$$\Gamma_\chi(g) = \int_K \gamma/_\chi(kg)\, dk \, .$$

It is a function on $K\backslash G/K$, hence determined by its restriction to A^{--}. Macdonald's formula is for this spherical function:

Theorem 5.1 (Macdonald)**.** *For χ an unramified character of A and $a \in A^{--}$*

$$\Gamma_\chi(a) = \mu_G \cdot \delta^{1/2}(a) \left(\sum_{w \in W} \frac{\prod_{\gamma>0} \big(1 - q^{-1}[w\chi](\varpi^{-\gamma})\big)}{\prod_{\gamma>0} \big(1 - [w\chi](\varpi^{-\gamma})\big)} \cdot [w\chi](a) \right) .$$

Here δ is the modulus character $|\det \mathrm{Ad}_{\mathfrak{n}}|$. This formula is proved by a careful analysis of the asymptotic behaviour of the spherical function and its relation to the Jacquet module.

The relationship between the Satake homomorphism and Macdonald's formula is very simple:

Lemma 5.2. *For $F = \mathfrak{char}(KaK)$ with a in A^{--}*

$$\mathfrak{S}_\chi(F) = |KaK/K|\, \Gamma_\chi(a) \, .$$

This is an easy exercise. It is to be paired with:

Lemma 5.3. *For a in A^{--}*

$$|KaK/K| = \frac{\mu_{M_a}}{\mu_G} \cdot \delta^{-1}(a) \, ,$$

where M_a is the Levi component centralizing a.

Here a corresponds to an element $\lambda^\vee$ of $X_*(A)$ and $M_a = M_\Theta = M_\lambda$, where

$$\Theta = \big\{\alpha \in \Delta \,\big|\, \langle \alpha, \lambda^\vee \rangle = 0\big\}.$$

The proof fibres $K/K \cap aKa^{-1}$ over the flag variety $G(\mathbb{F}_q)/P_\Theta(\mathbb{F}_q)$, whose size follows from the Bruhat decomposition.

Theorem 5.4. *For $F = \mathfrak{char}(K\varpi^\lambda K)$ with λ in $X^{++}(A)$*

$$\mathfrak{S}_\chi(F) = \mu_{M_\lambda} \cdot \delta^{-1/2}(\varpi^\lambda) \left(\sum_{w \in W} \frac{\prod_{\gamma>0} \big(1 - q^{-1}[w\chi](\varpi^{-\gamma})\big)}{\prod_{\gamma>0} \big(1 - [w\chi](\varpi^{-\gamma})\big)} \cdot [w\chi](\varpi^\lambda) \right).$$

This second version has a subtly different meaning from the first, since we are no longer talking about the asymptotic behaviour of a matrix coefficient on G, for a fixed value of χ, but now looking at a function of χ.

From now, I'll use additive notation. The character of $\widehat{A}$ corresponding to λ will be expressed as e^λ. Why is this reasonable? Through the exponential map, the group $\widehat{A}$ may be identified with a quotient of $\mathbb{C} \otimes X_*(\widehat{A})$, and then for $\mu^\vee$ in $X_*(\widehat{A})$, the dual of $X^*(\widehat{A})$,

$$\langle \lambda, x \otimes \mu^\vee \rangle = e^{x\langle \lambda, \mu^\vee \rangle}.$$

The set $A^{--}/A(\mathfrak{o})$ may be identified with the set $X^{++}(A)$ of dominant weights of $\widehat{A}$. Each dominant weight in $X^*(\widehat{A})$ gives rise to an irreducible representation σ_λ of $\widehat{G}$ with highest weight λ. Its image in the Grothendieck group of $\widehat{A}$ may be identified with its character. As we shall see in a moment, somewhat hidden in Macdonald's formula are instances of Weyl's character formula for representations of $\widehat{G}$.

The basic point is very simple—the image of $\mathfrak{char}(K\varpi^\lambda K)$ in $\mathcal{H}$ with respect to the Satake transform, considered as a function of χ, is a conjugation-invariant function of semi-simple classes in $\widehat{G}$, and in fact the Satake homomorphism is an isomorphism of $\mathcal{H}$ with $\mathbb{C} \otimes R_{\widehat{G}}$. The characters of finite-dimensional representations of $\widehat{G}$ are a basis of the representation ring $R_{\widehat{G}}$. This may be interpreted also as the ring of functions on $\widehat{G}$ that are invariant under conjugation. As I have already pointed out, Macdonald's formula tells us the relationship between two natural bases.

By definition of the dual torus $\widehat{A}$, the character $\chi\colon A \to \mathbb{C}^\times$ may be interpreted as an element $\widehat{a}_\chi$ of $\widehat{A}$. Thus $\chi(\varpi^\lambda)$ may be interpreted as $\lambda(\widehat{a}_\chi)$.

The factor $\delta^{-1/2}(\varpi^\lambda)$ can be expressed as $q^{\langle \lambda, \rho^\vee \rangle}$, in which $\rho^\vee$ is half the sum of positive coroots. The right hand side of Macdonald's formula can now be interpreted as an identity of functions on $\widehat{A}$:

$$\mathfrak{S}(F_\lambda) = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \left(\sum_{w \in W} \frac{\prod_{\gamma>0} \big(1 - q^{-1} e^{-w\gamma}\big)}{\prod_{\gamma>0} \big(1 - e^{-w\gamma}\big)} \cdot e^{w\lambda} \right).$$

I now expand the product and invert the order of sums.

$$\mathfrak{S}(F_\lambda) = \mu_{M_\lambda} q^{\langle \lambda, \rho^\vee \rangle} \left(\sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \sum_W \frac{e^{w(\lambda - \gamma_S)}}{\prod_{\gamma > 0} (1 - e^{-w\gamma})} \right).$$

Here, for $S \subseteq \Sigma^+$

$$\gamma_S = \sum_{\gamma \in S} \gamma.$$

One form of Weyl's character formula tells us that for λ dominant

$$\tau_\lambda = \sum_W \frac{e^{w\lambda}}{\prod_{\gamma > 0} (1 - e^{-w\gamma})},$$

in which τ_λ is the character of σ_λ. I therefore write the previous formula as

$$\mathfrak{S}(F_\lambda) = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \tau_{\lambda - \gamma_S} \,.$$

There is a small problem with this, since even when λ is dominant it may well happen that $\lambda - \gamma_S$ is not. It is important to take this into consideration at the same time as a matter of symmetry.

Another form of Weyl's formula asserts that

$$\tau_\lambda = \frac{\sum_w \operatorname{sgn}(w)\, e^{w(\lambda + \rho)}}{\sum_w \operatorname{sgn}(w)\, e^{w\rho}} \,.$$

This form possesses a certain symmetry that allows us to evaluate it even when λ is not dominant. It tells us that $\tau_\lambda = \operatorname{sgn}(w) \tau_\mu$ whenever $w(\lambda + \rho) = \mu + \rho$. This suggests that defining

$$\Pi_\lambda = \tau_{\lambda - \rho}, \quad \text{equivalently} \quad \Pi_{\lambda + \rho} = \tau_\lambda \,.$$

The symmetry now becomes $\Pi_{w\lambda} = \operatorname{sgn}(w)\, \Pi_\lambda$. With this new notation, Macdonald's formula becomes

$$\mathfrak{S}(F_\lambda) = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \Pi_{\lambda + (\rho - \gamma_S)} = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \Pi_{\lambda + \rho_S},$$

with $\rho_S = \rho - \gamma_S$. Thus, for example, $\rho_\emptyset = \rho$ and $\rho_{\Sigma^+} = -\rho$. In fact, every $w\rho$ is one of the ρ_S, since

$$\begin{aligned} 2\rho &= \textstyle\sum_{\gamma > 0} \gamma, \\ 2w\rho &= \textstyle\sum_{\gamma > 0} w\gamma \\ &= \textstyle\sum_{\gamma > 0, w^{-1}\gamma > 0} \gamma \quad - \sum_{\gamma > 0, w^{-1}\gamma < 0} \gamma \\ &= \textstyle\sum_{\gamma > 0} \gamma - 2 \sum_{\gamma > 0, w^{-1}\gamma < 0} \gamma \\ &= 2(\rho - \gamma_{S_w}) \quad (S_w = \{\gamma > 0 \,|\, w^{-1}\gamma < 0\}) \,. \end{aligned}$$

Let $\mathcal{C}_\rho$ be the set of all ρ_S. It is in fact the set of weights of the irreducible representation σ_ρ of highest weight ρ. It plays an important role in proving

Weyl's character formula, and I shall say more about it in the next section. For each μ in $\mathcal{C}_\rho$ define

$$P_\mu(x) = \sum_{S|\rho_S=\mu} x^{|S|} \,.$$

Thus $P_\rho(x) = 1$, $P_{w\rho}(x) = x^{\ell(w)}$, and if we set $x = 1$ this becomes the same as the multiplicity of the weight μ in π^ρ, the finite-dimensional representation of $\widehat{G}$ with highest weight ρ. Our formula now becomes

$$\mathfrak{S}(F_\lambda) = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \sum_{\mu \in \mathcal{C}_\rho} P_\mu(-q^{-1}) \Pi_{\lambda+\mu} \,.$$

There is still one more modification to come. In calculating with this formula, any Π_λ can be transformed to some $\pm\Pi_\mu$ with μ in X^{++}, by applying the familiar algorithm of W-reduction to the positive chamber. And then either μ is of the form $\nu + \rho$ with ν dominant, in which case $\Pi_\mu = \pi^\nu$, or it is singular and $\Pi_\mu = 0$.

There is one significant case in which one can use these observations to improve the formula we have so far. Suppose $\Theta \subseteq \Delta$ to be the subset of α with $\langle \lambda, \alpha^\vee \rangle = 0$. Let W_Θ be the subgroup of W generated by the simple corresponding to the simple roots in Θ. Then $w\lambda = \lambda$ for w in W_Θ, and $w(\lambda + \mu) = \lambda + w\mu$ for w in W_Θ. If μ is singular with respect to Θ then $\Pi_{\lambda+\mu} = 0$, so in the previous formula one can restrict to the μ that are not singular in this sense. Every such μ in $\mathcal{C}_\rho$ is equal to the W_Θ-transform of some unique ν in the subset

$$[W_\Theta \backslash \mathcal{C}_\rho] = \big\{ \nu \in \mathcal{C}_\rho \,\big|\, \langle \nu, \alpha^\vee \rangle > 0 \text{ for } \alpha \in \Theta \big\},$$

Therefore the previous formula can be rewritten once again to get my final version of Macdonald's formula:

Theorem 5.5. *Suppose λ dominant, and let Θ be the subset of simple roots α such that $\langle \lambda, \alpha^\vee \rangle = 0$. Then*

$$q^{-\langle \lambda, \rho^\vee \rangle} \mathfrak{S}(F_\lambda) = \mathfrak{S}(f_\lambda) = \sum_{\mu \in [W_\Theta \backslash \mathcal{C}_\rho]} \mathcal{M}_\mu(q^{-1}) \Pi_{\lambda+\mu} \,.$$

in which

$$\mathcal{M}_\mu(x) = \frac{\sum_{W_\Theta} \operatorname{sgn}(w)\, P_{w\mu}(-x)}{\sum_{w \in W_\Theta} x^{\ell(w)}} \,.$$

The denominator can be expressed as

$$\sum\nolimits_{W_\Theta} \operatorname{sgn}(w)\, P_{w\rho}(-q^{-1}) \,,$$

and this guarantees that the matrix of the Satake transformation is unipotent. We can now do a simple reality check. If $\lambda = 0$ then $f_\lambda = \mathfrak{char}(K)$ and $\mathfrak{S}(f_0) = 1$ identically, for trivial reasons. What does the formula tell us? If $\lambda = 0$ then $M_\lambda = G$ and the μ-term is $\sum q^{-\ell(w)}$. A basic property of the set $\mathcal{C}_\rho$ is that all weights in it are singular except the extremal weights $w\rho$. I shall recall the proof in the next section. This implies that the weight ρ is the smallest of the regular dominant weights. This is because if λ is in X^{++}, it must be either in $\rho + X^{++}$ or fixed by some s_α, since the intervening band has width 1. So all terms in the sum over $\mathcal{C}_\rho$ vanish except those for the $w\rho$, and for one of those $\Pi_{w\rho} = \mathrm{sgn}(w)\,\Pi_\rho = \mathrm{sgn}(w)$. So there is no contradiction with the trivial evaluation.

There is one thing about this formula that is not *a priori* evident. The denominator μ_{M_λ} implies that the coefficient of $\Pi_{\lambda-\mu}$ is at least a formal series in q^{-1}. In fact:

Lemma 5.6. *Suppose $\Theta \subseteq \Delta$ and μ to be an element of $[W_\Theta\backslash\mathcal{C}_\rho]$. Then the quotient*

$$\mathcal{M}_\mu(x) = \frac{\sum_{W_\Theta} \mathrm{sgn}(w)\, P_{w\mu}(-x)}{\sum_{w\in W_\Theta} x^{\ell(w)}}$$

is a polynomial in x.

Here x is a variable to be set equal to q^{-1}.

A proof of this Lemma was first published in [26, Section 3.3.8]. Other proofs appear in the discussion after the statement of [25, Theorem 6.6] and at the end of [16]. These are not as direct as I'd like. It ought to be possible to understand better how the action of W on $\mathcal{C}_\rho$ interacts with the polynomials $P_\mu(x)$, or perhaps how to interpret the function P_μ on $\mathcal{C}_\rho$.

Remark 5.7. There are two natural bases of W-invariant affine functions on $\widehat{A}$. One is made up of the $\sum_W w\lambda$ as λ ranges over $X^{++}(\widehat{A})$. The other is made up of the characters π_λ. If we set $q = 1$ in the previous Theorem we obtain an expression for the first in terms of the second, whereas the inverse transform amounts to a version of Weyl's character formula.

6. Programming matters

The set $\mathcal{C}_\rho$ is crucial in the calculation of the Satake transform. *A priori* this might be a serious bottleneck. That's because the obvious method involves examining every subset of the positive roots. For F_4, for example, there are 2^{24} such subsets, and each of these leads to quite a bit more computation. This is, even for modern computers, a large amount of work. But the set of data one finally needs is not nearly so large, and it ought not to be too surprising that a short cut exists.

I recall that for $S \subseteq \Sigma^+$

$$\gamma_S = \sum_{\gamma\in S} \gamma, \quad \rho_S = \rho - \gamma_S\,,$$

and that

$$w\rho = \rho_{S_w} \ \text{ with } \ S_w = \{\gamma > 0 \,|\, w^{-1}\gamma < 0\}\,.$$

Lemma 6.1. *For $S \subseteq \Sigma^+$, w in W*

$$w(\gamma_S) = \gamma_T$$

with

$$T = \big\{\gamma > 0 \,\big|\, w^{-1}\gamma \in S \text{ or } w^{-1}\gamma \in -(\Sigma^+ - S)\big\}.$$

Proof. From the formula

$$\rho_S = \frac{1}{2}\Big(\sum_{\gamma>0,\gamma\notin S} \gamma - \sum_{\gamma>0,\gamma\in S} \gamma\Big). \tag{6.1}$$

□

In particular, if α is a simple root then $s_\alpha(\gamma_S) = \gamma_T$ with

$$T = \left[\begin{array}{ll} s_\alpha(S - \{\alpha\}) & \alpha \in S \\ s_\alpha(S) \cup \{\alpha\} & \text{otherwise.} \end{array}\right.$$

(Keep in mind that s_α permutes the complement of α in the set of positive roots.) This generalizes how s_α acts on W itself, incrementing or decrementing length.

Lemma 6.2. *If μ lies in $\mathcal{C}_\rho$ and is not in the W-orbit of ρ, then it is singular.*

Proof. We may assume that μ is in the positive Weyl chamber X^{++}. But then if it is not in $\rho + X^{++}$ we must have $0 \le \langle \mu, \alpha^\vee \rangle < 1$ for some simple α. □

For every μ in $\mathcal{C}_\rho$, recall that

$$P_\mu(-x) = \sum_{S \mid \rho_S = \mu} (-x)^{|S|}\,.$$

Lemma 6.3. *The set $\mathcal{C}_\rho$ is contained in the convex hull of the W-orbit of ρ. Every μ in $\mathcal{C}_\rho$ is a weight of the irreducible representation and its multiplicity is $P_\mu(-1)$.*

Proof. The first claim follows from equation (6.1), the second from Weyl's character formula. □

The computation of the polynomials $P_\mu(x)$ for μ in $\mathcal{C}_\rho$ is by induction. Make the set of positive roots into an ordered list $\{\gamma_i\}$. Start by defining variants of P_μ, for certain subsets S of Σ^{++}. Let

$$S_\emptyset = \emptyset, \quad S_{i+1} = S_i \cup \{\gamma_{i+1}\}\,.$$

Set $Q_\emptyset(\mu)$ for all μ in $\rho + L_\Delta$:

$$Q_\emptyset(\mu, x) = \left[\begin{array}{ll} 1 & \mu = 0 \\ 0 & \text{otherwise.} \end{array}\right.$$

Then proceed inductively:

$$Q_{S\cup\{\gamma\}}(\mu, x) = Q_S(\mu, x) + (-x)Q_S(\mu - \gamma, x)\,.$$

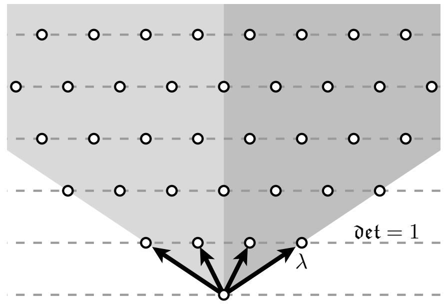

FIGURE 1. Lattice cone for σ_3

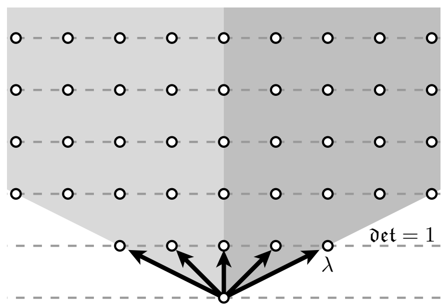

FIGURE 2. Lattice cone for σ_4

The final function $Q_{\Sigma^+}(\mu, x)$ is the same as $P_\mu(-x)$.

7. GL(n)

In this section and the next one, I shall exhibit a few explicit examples of basic functions. Later on, I'll look at GL_2 in detail. Before I do that, I explain first what we shall see.

The basic function always has support on $G_\sigma(\mathfrak{k}) \cap M_\sigma(\mathfrak{o})$, and is bi-$K$-invariant. It is therefore determined by its restriction to A^{--}, and may be identified with a function on the lattice cone in $A/A(\mathfrak{o}) = X_*(A)$ spanned by the weights of σ.

Suppose, for example, that σ is the representation σ_3 of GL_2. Figure 7, for example, is a picture of the lattice cone (with the dominant weights on the right).

The slices $\mathfrak{det} =$ constant are also shown. And in Figure 7 the group $\widehat{G}_\sigma = \mathbb{C}^\times \times \mathrm{PGL}_2$.

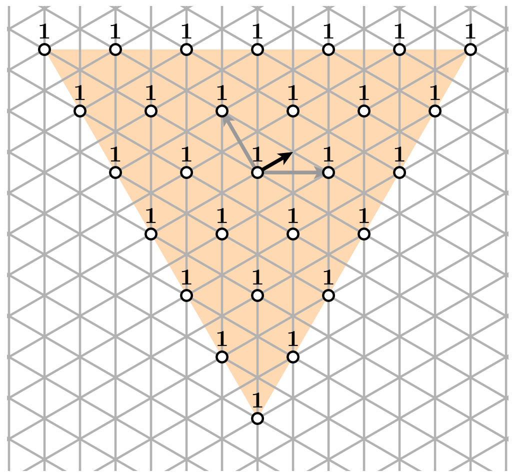

FIGURE 3. The slice $\mathfrak{det} = 6$

When the rank of $\widehat{G}_\sigma$ is more than two, we cannot visualize well the entire lattice cone. When the rank is three, at least, we can look at slices $\mathfrak{det} =$ constant. I haven't yet attempted to investigate groups of rank more than three.

One point to keep in mind, convenient in computations, is that it is not really necessary to consider the group $\widehat{G}_\sigma$. What is at stake, at least most of the time, is really a result about the original simply connected group $\widehat{G}_{\rm sc}$. The question now becomes this: suppose σ to be an irreducible representation of $\widehat{G}_{\rm sc}$ with highest weight λ. *What can one say about the value of the basic function $\Phi_{\lambda,m}$ at dominant weights $\mu \le m\lambda$ on the slice $\mathfrak{det} = m$?* The hope is that these values possess some uniformity and consequently behave well as $k \to \infty$ (and points on G_σ thus pass off to the singular locus of the monoid M_σ).

As we have seen, $P_{\mu,\lambda} = 1$ *for all* $0 \ll \mu \le \lambda$ *when* $G = {\rm GL}_n$, σ *is its standard representation (with highest weight* ε_1*), and* $\lambda = k\varepsilon_1$.

To get a better feel later for more complicated cases, suppose now that $\widehat{G} = {\rm SL}_3(\mathbb{C})$, $\sigma = \sigma_{\rm std}$. Figure 7 is a picture of the root system and weights on the slice $\mathfrak{det} = 6$, for $S^6(\sigma)$.

The weights are the monomials in three variables of degree 6. Each weight has multiplicity 1, which means indeed that we are looking here at the characteristic function of the integral matrices with $\mathfrak{det} = 6$. The entire support of f_σ is a cone with triangular slices like this at every level.

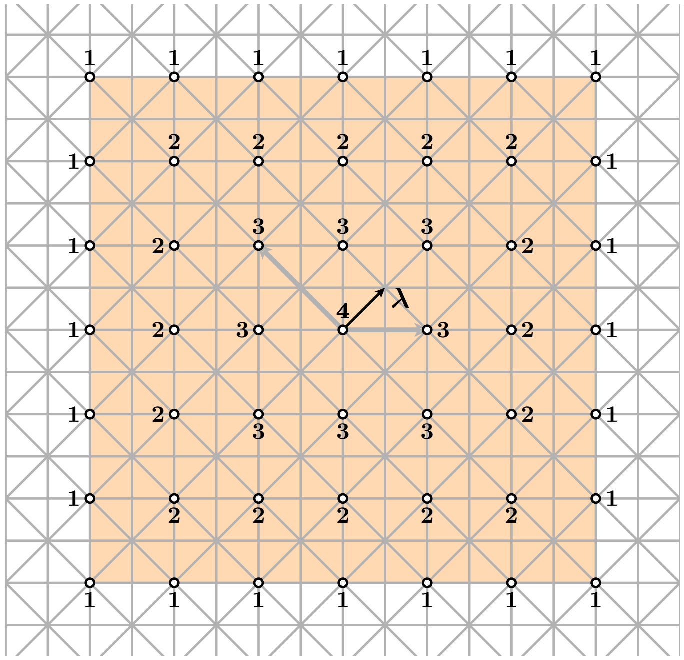

FIGURE 4. A slice for GSp_4

8. GSp(4)

As we have already seen, already around 1963 Satake and Shimura looked at the groups GSp_{2n} to see if they could obtain a formula like Tamagawa's. But, as I also said, they asked the wrong question.

It turns out that the basic functions for Sp_{2n} and σ_{std} are, according to extensive computations, apparently rather simple. This ought not to be too surprising, since in these cases the representations $S^m(\sigma)$ are irreducible. In fact, the groups SL_n and Sp_{2n} and their standard representations are unique in this regard.

The diagram for GSp_4 analogous to that for GL_3 and the basic function associated to σ_{std} is in Figure 8.

This just gives literally the numerical weights at level 6. But a label n signifies also the polynomial $P_n(q) = 1 + q^2 + \cdots + q^{2(n-1)}$. So in 3D we are looking at a sequence of square cones embedded in one large one, each with a fixed multiplicity, even q-weighted multiplicity. In the centre at level $2n$ is the polynomial $P_{n+1}(q)$. The boundary values are the constants 1, and increase by one monomial on inside layers.

In words, suppose σ to be the standard representation of GSp_4, with highest weight $\lambda = \varepsilon_1$. The representation $S^k(\sigma)$ is irreducible, with highest weight $n\lambda$.

Let α be the short simple root, β the long one, $\gamma = \beta + \alpha$. The dominant weights occurring in $S^n(\sigma)$ are those of the form $\mu = n\lambda - a\alpha - c\gamma$ with $0 \le c \le a \le n/2$. The multiplicity of this weight in $S^n(\sigma)$ is $c+1$, and its q-weight (the value of the basic function of the coset $K\varpi^\mu K$) is

$$1 + q^2 + \cdots + q^{2c} \,.$$

I should point out that this simplicity is exceptional. Most basic functions are far more complicated (and presumably more interesting).

9. **Vector partitions**

How might one be able to prove the formula for the basic function of GSp_4 suggested in the previous section? Perhaps by using the second feature of the polynomials $K_{\mu,\lambda}$, relating them to **q-weighted partition functions**.

The vector space $\mathbb{R}^n$ may be identified with $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{R})$. To each γ in $\mathbb{Z}^n$ associate a function on $\mathbb{R}^n$:

$$\langle e^\gamma, x\rangle = e^{\langle \gamma, x\rangle} \,.$$

Of course $e^{\lambda+\mu} = e^\lambda e^\mu$.

Suppose $\Gamma = \{\gamma_i\}_1^g$ to be any finite set of vectors in $\mathbb{Z}^n$ spanning an acute cone. For each n in $\mathbb{N}^g$ define

$$\sigma(n) = \sum_{i=1}^{g} n_i \gamma_i \,.$$

This is a linear map from $\mathbb{N}^g$ to $\mathbb{Z}^n$. Define now the infinite series

$$\begin{aligned} P_\Gamma &= \frac{1}{\prod_{\gamma>0} (1 - e^\gamma)} \\ &= \textstyle\sum_{\mathbb{N}^g} e^{\sigma(n)} \\ &= \textstyle\sum_{\mathbb{N}^d} P_{\Gamma,\lambda}\, e^\lambda \,. \end{aligned}$$

in which $P_{\Gamma,\lambda}$ is the number of points n in $\mathbb{N}^g$ such that $\sigma(n) = \lambda$. The support of the function P_Γ is the cone $\mathcal{C}_\Gamma$ spanned by Γ. It is called a **vector partition function**, being somewhat analogous to the usual partition function on the positive integers.

The **Kostant partition function** is what one gets with Γ equal to the set of positive roots in a root system. Its support is the cone in the lattice of weights spanned by the positive roots. Figure 9, for example, is a diagram of the Kostant partition function for SL_3.

On each of the domains separated by the dark line, it is the restriction of a polynomial to the root lattice.

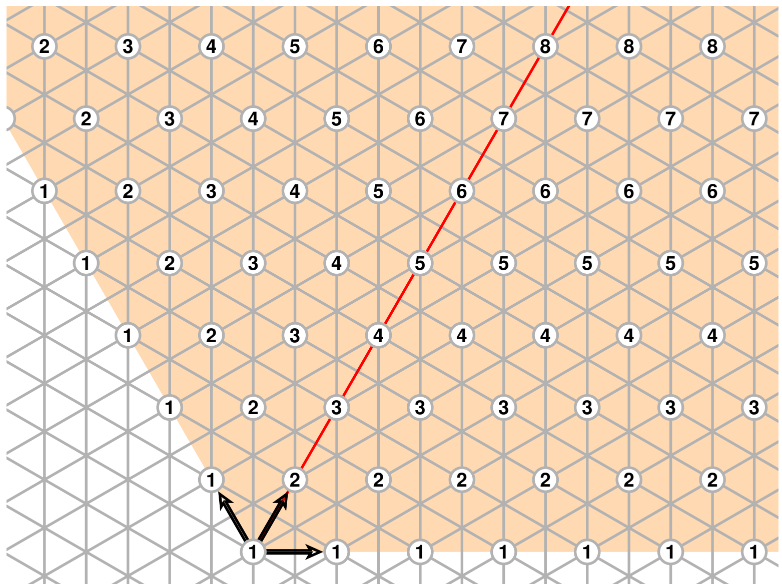

FIGURE 5. The Kostant partition function for SL_3

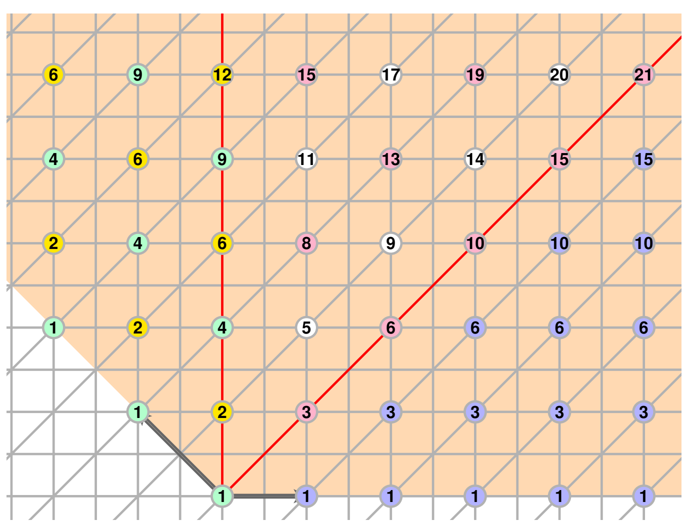

FIGURE 6. The Kostant partition function for Sp_4

In contrast is Figure 9, the diagram of the Kostant partition function for Sp_4.

This exhibits what is more commonly seen in the behaviour of vector partition functions. On each of three domains the function is the restriction of a **quasi-polynomial**, which is to say a polynomial plus a polynomial adjustment of lower degree depending on congruence conditions. It follows from [35] that this phenomenon is universal. Hales has pointed out to me that similar results are among the main theorems of motivic integration (for example, [13, Section 4]), which antedate Sturmfels' theorem.

This function occurs in a formula also due to Kostant for the weight multiplicities $m_{\mu,\lambda}$ of an irreducible representation with highest weight λ:

$$m_{\mu,\lambda} = \sum_{w\in W} (-1)^{\ell(w)} P_{\Sigma^+, w(\lambda+\rho)-(\mu+\rho)} \,.$$

Here ρ is the half-sum of positive roots. I'd say it is not entirely clear that this is a practical tool for computing weight multiplicities, since it calculates relative small numbers as a linear combination of large ones. There is nonetheless a large literature applying it in special cases. The point is to find the domains of quasi-polynomial behaviour, then its exact nature on each domain.

The lattice of weights occurring in an irreducible representation can be broken up into relatively simple domains on which it is a quasi-polynomial function. For some idea of how things go, look at [4] or [5].

There is one bright side, however—a well known asymptotic formula for the weight multiplicities of $\sigma_{n\lambda}$ as $n \to \infty$ can be found in [17]. Heckman's theorem shows that as n goes to infinity lattice geometry is replaced by ordinary Euclidean geometry. The simplest example of this phenomenon is that if Π is a bounded polytope then the number of lattice points inside $n\Pi$ is asymptotically the volume of $n\Pi$.

What does this have to do with basic functions? There exist a q-weighted version of vector partition functions. For n in $\mathbb{N}^g$ let $|n| = \sum n_i$. Define the infinite series

$$\begin{aligned}\mathcal{P}_\Gamma &= \textstyle\prod_\Gamma \dfrac{1}{1-qe^\gamma} \\ &= \textstyle\sum_{\mathbb{N}^g} q^{|n|} e^{\sigma(n)} \\ &= \textstyle\sum \mathcal{P}_{\Gamma,\lambda}(q) e^\lambda,\end{aligned}$$

where now $\mathcal{P}_{\Gamma,\lambda}(q)$ is a polynomial in q that counts the points n in the inverse image of λ, but weighting them by a factor $q^{|n|}$. Of course $\mathcal{P}_\Gamma(1) = P_\Gamma$.

Let $\mathcal{P}$ be the q-weighted version of Kostant's function. A remarkable formula conjectured by Lusztig and proved in [21] asserts that if $\mathcal{P}$

$$K_{\mu,\lambda}(q) = \sum_{w\in W} (-1)^{\ell(w)} \mathcal{P}_{w(\lambda+\rho)-(\mu+\rho)}(q) \,.$$

If q is set to 1, this becomes Kostant's formula for weight multiplicities. If $w\lambda = \lambda$, this takes a particularly simple form, and $K_{\mu,\lambda}$ is the difference between two slightly shifted partition functions. If $G = \mathrm{GL}_n$ or GSp_{2n} and λ is the highest weight of the standard representation, then for dominant weights μ things are especially simple, and we presumably recover rigourously the computer's conjecture. The q-weighted partition functions also occur in motivic integration, and in [13, Section 4] there is a version of Sturmfels' theorem for them.

In [24] it is proved that basic functions are q-weighted vector partition functions. Among other things, it suggests that basic functions have significance beyond the applications one has in mind to automorphic forms. For this, suppose given $\widehat{G}_{\mathrm{sc}}$, σ etc. Let S be the set of weights of σ. Let $\mathcal{W}$ be the series

$$\prod_{\gamma\in\Sigma^-}\frac{1}{1-qe^{\gamma}}\cdot\prod_{\gamma\in S}\frac{1}{1-e^{\gamma}}=\sum_{\mu}\mathcal{W}_\mu(q)\,e^{\mu}\,.$$

Its coefficients $\mathcal{W}_\mu$ have support on the lattice cone spanned by Σ^- and S.

Theorem 9.1. (Wen-wei Li) *For dominant weights μ*

$$\phi_\mu=\sum_{w\in W}(-1)^w\mathcal{W}_{\mu+(\rho-w\rho)}(q^{-1})\,.$$

Wen-wei Li's proof uses algebraic geometry. Another, somewhat more direct, proof is suggested in [32], using methods also used by [21]. I suspect that there is a simpler derivation that follows even more directly from Kato's main formula. I do not see how to use this formula for practical computation, although some part of it is incorporated in my programs.

10. Decomposing symmetric powers

As we have seen, there are two principal steps to computing basic functions. One involves the decomposition of symmetric powers of an irreducible representation into irreducible components. The other involves finding the Satake transform and its inverse. I have explained an apparently efficient way how to do the second. What about the first?

The program `LiE` (which can be downloaded from

`http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`)

computes decomposition multiplicities of symmetric powers. The algorithm it uses—which incorporates Adams operations—looks very reasonable, but in practice it balks at symmetric powers of even moderately high degree.

The method I use currently seems to be acceptably efficient, although for various reasons one might hope that something better will come along. It uses Molien's formula! First it finds the denominator as a polynomial in the single variable t (which replaces q^{-s} in the form we have seen previously). The amount of work involved in this is not at all great, especially in comparison with what is involved in all other parts of the computation. It is very, very roughly proportional to the square of the dimension of the representation. The inverse of this polynomial is an infinite series in t whose coefficients are linear combinations of characters. They satisfy a linear difference equation that may be solved for successive terms, at least up to a given degree. According to Molien's formula the coefficients of powers of t in this series give us the weight multiplicities of symmetric powers. One unfortunate feature of this method, and indeed all methods I am aware of, is that it produces all weights of the symmetric powers, even though only those in a small subset are necessary. We shall see later a method for GL_2 that does not have this problem.

From this, one deduces the decomposition into irreducibles. For this step there are, at least in principle, a number of choices.

The paper [22] gives a very explicit formula for the character of a symmetric product, which of course gives implicitly the weight multiplicities. But in §6 of this paper Kousidis expresses doubts that anything practical will come of it.

Once one knows the weight multiplicities of a representation, there are several ways to find the multiplicities in the decomposition into irreducibles.

The method I used at first applies an algorithm of [27] for finding the dominant weight multiplicities of an irreducible representation. One scans through the list of weight multiplicities given, starting with a maximal one λ and peeling off the multiplicities of the irreducible with highest weight λ. This has the advantage that it only requires knowing the dominant weight multiplicities. But more recently I came across a formula to be found as Corollary III.1 or in [12] or in [11, Section 6]. Its basic idea is suggested in the well known [34].

Proposition 10.1. *If we are given weight multiplicities m_μ for a representation σ, then for λ dominant the multiplicity of the representation with highest weight λ in σ is*

$$\sum_W (-1)^{\ell(w)} m_{\lambda+(\rho-w\rho)} \,.$$

Here ρ is the half-sum of positive roots. The Proposition is an easy consequence of Weyl's character formula.

When W is small, this is undoubtedly the method of choice. It does have one drawback, in that it requires knowing multiplicities outside the positive chamber, but that is not difficult to deal with efficiently. Besides, when W is large, for example when G is of type F_4 and $|W| = 1152$, my programs generally fail for many reasons. It is not clear to me which of the two principal steps are the main bottleneck in my programs, nor what one might hope to come up with eventually.

In the rest of this paper I'll discuss in detail what seems to be the best one can do for GL_2. But even in this case there are interesting open questions.

11. Basic functions for GL(2)

From now on in this paper, let $\widehat{G}_{\mathrm{sc}}$ be $\mathrm{SL}_2(\mathbb{C})$. Recall that σ_m is the representation of $\mathrm{GL}_2(\mathbb{C})$ on $S^m(\sigma_{\mathrm{std}})$, which has dimension $m+1$.

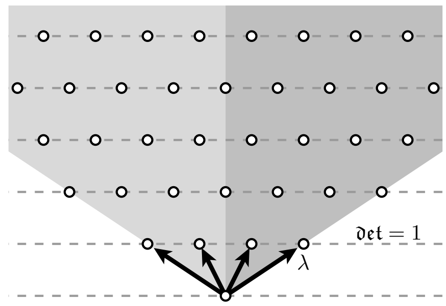

FIGURE 7. σ_3 again

The irreducible representations of $\widehat{G}_{\rm sc}$ are the restrictions of the σ_m. If $m = 2n+1$ the group G_σ is then $\mathrm{GL}_2(\mathbb{C})$, and the extended representations are the $\sigma_{2n+1}\cdot\det^{-n}$. If $m = 2n$, we have $\widehat{G} = \mathrm{PGL}_2(\mathbb{C})$, the group $\widehat{G}_\sigma$ is equal to $\mathbb{C}^\times \times \mathrm{PGL}_2(\mathbb{C})$, and the admissible representations of $\widehat{G}_\sigma$ are the $\sigma_{2n}\cdot \det^{-n}$.

In all cases, let $\widetilde{\sigma}_n$ be the relevant representation of $\widehat{G}_\sigma$. Let λ be its highest weight, with $\mathfrak{det}(\lambda) = 1$.

The basic function is bi-K-invariant, hence determined by its values on the dominant weights μ in the cone spanned by the weights of σ, which are of the form $n\lambda - c_\alpha\alpha$, for c_α with $0 \le c_\alpha \le \langle\lambda, \alpha^\vee\rangle$.

Figure 11 recalls the diagram for $\widetilde{\sigma}_3$ (with dominant weights shaded more darkly).

There are a number of features of basic functions that occur for SL_2 but do not occur for other groups. This ought not to be too surprising, since the Satake homomorphism in this case is very simple. All the complexity of the basic function follows from complexity of symmetric power decomposition, and this turns out to be amenable to reasonable analysis.

For SL_2, the relation between symmetric power decompositions and basic functions is very simple. Here, for example, is the slice of the basic function for $\mathfrak{det} = 6$ with $\sigma = \widehat{\sigma}_3$ (i.e. with $\lambda = 3$):

n	$\Phi_{6\lambda - n\alpha}$
0	$[1]$
1	$[1,0]$
2	$[1,0,1]$
3	$[1,0,1,1]$
4	$[1,0,1,1,1]$
5	$[1,0,1,1,1,1]$
6	$[1,0,1,1,1,1,2]$
7	$[1,0,1,1,1,1,2,0]$

8 [1, 0, 1, 1, 1, 1, 2, 0, 1]
9 [1, 0, 1, 1, 1, 1, 2, 0, 1, 0].

Here I continue to interpret arrays as polynomials, indexed from low degree to high, so that the array for $n = 3$ is the polynomial $1 + q^2 + q^3$. In conformity with Proposition 4.1, the last entries record the coefficients in symmetric power decompositions. What we see in this example is always true—when $\widehat{G}_{\rm sc} = {\rm SL}_2$, the polynomial values of the basic function on a slice $\mathfrak{det}$ = constant are initial segments of the central values of the slice, which are just the multiplicities in the symmetric power decomposition.

Proposition 11.1. *Suppose λ a dominant weight for* ${\rm GL}_2$*, and that*

$$S^m(\sigma_\lambda) = \sum m_i \sigma_{m\lambda - i\alpha} \,.$$

Then the basic function evaluated at $m\lambda - i\alpha$ is

$$\Phi_{m\lambda - i\alpha} = \sum_{0 \le \ell \le i} m_\ell q^\ell \,.$$

Proof. Suppose that

$$S^m(\sigma_\lambda) = \sum m_i \sigma_{m\lambda - i\alpha} \,.$$

From equation (2.1) we know that

$$\mathfrak{S}^{-1}\tau_{m\lambda - i\alpha} = \sum q^{-j} f_{m\lambda - i\alpha - j\alpha}$$

which leads to

$$S^m(\sigma_\lambda) = \sum_{i,j} m_i f_{m\lambda - i\alpha - j\alpha} = \sum_\ell f_{m\lambda - \ell\alpha} \sum_i q^{-(\ell - i)} m_i,$$

in which the first sum is over all ℓ for which $m\lambda - \ell\alpha \ge 0$, the second over $0 \le i \le \ell$. Dualize. ∎

In other words, because the Satake transform for ${\rm SL}_2$ is so simple, evaluating basic functions reduces to evaluating symmetric power decompositions.

In the rest of this paper I'll explain a well known formula for symmetric power decompositions when $\widehat{G}_{\rm sc} = {\rm SL}_2$, and another conjectured formula which might be valuable in dealing with groups of higher rank. It is not clear how much these formulas will tell what we want to know about basic functions, but they will make it easier to compute many examples that brute-force methods will not.

12. q-ology

In the remaining sections, I'll discuss the decomposition of symmetric powers of finite-dimensional representations of $G = \mathrm{GL}_2(\mathbb{C})$.

Let

$$\gamma = \left[\begin{array}{cc} 1 & 0 \\ 0 & q \end{array} \right] .$$

If π is any finite-dimensional representation of G, I define its **trace polynomial** to be the trace of $\pi(\gamma)$ expressed as a Laurent polynomial in the variable q. If the restriction of π to scalar matrices is a single character, then the trace polynomial and that character determine π, since

$$\left[\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right] = \left[\begin{array}{cc} \alpha & 0 \\ 0 & \alpha \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & \beta/\alpha \end{array} \right] ,$$

so that the entire character of π is determined. For example, the trace polynomial of σ_k is

$$1 + q + \cdots + q^k = \frac{q^{k+1} - 1}{q - 1} .$$

In this section I'll find explicit formulas for the trace polynomials of all $\bigwedge^p(\sigma_k)$ and $S^p(\sigma_k)$, thus in effect enabling the computation of their decomposition into irreducible representations.

For any $n \geq 0$ define

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} .$$

If we set $q = 1$ then $[n]_q$ becomes n, and $[n]_q$ is known as the q-analogue of the function $f(n) = n$. As we have seen, the trace polynomial of σ_k is $[k+1]_q$.

The q-analogue of the factorial function is

$$[n]_q^! = [1]_q \cdot [2]_q \ldots [n]_q .$$

These are easy to compute inductively:

$$\begin{aligned} [0]_q^! &= 1 \\ [1]_q^! &= 1 \\ [n]_q^! &= [n]_q [n-1]_q^! . \end{aligned}$$

The q-analogue of the binomial coefficient is

$$\left[\begin{array}{c} n \\ m \end{array} \right]_q = \begin{cases} \dfrac{[n]_q^!}{[m]_q^! [n-m]_q^!} & 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This is also

$$\frac{[n-m+1]_q \ldots [n]_q}{[m]_q^!} \quad (0 \leq m \leq n).$$

It is symmetric in m and $n-m$:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q .$$

Special cases are

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \ (n \ge 0) \quad \text{and} \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_q = [n]_q \ (n \ge 1) .$$

These fit into a q-analogue of Pascal's triangle:

n						
$0:$	1					
$1:$	1	1				
$2:$	1	$1+q$	1			
$3:$	1	$1+q+q^2$	$1+q+q^2$	1		
$4:$	1	$1+q+q^2+q^3$	$1+q+2q^2+q^3+q^4$	$\ldots$	1	
$5:$	1	$1+q+q^2+q^3+q^4$	$1+q+2q^2+2q^3+2q^4+q^5+q^6$	$\ldots$	$\ldots$	1
	$\ldots$					

This illustrates the truth of the following, which is easily verified:

Proposition 12.1. *We have*

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n \\ m \end{bmatrix}_q .$$

Which is to say that, as in Pascal's triangle, the expression at $(n+1, m)$ is a simple linear combination of those at (n, m) and $(n, m-1)$. This can be combined with the evaluation of the first row:

$$\begin{bmatrix} 0 \\ m \end{bmatrix}_q = \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

to recover by induction:

Corollary 12.2. *The function*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q$$

is a polynomial in q.

Here is the point. From now on, for $n \le m$ let

$$\lambda_n^m = \text{ the trace polynomial of } \bigwedge^m (\sigma_{n-1}) .$$

Since the dimension of σ_{n-1} is n, the dimension of $\bigwedge^m(\sigma_{n-1})$ is $\binom{n}{m}$, and therefore the following should not be too surprising:

Corollary 12.3. *For $0 \le m \le n$,*

$$\lambda_n^m = q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q .$$

Proof. Let the e_i for $0 \le i \le n-1$ be an eigenbasis of σ_{n-1} with respect to γ, and for an ordered subset $I = \{i_j\}$ of size $|I| = m$ with $0 \le i_1 < \ldots < i_p \le n-1$ let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$. The e_I with $|I| = m$ form a basis of $\bigwedge^m(\sigma_{n-1})$. We can partition these into the $e_0 \wedge e_I$ with $I \subseteq [1, n-1]$ of size $m-1$ and the e_I with $I \subseteq [1, n-1]$, $|I| = m$. This gives us

$$\lambda^m_{n+1} = q^{m-1}\lambda^{m-1}_n + q^m \lambda^m_n .$$

But if we multiply the recursion equation of Proposition 12.1 by $q^{m(m-1)/2}$ we get the same relation. Initial conditions are obviously the same. □

Corollary 12.4. *The trace polynomial of $S^m(\sigma_n)$ is equal to*

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q .$$

This is apparently well known. It seems to have been rediscovered often, and is sometimes formulated equivalently in terms of Young diagrams. For example, it is Lemma 4.1.22 of [15], proved by an application of Molien's formula. It ought perhaps to be thought of as a generalization of Weyl's character formula, although no generalization for other reductive groups seems to be known or even conjectured.

Proof. It suffices to prove that the polynomial λ^{n+m}_m is equal to the trace polynomial of $S^m(\sigma_n)$ multiplied by $q^{m(m-1)/2}$.

There is a simple bijection of eigenvectors for γ in the two spaces $\bigwedge^m(\sigma_{n-1})$ and $S^m(\sigma_{n-m})$. The exterior product $e_{i_1} \wedge \cdots \wedge e_{i_m}$ with $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ maps to the symmetric product $e_{i_1} e_{i_2-1} \ldots e_{i_m-(m-1)}$. □

The oldest result about the representation $S^m(\sigma_n)$ that I am aware of is *Hermite reciprocity* (from [18]). It follows from the previous corollary, since if π is either $S^m(\sigma_n)$ or $S^n(\sigma_m)$ then

$$\pi\colon \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \longmapsto \alpha^{mn} \cdot I .$$

Corollary 12.5. *The representation $S^n(\sigma_m)$ is isomorphic to $S^m(\sigma_n)$.*

This can be used to reduce work in computation by swapping m and n.

According to Proposition 10.1, Corollary 12.4 has as consequence a formula for the decomposition of $S^m(\sigma_k)$. For GL_2 this result is elementary, as we shall see in a moment.

13. The irreducible decomposition

The highest weight of $S^m(\sigma_k)$ is km, and the center acts on it by a single character. Its decomposition into irreducible components is therefore of the form

$$S^m(\sigma_k) = \sum_{i=0}^{\lfloor km/2 \rfloor} c_i \cdot \sigma_{km-2i} \cdot \det{}^i \,.$$

In these circumstances I define its **decomposition polynomial** to be

$$\delta_{m,k}(q) = \sum_{i=0}^{\lfloor km/2 \rfloor} c_i q^i \,.$$

I'll often assume m, k to be fixed, and ignore them in subscripts.

The trace polynomial of $S^m(\sigma_k)$ is then

$$\sum_{i=0}^{\lfloor km/2 \rfloor} c_i \cdot q^i \cdot \frac{q^{km-2i+1} - 1}{q-1} \quad = \frac{\sum_{i=0}^{\lfloor km/2 \rfloor} c_i q^i - \sum_{i=0}^{\lfloor km/2 \rfloor} c_i q^{km-i+1}}{1-q} \,.$$

If P has degree $\le d$ and it is assigned nominal degree d, define its dual to be the polynomial $q^d P(q^{-1})$. Thus

$$\text{if } \ P(q) = p_0 + p_1 q + \cdots + p_d q^d \ \text{ then } \ P^{\#}(q) = p_d + p_{d-1} q + \cdots + p_0 q^d \,.$$

This gives us a convenient form:

$$(1-q)\tau_{m,k} = \delta(q) + q^{km - \lfloor km/2 \rfloor + 1} \delta^{\#}(q) \,. \tag{13.1}$$

The first term has degree $\lfloor km/2 \rfloor$. The second term starts off with a term of degree $km - \lfloor km/2 \rfloor + 1$. Since $km \ge 2\lfloor km/2 \rfloor$, there is no overlap in the two. That is to say, $\delta(q)$ is a truncation of $(1-q)\tau$ at its midpoint. Hence:

Proposition 13.1. *The decomposition polynomial $\delta_{m,k}(q)$ of $S^m(\sigma_k)$ is the polynomial obtained from $(1-q)\tau_{m,k}$ by truncating all terms of degree more than $\lfloor km/2 \rfloor$.*

This provides a simple way to find the decomposition polynomial, but for various reasons I shall now explore variants of this formula.

Since $[n]_q = (q^n - 1)/(q-1)$, Corollary 12.4 implies that

$$\tau_{m,k} = \frac{(q^{m+1} - 1) \ldots (q^{m+k} - 1)}{(q-1) \ldots (q^k - 1)} \,. \tag{13.2}$$

Let

$$P_k(q) = \prod_2^k (q^i - 1), \quad Q_k(q) = (q-1)P_k(q) = \prod_1^k (q^i - 1) \,.$$

Expanding the product in the numerator of (13.2), we see that it is

$$\begin{aligned} Q_k(q)\tau(q) &= 1 - q^{m+1}(q^0 + q^1 + \cdots + q^{k-1}) \\ &\quad + q^{2(m+1)}(q^{0+1} + q^{0+2} + \cdots q^{(k-2)+(k-1)}) \\ &\quad - \cdots \pm q^{k(m+1)} \cdot q^{1+2+k-1} \\ &= \textstyle\sum_{p=0}^{k} (-1)^p \left(q^{(m+1)p} \sum_{0 \le i_1 < i_2 < \ldots i_p \le k-1} q^{i_1 + \cdots + i_p} \right) \\ &= \textstyle\sum_{p=0}^{k} (-1)^p \cdot q^{p(m+1)} \cdot \text{ trace polynomial of } \bigwedge^p(\sigma_{k-1}) \\ &= \textstyle\sum_{p=0}^{k} (-1)^p \cdot \text{ trace polynomial of } \bigwedge^p(\det^{m+1} \cdot \sigma_{k-1}) \,. \end{aligned}$$

We deduce the following curious and suggestive reformulation of (13.2):

Proposition 13.2. *We have*

$$\tau_{m,k} = \frac{\sum_{p=0}^{k} (-1)^p \cdot \textit{ trace polynomial of } \bigwedge^p(\det^{m+1} \cdot \sigma_{k-1})}{\sum_{p=0}^{k} (-1)^p \cdot \textit{ trace polynomial of } \bigwedge^p(\det \cdot \sigma_{k-1})} \,.$$

This has degree of the numerator is $d = km + k(k+1)/2$. From now on, let

$$\lambda_k^p = \text{ trace polynomial of } \bigwedge^p(\sigma_{k-1}) \,.$$

The degree of λ_k^p is

$$\begin{aligned} (k-1) + (k-2) + \cdots + (k-p) &= pk - p(p+1)/2 \\ &= p\big(k - (p+1)/2\big) \,. \end{aligned}$$

Set

$$\begin{aligned} \lambda_k^{<} &= \textstyle\sum_{2p<k} (-1)^p q^{p(m+1)} \lambda_k^p \,, \\ \lambda_k^{\le} &= \textstyle\sum_{2p \le k} (-1)^p q^{p(m+1)} \lambda_k^p \,, \\ \lambda_k^{\ge} &= \textstyle\sum_{2p \ge k} (-1)^p q^{p(m+1)} \lambda_k^p \,. \end{aligned}$$

The numerator in (13.2) can hence be expressed as $\lambda_k^{<} + \lambda_k^{\ge}$.

Let $a = \lfloor km/2 \rfloor$, $b = k(k-1)/2 - 1$. Then $\delta_{m,k}(q)$ has degree at most a, and P_k has degree b. Let

$$X(q) = P_k(q)\delta_{m,k}(q) \,,$$

which has degree at most $a + b$.

Lemma 13.3. *In these circumstances,*

$$X(q) = \lambda^{<}(q) + q^{a+1}R(q) \,,$$

in which $R(q)$ has degree at most $b - 1$.

Assign the product $X(q)$ the nominal degree $d = a + b$.

$$X^{\#}(q) = P^{\#}(q)\delta^{\#}(q) = q^{a+b}X(q^{-1}) = q^{a+b}\big(\Pi(q^{-1}) + q^{-a-1}R(q^{-1})\big) = \Pi^{\#}(q) + R^{\#}(q) \,.$$

As a consequence:

Proposition 13.4. *The dual $\delta^{\#}(q)$ of the decomposition polynomial is the 'integral quotient' of $\Pi^{\#}(q)$ by $P^{\#}(q)$.*

Proof. (Of the Lemma.) Let

$$Y(q) = P_k(q)\, q^{km-\lfloor km/2\rfloor+1}\delta^{\#}\,.$$

Then

$$(1-q)P_k(q)\tau(q) = \lambda_k^{<}(q) + \lambda_k^{\geq}(q) = X(q) + Y(q)\,,$$

The Lemma now follows from the following facts:

• *The coefficients of $\lambda_k^{\geq}$ vanish in degrees $\leq \lfloor km/2\rfloor$.*

The lowest degree of $\lambda^{\leq}$ is $p(m+1)$ where p is the smallest p such that $2p \geq k$. The two cases k odd and k even are slightly different. If $k = 2\ell$ then the minimal p is equal to ℓ, $\lfloor km/2]\rfloor = \ell m$, and

$$(k/2)(m+1) = km/2 + k/2 = \ell m + \ell \geq \lfloor km/2\rfloor\,.$$

Otherwise, say $k = 2\ell+1$. The minimal p is equal to $\ell+1$, $\lfloor km/2\rfloor = \ell$, and

$$p(m+1) = (\ell+1)(m+1) = \ell m + \ell(m+1) + 1 \geq \lfloor km/2\rfloor + 1\,.$$

• *So do those of $Y(q)$.* The lowest degree of $Y(q)$ is $(km - \lfloor km/2\rfloor + 1) + k(k-1) - 1$, which is certainly $\geq \lfloor km/2\rfloor + 1$.

• *The degree of $X(q)$ is at most $\lfloor km/2\rfloor + k(k-1)/2 - 1$.* Trivial.

• *So is that of $\lambda^{\geq}$.* I claim that $\lambda^{\leq}(q)$ *has degree at most $a+b$.* The degree of each term in the defining sum is

$$\begin{aligned} p(m+1) + (k-1) + (k-2) + \cdots + (k-p) \quad &= p(m+1) + pk - p(p+1)/2 \\ &= p\big(m+1+k-(p+1)/2\big)\,. \end{aligned}$$

As one can see by calculating derivatives, this is a monotonic function of p. It suffices to show that $2p \leq k$ implies that

$$2p(m+1) + 2p + 2pk - p(p+1) \leq km + k(k+1) - 2\,.$$

But if $2p \leq k$ then

$$2p(m+1) + 2p + 2pk - p(p+1) \leq km + k + k^2 - p(p+1)\,,$$

so all is all right except possibly if $k = 1$. But in this case also everything is all right.

We are done. ∎

The point of the Proposition is that we frequently want to compute the decomposition of $S^m(\sigma_k)$ for a fixed k and many m. It is easy enough to compute λ_k^p for the small number of p in the range $2p < k$, and then easy to compute $\lambda^{<}$ for many m. The Proposition then reduces by a factor of roughly two the amount of work one might expect.

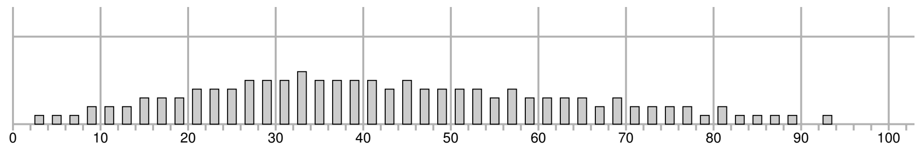

FIGURE 8. Decomposition of $S^{31}(\sigma)$

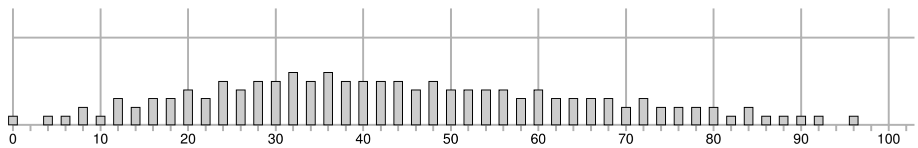

FIGURE 9. Decomposition of $S^{32}(\sigma)$

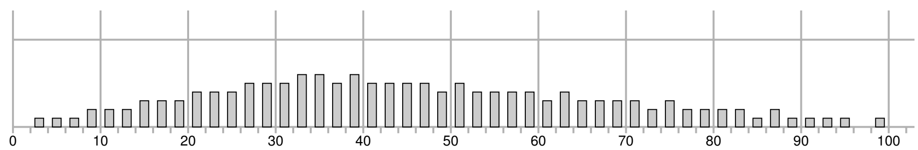

FIGURE 10. Decomposition of $S^{33}(\sigma)$

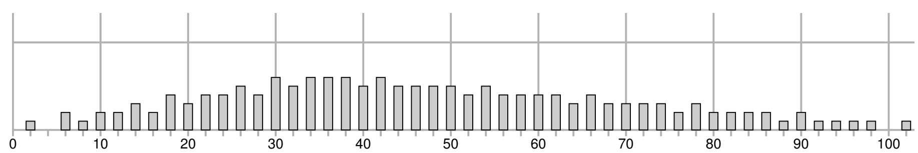

FIGURE 11. Decomposition of $S^{34}(\sigma)$

14. **Pictures**

There are still a few unsolved problems regarding the representations $S^m(\sigma_k)$. I'll exhibit in the section some of them for $\sigma = \sigma_3$. Figures 8–13 illustrate graphically the decomposition of $S^n(\sigma)$ for $n = 31$ through 36. Indexing is by the highest weight of components. The highest weight is always $3n$, at the far right.

Some kind of periodicity appears at the left as well as on the right, and we see also an asymptotic approach to a piecewise linear function. The decompositions of all of the $S^m(\sigma_3)$ are shown at once in the more striking image of Figure 14.

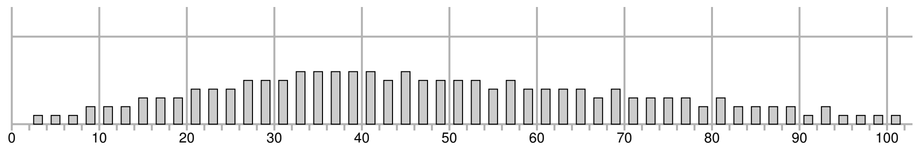

FIGURE 12. Decomposition of $S^{35}(\sigma)$

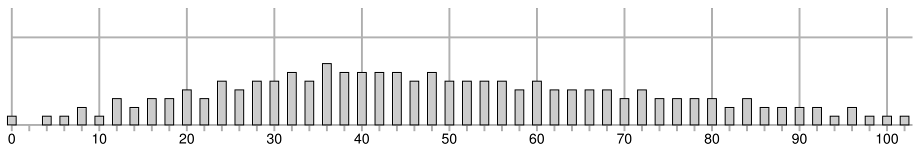

FIGURE 13. Decomposition of $S^{36}(\sigma)$

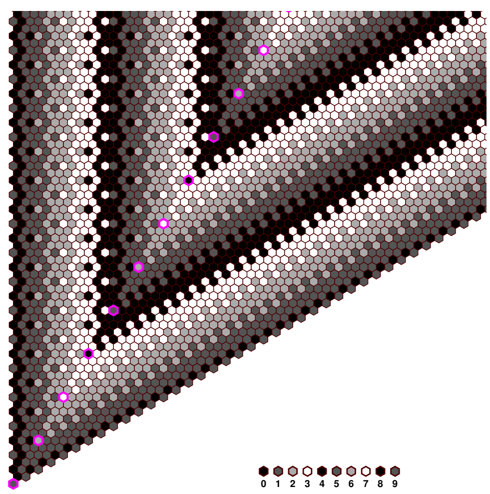

FIGURE 14. Decompositions of all of the $S^m(\sigma_3)$

Let me at least explain what the picture means. First of all, I have scaled the lattice of points (i, m) so as to accommodate hexagons nicely. The vertical coordinate is m, the horizontal one i. This is illustrated in Figure 15.

The hexagon in Figure 14 at (i, m) indicates the multiplicity of σ_i in $S^m(\sigma_3)$. But it does this by shading, as the key shows. The most obvious symmetry is the shift diagonally up, indicated by special marks on certain hexagons. I have only vaguely explained this. As I have mentioned, vector partition functions

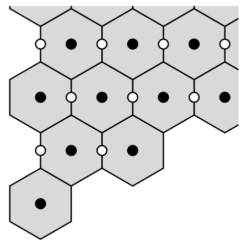

FIGURE 15. How Figure 14 is constructed

are piece-wise quasi-polynomial. This mean that you can divide up the support of the function into a finite number of polyhedra on which the function is polynomial plus some lower order correction terms that depend on lattice congruence conditions. Here, the cone is divided into two such regions by the diagonal line, and that the polynomials are linear functions. The congruence conditions are visible. For the σ_k with $k > 3$, the images are similar but more complicated. In the literature are many examples for which such decompositions have computed, but it is not yet clear to me how valuable they are here.

Final remarks. Since writing this, I have found (and proved) a formula for the decomposition of the $S^m(\sigma_3)$ that explains clearly what one sees in the figures above. I have also found extraordinarily simple formulas for the asymptotic behaviour of all $S^m(\sigma_k)$ for a fixed k, as $m \to \infty$. Explaining these things would require a major revision of this paper which I do not have time to carry out. They will be covered in a sequel, along with speculations about, if not proofs of, what one might consequently expect for groups of higher rank.

Added in proof: Aaron Pollack has proved my conjecture about the basic function associated to $GSp(4)$, and in fact found a formula for all groups $GSp(2n)$ and the standard representation.

References

[1] G. Almkvist, Some formulas in invariant theory, *J. Algebra* **77** (1982) 338–359.

[2] M. Asgari, Local L-functions for split spinor groups, *Canad. J. Math.* **54** (2002) 673–693.

[3] A. Barvinok, A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed, *Math. Oper. Res.* **19** (1994) 769–779.

[4] M. Beck, The partial fractions method for counting solutions to integral linear systems, *Discrete Comput. Geom.* **32** (2004), no. 4, 437–446.

[5] T. Bliem, Towards computing vector partition functions by iterated partial fraction decomposition, Arxiv:0912.1131 [math.CO].

[6] A. Borel, Automorphic L-functions, in: Automorphic Forms, Representations, and L-functions, (Proc. Sympos. Pure Math. Oregon State Univ. Corvallis, Ore. 1977), part 2, pp. 27–61, Proc. Sympos. Pure Math. 33, Amer. Math. Soc. Providence, RI, 1979.

[7] A. Bouthier, B.C. Ngô and Y. Sakellaridis, On the formal arc space of a reductive monoid, Arxiv:1412.6174 [math.AG].
[8] M. Brion, Stable properties of plethysm, Manuscripta Math. **80** (1993) 347–371.
[9] M. Brion, and M. Vergne, Residue formulae, vector partition functions and lattice points in rational polytopes, *J. Amer. Math. Soc.* **10** (1997), no. 4, 797–833.
[10] R. Brylinski, Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits, *J. Amer. Math. Soc.* **2** (1989) 517–533.
[11] M. Christandl, B. Doran, S. Kousidis and M. Walter, Eigenvalue distributions of reduced density matrices, *Comm. Math. Phys.* **332** (2014) 1–52.
[12] M. Christandl, B. Doran and M. Walter, Computing multiplicities of Lie group representations, in: 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science (FOCS 2012), pp. 639-648, IEEE Computer Society, 2012.
[13] R. Cluckers and F. Loeser, Constructible motivic functions and motivic integration, *Invent. Math.* **173** (2008) 23–121.
[14] R. Godement and H. Jacquet, Zeta Functions of Simple Algebras, Lecture Notes in Math. 260, Springer-Verlag, 1972.
[15] R. Goodman and N. Wallach, Symmetry, Representations, and Invariants, Springer, 2009.
[16] T. Haines, R. Kottwitz and A. Prasad, Iwahori-Hecke algebras, preprint, 2009.
[17] G. Heckman, Projection of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Invent. Math.* **67** (1982) 333–356.
[18] C. Hermite, Sur le théorie des fonctions homogène à deux indéterminés, *The Cambridge and Dublin Mathematical Journal* **9** (1854) 172–216.
[19] H. Jacquet, Principal L-functions of the linear group, in: Automorphic Forms, Representations, and L-functions, (Proc. Sympos. Pure Math. Oregon State Univ. Corvallis, Ore. 1977), paret 2, pp. 63–86, Proc. Sympos. Pure Math. 33, Amer. Math. Soc. Providence, RI, 1979.
[20] T. Kahle and M. Michałek, Plethysm and lattice point counting, Arxiv:1408.5708 [math.RT].
[21] S. Kato, Spherical functions and a q-analog of Kostant's weight multiplicity formula, *Invent. Math.* **66** (1982), 461–468.
[22] S. Kousidis, A closed character formula for symmetric powers of irreducible representations, in: S. Billey and V. Reiner (eds.), 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), pp. 833–844, Discrete Math. Theor.Comput. Sci. Proc., AN Assoc. Discrete Math. Theor. Comput. Sci. Nancy, 2010.
[23] L. Lafforgue, Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires, *Jpn. J. Math.* **9** (2014) 1–68.
[24] W.W. Li, Basic functions and unramified local L-factors for split groups, Arxiv:1311.2434 [math.RT].
[25] G. Lusztig, Singularities, character formulas, and a q-analogue of weight multiplicities, *Astérisque* **101–102** (1983) 208–227.
[26] H. Matsumoto, Analyse Harmonique dans les Systemes de Tits Bornologiques de Type Affine, Lecture Notes in Math. 590, Springer, 1977.
[27] R. Moody and J. Patera, Fast recursion for weight multiplicities, *Bull. Amer. Math. Soc.* **7** (1982) 237–242.
[28] B.C. Ngô, On automorphic L-functions, slides of a talk in Hué.
[29] B.C. Ngô, Semi-groups and basic functions, preprint, 2012.
[30] B.C. Ngô, On certain sums of automorphic L-functions, in: Automorphic Forms and Related Geometry: Assessing the Legacy of I.I. Piatetski-Shapiro, pp. 337–343, Contemp. Math. 614, Amer. Math. Soc. Providence, RI, 2014.
[31] L. Renner, Linear Algebraic Monoids, Springer-Verlag, 2005.

[32] Y. Sakellaridis, Inverse Satake transforms, Arxiv:1410.2312 [math.RT].
[33] I. Satake, Theory of spherical functions on reductive algebraic groups over p-adic fields, *Publ. Math. Inst. Hautes Études Sci.* **18** (1963) 5–69.
[34] R. Steinberg, A general Clebsch-Gordan theorem, *Bull. Amer. Math. Soc. (N.S.)* **67** (1961) 406–407.
[35] B. Sturmfels, On vector partition functions, *J. Combin. Theory Ser. A* **72** (1995) 302–309.
[36] T. Tamagawa, On the ζ-function of a division algebra, *Ann. of Math.* **77** (1963) 387–405.
[37] E.B. Vinberg, On reductive algebraic semigroups, in: Lie Groups and Lie Algebras: E.B. Dynkin's Seminar, pp. 145–182, Amer. Math. Soc. Transl. Ser. 2, 169, Adv. Math. Sci. 26, Amer. Math. Soc. Providence, RI, 1995.

(Bill Casselman)
E-mail address: wacasselman@gmail.com