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GLOBALLY ANALYTIC p-ADIC REPRESENTATIONS OF THE PRO–p–IWAHORI SUBGROUP OF $GL(2)$ AND BASE CHANGE, I : IWASAWA ALGEBRAS AND A BASE CHANGE MAP

L. CLOZEL

A Freydoon Shahidi, à l'occasion de son soixante-dixième anniversaire

Abstract. This paper extends to the pro-p Iwahori subgroup of $GL(2)$ over an unramified finite extension of $\mathbb{Q}_p$ the presentation of the Iwasawa algebra obtained earlier by the author for the congruence subgroup of level one of $SL(2,\mathbb{Z}_p)$. It then describes a natural base change map between the Iwasawa algebras or more correctly, as it turns out, between the global distribution algebras on the associated rigid-analytic spaces. In a forthcoming paper this will be applied to p-adic representation theory.
Keywords: Iwasawa theory, automorphic representations, rigid analytic geometry.
MSC(2010): Primary: 11R23; Secondary: 11F70, 14G22.

1. Introduction

This is the first part of a study of base change for globally analytic representations of the pro-p-Iwahori subgroup of $GL(2)$ over an unramified extension of $\mathbb{Q}_p$. This paper contains an extension to such fields, and to the pro-p-Iwahori, of the presentation given in [6] of the Iwasawa algebra. We are led naturally (Section 3.3) to a base change map; however, this map makes sense only for the algebras of global distributions on the rigid-analytic spaces; it is constructed in Chapter 4.

In the second part, we will follow the implications of this construction for certain p-adic representations of our groups. This Introduction concerns both papers.

1.1. We first recall some well-known facts about admissible complex representations of $GL(n,F)$ when F is a local field.

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Assume first that F is p–adic. Let E/F be an unramified extension of degree r. There exists a base change map which to each admissible irreducible representation π of $GL(n,F)$ associates an admissible representation π_E of $GL(n,E)$ [1, Section 2.2].

The local Langlands conjecture is known [7, 8] and we know that π is associated to a representation r of degree n of the Weil–Deligne group WD_F of F. (Here a systematic normalization of the correspondence must be specified, but this is inessential for us. For instance adopt the normalization of [7, Introduction]). We can restrict r to WD_E, to r_E, and it is known (indeed part of the proof) that r_E is associated to π_E.

There are two opposite kinds of representations. We can first consider the unramified principal series. Here

$$\pi = \mathrm{ind}_B^G(\chi_1, \ldots, \chi_n)$$

(unitary induction), where $G = GL(n,F)$, B is a Borel subgroup, and the χ_i are unramified characters of $F^\times$. Assume the χ_i unitary, so the induced representation is irreducible. Then

$$\pi_E = \mathrm{ind}_{B_E}^{G_E}(\eta_1, \ldots \eta_n)$$

where $G_E = GL(n,E)$, B_E is its Borel subgroup, and $\eta_i = \chi_i \circ N_{E/F}$.

In this case, base change is naturally associated to a homomorphism of Hecke algebras [9]. Let $\mathcal{H}_F$, $\mathcal{H}_E$ be the unramified Hecke algebras of compactly supported functions invariant by $GL(n,\mathcal{O}_F)$, $GL(n,\mathcal{O}_E)$. There is a homomorphism

$$b : \mathcal{H}_E \longrightarrow \mathcal{H}_F,$$

and the action of $\varphi \in \mathcal{H}_E$ on the spherical vector in π_E is given by $\lambda_F \circ b$, $\lambda_F : \mathcal{H}_F \to \mathbb{C}$ being the character associated to the spherical vector in π_F.

On the other hand, we can consider a supercuspidal representation π of $GL(n,F)$, associated to an irreducible representation r of W_F (thus also of WD_F.) In this case π has a minimal K–type – a type in the terminology of [4], which defines a representation τ of $GL(n,\mathcal{O}_F)$ [12].

This is uniquely associated to the *inertial type* of π, i.e., its equivalence class under twist by unramified characters of the determinant. This class corresponds bijectively to the restriction of r to the inertia $I_F \subset W_F$, an irreducible representation of I_F. Thus we can see τ, a representation of $GL(n,\mathcal{O}_F)$, as being associated to $r|_{I_F}$.

The restriction of r to I_E is not necessarily irreducible – of course, the cases of reducibility are easily understood. When it is, however, we see that the representation τ_E occurring in π_E is naturally associated to τ.

In this case, however, although the map $\pi \rightsquigarrow \pi_E$ is compatible, via the base change "transfer" [1, Section 1.3], to a *correspondence* of functions $\varphi \rightsquigarrow f$, there does not seem to exist, in general, a *homomorphism* of Hecke algebras realising this correspondence.

We end this introductory description with the case when $F = \mathbb{R}$ and $E = \mathbb{C}$. In this case the local correspondence, with representations of degree n of $W_{\mathbb{R}}$ (resp. $W_{\mathbb{C}}$), has been proved by Langlands [10] (completed by Speh [15]). We can consider the restricted class of the finite–dimensional, algebraic, irreducible representations of $GL(n, \mathbb{R})$.

If π is such a representation, on a complex vector space V, we can extend it holomorphically to $GL(n, \mathbb{C})$, identifying the extension $\mathbb{C}$ of $\mathbb{R}$ with the complex field of coefficients : we still denote it by π. Let $\bar{\pi}$ be the complex conjugate. Then [5, Proposition 3.2] $\pi \otimes \bar{\pi} = \pi_{\mathbb{C}}$ is the representation of $GL(n, \mathbb{C})$ associated by base change to π.

Finally, assume F is a finite field, E/F an extension of degree r, and fix an embedding of E into an algebraic closure $\bar{F}$ of F. Then Steinberg's theorem [16] tells us that any algebraic, irreducible representation π_E of $GL(n, E)$ (seen as an F-group) over $\bar{F}$ can be written

$$\pi_E = \pi_1 \otimes \pi_2^{\varphi} \otimes \cdots \otimes \pi_r^{\varphi^{r-1}},$$

where π_i is an irreducible algebraic representation of $GL(n, F)$, extended E–linearly to $GL(n, E)$, and φ is the Frobenius automorphism acting in $GL(n, E)$. If the π_i are isomorphic to π, we can see π_E as the base change of π. This is the construction that we will extend.

1.2. This paper arose from the natural attempt at extending to a p–adic field L the presentation given in [6] of the Iwasawa algebra of the level–1 subgroup

$$\Gamma_1 SL(2, \mathcal{O}_L) = \ker(SL(2, \mathcal{O}_L) \longrightarrow SL(2, k_L)) .$$

We extend this in two ways. First, following a suggestion of Gaëtan Chenevier (whom we heartfully thank) we consider instead the pro–p–Iwahori subgroup, denoted by I in this introduction :

$$I = \{g \in GL(2, \mathcal{O}_L) : \bar{g} \in N(k_L)\},$$

where $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$. This is a maximal pro–p– subgroup of $GL(2, L)$.

If L is unramified over $\mathbb{Q}_p$, we will give (for $p > 5$, see Chapter 2) a presentation of the Iwasawa algebra entirely similar to that given in [6].

To be more specific in this Introduction, assume now that k_r is an extension of degree r of $\mathbb{F}_p$ and that L is the fraction field of $\mathcal{O}_L = W(k_r)$, the ring of Witt vectors.

Let Λ_1, Λ_r be the Iwasawa algebras for the pro-p-Iwahori subgroups, respectively in $GL(2, \mathbb{Q}_p)$ and $GL(2, L)$. Then it appears immediately in the presentation obtained for Λ_1, Λ_r that there is a *formal* homomorphism

$$b : \Lambda_r \longrightarrow \Lambda_1 ,$$

exactly similar to that existing, in the classical situation between unramified Hecke algebras.

Here "formal" is taken in the colloquial, or *informal* sense. To explain this, note that, just as in [6], the algebra Λ_1 can be seen as a *space* of formal series

$$f = \sum_n a_n Y^{n_1} H^{n_2} X^{n_3} Z^{n_4} \tag{1.1}$$

($n = (n_1, n_2, n_3, n_4) \in \mathbb{N}^4$) with $a_n \in \mathbb{Z}_p$. (See §2.) Similarly, an element $\varphi \in \Lambda_r$ can be seen as a formal series, with coefficients in $\mathbb{Z}_p$, in $4r$ variables. (The variables, in each case, do not commute! Their relations are given before Theorem 3.1) Given φ we can naturally define an expansion (1.1) for a "distribution" f, but the coefficients a_n do not belong to $\mathbb{Z}_p$, and are not even integral.

It is then natural to inquire for which spaces of distributions φ, f (subspaces of Λ_r, Λ_1) this map is actually defined. It turns out that these spaces are the spaces $\mathcal{D}_r$, $\mathcal{D}_1$ of **global distributions** on the pro–p–Iwahori subgroups I_r, I_1.

Recall that on a p–adic Lie group such as I_r, we can consider the space $\mathcal{D}^{loc}$ introduced by Schneider and Teitelbaum (who denote it by $\mathcal{D}$) of distributions, dual to the space $\mathcal{C}_{an}$ of locally analytic functions. Here, for $p > 5$, our pro–p–Iwahori subgroups have a richer structure: they are saturated pro–p–groups in the sense of Lazard, i.e., group objects in the category of rigid–analytic varieties, respectively over $\mathbb{Q}_p$ and L, cf. [11, III. 1.3.4, III. 3.3.2]. We can then consider the $\mathbb{Q}_p$–vector space $\mathcal{A}_1$ of globally analytic functions on I_1 ($\cdots$ seen as a rigid–analytic space) and the $\mathbb{Q}_p$–vector space $\mathcal{A}_r$ of globally analytic function on I_r, seen by restriction of scalars as a rigid analytic space over $\mathbb{Q}_p$. The map which appeared formally in Section 3.3 is then obtained naturally in Section 4 as a map between the duals $\mathcal{D}_r$, $\mathcal{D}_1$ of $\mathcal{A}_r$ and $\mathcal{A}_1$. (Note that the spaces $\mathcal{A}$ and $\mathcal{D}$ are Banach spaces, unlike the topologically more complicated spaces of Schneider–Teitelbaum.) We must however, for this, extend scalars, so we obtain

$$b : \mathcal{D}_r \longrightarrow \mathcal{D}_1 \otimes_{\mathbb{Q}_p} L.$$

2. **Iwasawa algebra : the case of $\mathbb{Q}_p$**

2.1. Let $L/\mathbb{Q}_p$ be a finite extension, and e its ramification index. We will assume, in this paragraph, that L is **mildly ramified**, i.e.

$$2e < p - 1\,. \tag{2.1}$$

In particular $p \geq 5$. (Note that the condition (2.1) is adequate only because we are working with $GL(2)$. For $GL(n)$ it would be $ne < p - 1$, cf. [11, III. 3.2.7.5]).

Let $\mathfrak{p}$, $\mathcal{O} = \mathcal{O}_L$ denote the prime ideal, the integers of L. We denote by G the pro–p–Iwahori subgroup of $GL(2, L)$, i.e.

$$G = \left\{ g \in GL(2, \mathcal{O}_L) : g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} [\mathfrak{p}] \right\}.$$

Proposition 2.1 (Lazard)**.** *G is a saturated p–valued pro–p–group.*

We recall Lazard's proof. (Cf. also [14, p. 172].) Denote by v or v_p the canonical p–valuation on any finite extension of $\mathbb{Q}_p$, i.e., $v(p) = 1$.

By (2.1) we can choose a rational number α such that

$$\frac{1}{p-1} < \alpha < \frac{1}{e} - \frac{1}{p-1}.$$

Choose an extension L' of L containing an element a of (canonical) valuation α. Let $D \in GL(2, L')$ be the matrix $\begin{pmatrix} 1 & \\ & a \end{pmatrix}$; let w be the natural valuation $\inf\, v(x_{ij})$ on $M_2(L')$. We set, for $g \in G$,

$$v_G(g) = w(Dg\, D^{-1} - 1).$$

For $g = \begin{pmatrix} t & x \\ y & u \end{pmatrix} \in G$, $DgD^{-1} = \begin{pmatrix} t & ax \\ a^{-1}y & u \end{pmatrix}$.

This yields the formula

$$v_G(g) = \inf(v(x) + \alpha, v(y) - \alpha,\ v(t-1), v(u-1))\,. \tag{2.2}$$

Moreover, $v(t-1) \geq \frac{1}{e} > \frac{1}{p-1}$, $v(u-1) \geq \frac{1}{p-1}$,

$$\begin{aligned} v(x) + \alpha \quad &\geq \alpha > \frac{1}{p-1}, \\ v(y) - \alpha \quad &\geq \frac{1}{e} - \alpha > \frac{1}{p-1}. \end{aligned}$$

This implies [11, III. 3.2] that g is sent into the subgroup of $GL(2, L')$, where $w(g-1)$ is a valuation.

Moreover, it is saturated (if $v_G(g) > \frac{p}{p-1}$, g is a p–th power) because this is true in $GL(2, L')$ [11, III. 3.2].

2.2. The group G has a triangular decomposition

$$G = UTN\,,$$

$$\begin{aligned} U &= \left\{ \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} \quad : y \in \mathfrak{p} \right\}, \\ T &= \left\{ \begin{pmatrix} t & \\ & v \end{pmatrix} \quad : t, v \in \mathcal{O},\ t,\ v \equiv 1[\mathfrak{p}] \right\}, \\ N &= \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad : x \in \mathcal{O} \right\}. \end{aligned}$$

For $g = uhn$, we obtain the expression

$$\begin{pmatrix} t & tx \\ ty & txy + v \end{pmatrix}. \tag{2.3}$$

Since $p \geq 5$, the torus T is the direct product of the centre $Z_G = \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} \right\}$ $(z \equiv 1)$ and of the torus T_S in $SL(2)$:

$$T_S = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\} \qquad (t \equiv 1).$$

The valuation v_G restricts to a valuation v_S on the group $S = G \cap SL(2, L)$. In this group we have the expression

$$g = \begin{pmatrix} t & tx \\ ty & txy + t^{-1} \end{pmatrix}.$$

Lemma 2.2. *For $g \in S$,*

$$v_S(g) = \inf(v(x) + \alpha, v(y) - \alpha,\ v(t-1)).$$

We first prove :

Lemma 2.3. *For $x \in \mathcal{O}$, $y \in \mathfrak{p}$,*

$$v(xy) > \inf(v(x) + \alpha,\ v(y) - \alpha).$$

Assume first $v(y) - \alpha \leq v(x) + \alpha$. Since

$$v(xy) = (v(x) + \alpha) + (v(y) - \alpha)$$

it suffices to prove that $v(x) + \alpha > 0$, which is clear; if

$$v(x) + \alpha \leq v(y) - \alpha,$$

we must check that $v(y) > \alpha$; since $y \in \mathfrak{p}$,

$$v(y) \geq \frac{1}{e} > \alpha.$$

We can now prove Lemma 2.2. Write $z = txy + t^{-1}$. Since $|t| = 1$, we must check:

$$\inf(v(x) + \alpha,\ v(y) - \alpha,\ v(z-1)) = \inf(v(x) + \alpha,\ v(y) - \alpha,\ v(t-1)). \tag{2.4}$$

Write

$$v = v(t-1),\ v_1 = v(z-1)\,,$$
$$X = \inf(v(x) + \alpha,\ v(y) - \alpha).$$

Then (2.4) is equivalent to

$$\begin{aligned} v_1 \geq X \quad &\text{if} \quad v \geq X, \\ v_1 = v \quad &\text{if} \quad v < X. \end{aligned}$$

If $v = v(t^{-1} - 1) \geq X$, $v(xy) > X$ by the previous lemma, so $v(z-1) \geq X$. If $v < X < v(xy)$, $v(z-1) = v$, which proves the Lemma.

2.3. From now on we assume L unramified over $\mathbb{Q}_p$. Let $e_1, \dots e_r$ be a basis of $\mathcal{O}_L$ over $\mathbb{Z}_p$. If $x = \sum x_i e_i \in \mathcal{O}_L$ $(x_i \in \mathbb{Z}_p)$, we then have

$$v(x) = \inf v(x_i). \tag{2.5}$$

The elements e_i define generators

$$\mathbf{y}_i = \begin{pmatrix} 1 & \\ pe_i & 1 \end{pmatrix}$$

of U, and

$$\mathbf{x}_i = \begin{pmatrix} 1 & e_i \\ & 1 \end{pmatrix}$$

of N. We can write the torus T as $T = Z_G T_S$, the product being direct since L contains no square root of 1. We have generators for T_S:

$$\mathbf{h}_i = \begin{pmatrix} (t'_i)^{-1} & \\ & (t'_i) \end{pmatrix}, \; t'_i = (1 + pe_i)^{1/2}$$

and for Z_G:

$$\mathbf{z}_i = \begin{pmatrix} 1 + pe_i & \\ & 1 + pe_i \end{pmatrix}.$$

From Lemma 2.2 and (2.5) we immediately deduce (note that $v_G(\mathbf{y}_i) = 1 - \alpha$):

Proposition 2.4. *The elements $\boldsymbol{y}_i$, $\boldsymbol{h}_i$, $\boldsymbol{z}_i$, $\boldsymbol{x}_i$ $(1 \le i \le r)$ form an ordered basis of G, in the sense of Lazard: any element of G can be written uniquely*

$$g = \prod \boldsymbol{y}_i^{\eta_i} \cdot \prod \boldsymbol{h}_i^{\tau_i} \cdot \prod \boldsymbol{z}_i^{\zeta_i} \cdot \prod \boldsymbol{x}_i^{\xi_i}$$

for exponents $\eta_i, \dots, \xi_i \in \mathbb{Z}_p$. Moreover

$$v_G(g) = \inf(v(\eta_i) + 1 - \alpha, \; v(\tau_i), v(\zeta_i), \; v(\xi_i) + \alpha). \tag{2.6}$$

Remark 2.5. Assume $L = \mathbb{Q}_p$ and G is replaced by the group of [6]: $\{g \in SL(2, \mathbb{Z}_p) : g \equiv 1[p]\}$. The same argument shows that the subgroups U, N, T_S (with now $x \in p\mathbb{Z}_p$) form an ordered basis; furthermore, the formula similar to (2.6) is true. (This was omitted in [6].)

2.4. We now consider first the case where $L = \mathbb{Q}_p$. Thus

$$g = \mathbf{y}^\eta \mathbf{h}^\tau \mathbf{z}^\zeta \mathbf{x}^\xi$$

with obvious notations. Let Λ_G be the Iwasawa algebra of G (with coefficients in $\mathbb{Z}_p$). Recall that the Iwasawa algebra of a p-adic group G is the $\mathbb{Z}_p$-module of continuous linear forms $\mathcal{C}(G) \to \mathbb{Z}_p$, where $\mathcal{C}(G)$ is the space of continuous functions $G \to \mathbb{Z}_p$, endowed with the convolution product. For simplicity, we will consider the group S: this is harmless as G is a direct product. We set, δ denoting a Dirac measure:

$$Y = \delta(\mathbf{y}) - 1, \; H = \delta(\mathbf{h}) - 1, \; X = \delta(\mathbf{x}) - 1.$$

The (ordered) decomposition of S as a product:

$$g = uhn \qquad (g \in S)$$

then, as in [6], implies that we can consider Y, H, X as distributions on S. Moreover, as a space $\Lambda = \Lambda_S$ is the completed tensor product

$$\Lambda_U \widehat{\otimes} \Lambda_T \widehat{\otimes} \Lambda_N$$

with basis over $\mathbb{Z}_p$ the elements $Y^{n_1}H^{n_2}X^{n_3}$ $(n_i \geq 0)$. This is **equal** to the product $Y^{n_1} \cdot H^{n_2} \cdot X^{n_3}$ in the Iwasawa algebra.

We determine the commutation relation between these elements. The computation is the same as in [6]; we have changed the generator $\mathbf{h}$ in order to simplify the formulas. We have, with $q = 1 + p$:

(a) $(1 + X)(1 + H) = (1 + H)(1 + X)^q$,

(b) $(1 + H)(1 + Y) = (1 + Y)^q(1 + H)$.

Moreover,

$$\mathbf{xy} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1+p & 1 \\ p & 1 \end{pmatrix}.$$

Given the expression

$$g = \begin{pmatrix} t & tx \\ ty & txg + t^{-1} \end{pmatrix},$$

we have $t = 1 + p = ((1 + p)^{-1/2})^{-2}$, $tpy' = p$ so $y' = t^{-1} = (1 + p)^{-1}$ and likewise $x = (1 + p)^{-1}$. Thus

(c) $(1 + X)(1 + Y) = (1 + Y)^Q(1 + H)^{-2}(1 + X)^Q$ $(Q = (1 + p)^{-1})$.

The elements X, H, Y therefore verify in Λ the relations (a), (b), and (c).

We now consider the universal non–commutative p–adic algebra $\mathcal{A} = \mathbb{Z}_p\{\{Y, H, X\}\}$ over Y, H, X (cf. [6]), with maximal ideal $\mathcal{M}_A$ generated by (p, Y, H, X) and a prime ideal $\mathcal{P}_A$ generated by (Y, H, X). It contains the dense subalgebra $A = \mathbb{Z}_p\{Y, H, X\}$ of non–commutative polynomials. Let $\mathcal{R} \subset \mathcal{A}$ be the closed two–sided ideal generated by the relations (a), (b) and (c); let $\bar{\mathcal{A}} = \mathbb{F}_p\{\{Y, H, X\}\}$ and $\bar{\mathcal{R}}$ the image of $\mathcal{R}$ in $\bar{\mathcal{A}}$. As in [6] we have:

Lemma 2.6. *$\bar{\mathcal{R}}$ is the closed two–sided in $\bar{\mathcal{A}}$ generated by the relations* (a), (b) *and* (c)

There is a natural map $A \to \Lambda$.

Lemma 2.7. *This extends to a continuous map $\mathcal{A} \to \Lambda$.*

It suffices to show that if a sequence in A converges to 0 in the topology of $\mathcal{A}$, the image converges to 0 in Λ.

The topology of $\mathcal{A}$ is given by the valuation $v_{\mathcal{A}}$; write

$$F = \sum_n \sum \lambda_i x^i\,,$$

$n \geq 0, i$ ranging over the maps $\{1, \ldots, n\} \to \{1, 2, 3\}$, and

$$x^i = x_{i_1} x_{i_2} \cdots x_{i_n}\,; \tag{2.7}$$

we set $x_1 = Y$, $x_2 = H$, $x_3 = X$. For $F \neq 0$,

$$v_{\mathcal{A}}(F) = \inf_{n,i}(v_p(\lambda_i) + n).$$

Lazard [11, III. 2.3] shows that Λ has an additive valuation ($v(\lambda\mu) = v(\lambda) + v(\mu)$) given by $v(\lambda) = v_p(\lambda)(\lambda \in \mathbb{Z}_p)$, $v(Y) = 1 - \alpha$, $v(H) = 1$, $v(X) = \alpha$. For $F \in A$, with image $\mu \in \Lambda$, we have

$$\begin{aligned} v(\mu) \quad &= v\Big(\sum_n \sum \lambda_i x^i\Big) \\ &\geq \inf_{n,i}(v_p(\lambda_i) + (1-\alpha)n(1) + n(2) + \alpha n(3)) \end{aligned}$$

where $n(1)$ is the number of elements in $[1 \ldots n]$ with image 1, etc...

$$\geq \inf(\alpha, 1-\alpha)\; v_{\mathcal{A}}(F).$$

This proves the continuity, whence the existence of the map $\mathcal{A} \to \Lambda_G$. Note that this is surjective, since the ordered monomials $Y^{n_1} H^{n_2} X^{n_3}$ are obtained.

Theorem 2.8. *The map $\mathcal{A} \to \Lambda$ gives, by passing to the quotient, an isomorphism $\mathcal{B} = \mathcal{A}/\mathcal{R} \xrightarrow{\sim} \Lambda$*

As in our previous paper, we will use for the proof the corresponding algebras with finite coefficients. Thus let $\Omega = \Lambda \bigotimes_{\mathbb{Z}_p} \mathbb{F}_p$ be the algebra of distributions with values in $\mathbb{F}_p$. We now have a natural map $\bar{A} = A \otimes \mathbb{F}_p \to \Omega$. The previous proof shows that it extends naturally to a map $\bar{\mathcal{A}} \to \Omega$, whose kernel contains $\bar{\mathcal{R}}$. We will show:

Theorem 2.9. *The map $\bar{\mathcal{A}} \to \Omega$ gives an isomorphism $\bar{\mathcal{B}} = \bar{\mathcal{A}}/\bar{\mathcal{R}} \xrightarrow{\sim} \Omega$.*

Note that $\bar{\mathcal{A}}$ has again an natural valuation, given on $F \neq 0$ by $\inf_n n$, the infimum being taken on the integers n such that there exists a non–zero monomial of degree n in the expression of F.

Exactly as in [6], Theorem 2.8 follows from Theorem 2.9. We now begin the proof of this last result. Note that

$$\frac{1}{p-1} < \frac{1}{2} < \frac{p-2}{p-1}$$

since $p > 3$. We can therefore take $\alpha = 1/2$. Then

$$v(Y) = v(X) = \frac{1}{2}\,,\; v(H) = 1.$$

The Iwasawa algebra Λ is filtered by $\frac{1}{2}\mathbb{N}$. As an $\mathbb{F}_p$–vector space, $gr^\nu\,\Lambda$ is generated by the independent elements

$$p^\ell Y^{n_1} H^{n_2} X^{n_3}\,,\; \ell + \frac{1}{2}n_1 + n_2 + \frac{1}{2}n_3 = \nu.$$

(Cf. [14, p. 199].) The filtration of Λ defines a filtration of Ω by $Fil^\nu\Omega = Fil^\nu\Lambda \otimes \mathbb{F}_p$; we see that $gr^\nu\ \Omega$ is then generated by the elements

$$Y^{n_1}H^{n_2}X^{n_3}\ ,\ \frac{1}{2}n_1 + n_2 + \frac{1}{2}n_3 = \nu\ ,$$

and these elements are linearly independent.

We do not change the filtration by replacing $\nu \in \frac{1}{2}\mathbb{N}$ by $n = 2\nu \in \mathbb{N}$, the valuations of (Y, H, X) being now (1,2,1).

In particular, we now have (for $n \in \mathbb{N}$)

Lemma 2.10. *The dimension of $gr^n\Omega$ $(n \geq 0)$ over $\mathbb{F}_p$ is equal to the dimension of the space of homogeneous symmetric polynomials of degree n in (Y, H, X), the variables having degrees $(1, 2, 1)$.*

Note that this dimension is equal to $(\mu + 1)(n + 1 - \mu)$ where $\mu = [\frac{n}{2}]$. We denote it by d_n.

We must now consider on $\bar{\mathcal{A}}$ the filtration of $\bar{\mathcal{A}}$ obtained by assigning degrees (1,2,1) to the (formal) variables Y, H, X. Then $gr^n\bar{\mathcal{A}}$ is isomorphic to the space of non–commutative polynomials of degree n. We endow $\bar{\mathcal{B}} = \bar{\mathcal{A}}/\bar{\mathcal{R}}$ with the induced filtration, so

$$gr^n\bar{\mathcal{B}} = Fil^n\bar{\mathcal{A}}/Fil^{n+1}\bar{\mathcal{A}} + (Fil^n\bar{\mathcal{A}} \cap \bar{\mathcal{R}}).$$

The map $gr^n\bar{\mathcal{B}} \to gr^n\Omega$ is surjective by construction. We now have:

Lemma 2.11. $\dim gr^n\bar{\mathcal{B}} \leq d_n$ $(n \geq 0)$.

The Lemma is obvious for $n = 0, 1$. We also note that the generators of $\bar{\mathcal{R}}$ given by the relations (a, b, c) belong to $Fil^2\bar{\mathcal{A}}$. This is obvious for (a, b) since $q \equiv 1$ $[p]$; for (c) we obtain modulo terms of *ordinary* degree 2 in Y, H, X:

$$(1 + X)(1 + Y) - (1 + Y)(1 + H)^{-2}(1 + X) = 2H$$

and $H \in Fil^2\bar{\mathcal{A}}$.

Consider now $gr^2\bar{\mathcal{B}}$, a quotient of the space $Fil^2\bar{\mathcal{A}}/Fil^3\bar{\mathcal{A}}$ with basis XY, YX, H, X^2, Y^2. Since $Q \equiv 1$ $[p]$, $Q - 1 \equiv 0$ $[p]$ and H is of degree 2, the relation (c) yields

$$1 + X + Y + XY = 1 + Y + X + YX - 2H$$

modulo terms of degree ≥ 3. Thus $\dim gr^2\bar{\mathcal{B}} \leq d_2 = 4$.

In order to prove the general case, we state the necessary relations in $\bar{\mathcal{B}}$:

Lemma 2.12. (i) $XY - YX + 2H = 0 \quad (\bmod Fil^3)$
(ii) $XY - HX = 0 \quad (\bmod Fil^4)$
(iii) $HY - YH = 0 \quad (\bmod Fil^4)$

Part (i) has just been proven. For (ii) consider the identity (b). We have

$$\begin{aligned}(1 + X)^q &= 1 + qX + \frac{q(q-1)}{2}X^2 + \frac{q(q-1)(q-2)}{6}X^3 + \cdots \\ &= 1 + X \ (\bmod X^4),\end{aligned}$$

since $q = 1 + p$, $p > 3$. Thus

$$(1 + X)(1 + H) = (1 + H)(1 + X) \mod Fil^4,$$

since the filtration is multiplicative on $\bar{\mathcal{A}}$. Whence (ii), and similarly (iii).

We can now prove Lemma 2.11. Consider a non–commutative monomial cf., (2.7):

$$x^i = x_{i_1} x_{i_2} \cdots x_{i_t}\,. \tag{2.8}$$

Assume the homogeneous degree of x^i is equal to n (H being of course of degree 2). As in [6, Lemma 3.2] we can change x^i into a well–ordered monomial ($\alpha \mapsto i_\alpha$ increasing) by a sequence of transpositions. Consider a move $(\alpha, \alpha+1) \mapsto (\alpha+1, \alpha)$ and assume $i_\alpha > i_{\alpha+1}$. Write

$$x^i = x^j \; x_\alpha \; x_{\alpha+1} \; x^k$$

write $\deg(i) = n$, $\deg(j) = r$, $\deg(k) = s$. There are three possibilities

$$(x_\alpha, x_{\alpha+1}) = (X, H) \tag{2.9}$$

Note that the degree of HX is 3. By Lemma 2.12(ii), $XH = HX$ mod Fil^4. Multiplying by x^j and x^k, we obtain $x^j HX x^k$ (mod $Fil^{r+4+s} = Fil^{n+1}$).

(2.10) The case (H, Y) is similar.

$$(x_\alpha, x_{\alpha+1}) = (X, Y). \tag{2.11}$$

Now $XY = YX + 2H$ (mod Fil^3), all elements being of degree 2. The first term yields again (mod Fil^{n+1}) a monomial with fewer inversions.

The second term yields a monomial with the same homogeneous degree, but with trivial degree - the number t in (2.8) - lowered by 1. We have to show that we have decreased the number of inversions in the monomial. For x^i, it was

$$inv = \sum_{\substack{\beta<\gamma \\ i_\beta > i_\gamma}} 1 + \sum_{\substack{\gamma>\alpha+1 \\ i_\alpha = 3 > i_\gamma}} 1 + \sum_{\substack{\beta<\alpha \\ i_\beta > i_{\alpha+1} = 1}} 1 + 1$$

where the indices β, γ are different from α, $\alpha + 1$. After contraction of XH into (2 times) H, we obtain

$$inv' = \sum_{\substack{\beta<\gamma \\ i_\beta > i_\gamma}} 1 + \sum_{\substack{\gamma>\alpha+1 \\ i_\alpha = 2 > i_\gamma}} 1 + \sum_{\substack{\beta<\alpha \\ i_\beta > 2}} 1$$

Clearly $inv' < inv$. By performing the induction on the number of inversions (independently of the trivial degree), the proof of Lemma 2.11 is now completed.

Since $gr^n \bar{\mathcal{B}} \to gr^n \Omega$ was surjective, the proof of Theorem 2.9 is complete.

Corollary 2.13. *The Iwasawa algebra Λ_G is a quotient $\mathcal{A}/\mathcal{R}$, with $\mathcal{A} = \mathbb{Z}_p\{\{Z, Y, H, X\}\}$ and $\mathcal{R}$ defined by the relations (a, b, c) and*

(Comm) *Z commutes with Y, H, X.*

3. Iwasawa algebras in the unramified case

3.1. We return to the unramified extension L of $\mathbb{Q}_p$, of degree r; notations are as in Section 2.3. In particular, the Lazard basis of G is given by Proposition 2.4. This depends on the choice of a basis (e_i) of $\mathcal{O}_L = W(\mathbb{F}_{p^r})$, the ring of Witt vectors, over $\mathbb{Z}_p$[1].

As in Section 2.4, we will in fact compute the Iwasawa algebra Λ_S (with $\mathbb{Z}_p$–coefficients) of $S = G \cap SL(2, L)$, since G is a direct product. Associated to the (ordered) decomposition $g = uhn$, we have a decomposition $\Lambda_S = \Lambda_U \widehat{\otimes} \Lambda_T \widehat{\otimes} \Lambda_N$ (completed tensor product.) Each factor is a commutative Iwasawa algebra in r variables, respectively Y_i, H_i and X_i, defined as in Section 2.4. In particular $\Lambda = \Lambda_S$ has a topological basis.

$$Y^\ell \; H^m \; X^n \tag{3.1}$$

with $\ell = (\ell_1, \dots \ell_r)$, $Y^\ell = Y_1^{\ell_1} \cdots Y_r^{\ell_n}$, etc.

As recalled, the elements Y^ℓ (*resp.* H^m, X^n) commute. We want to write the other commutation relations. We have

$$\mathbf{h}_i \; \mathbf{y}_j \; \mathbf{h}_i^{-1} = \begin{pmatrix} 1 & \\ p(1+pe_i)e_j & 1 \end{pmatrix}.$$

Write

$$(1+pe_i)e_j = \sum_k q_{ijk} e_k \quad (q_{ijk} \in \mathbb{Z}_p). \tag{3.2}$$

Then

$$\begin{aligned} q_{ijj} &\equiv 1 \qquad [p] \\ q_{ijk} &\equiv 0 \qquad [p] \quad (k \neq j). \end{aligned} \tag{3.3}$$

The commutation relation implies

$$\text{(A)} \quad (1+X_j)(1+H_i) = (1+H_i) \prod_{k=1}^{r} (1+X_k)^{q_{ijk}}$$

$$\text{(B)} \quad (1+H_i)(1+Y_j) = \prod_{k=1}^{r} (1+Y_k)^{q_{ijk}} (1+H_i)$$

with the same exponents.

We must now compute the "triangular" relation, the analogue of (c) in Section 2.4. We have

$$\begin{aligned} \mathbf{x}_i \mathbf{y}_j &= \begin{pmatrix} 1 & e_i \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ pe_j & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+pe_ie_j & e_i \\ pe_j & 1 \end{pmatrix}. \end{aligned}$$

[1] We do not see how to obtain a natural basis of $W(\mathbb{F}_{p^r})$. In particular, the structure constants q_{ijk}, Q_{ijk} of Λ_G do not seem to have a natural expression.

From the expression of the triangular decomposition (before Lemma 2.2):

$$g = \begin{pmatrix} t & tx \\ ty & txy + t^{-1} \end{pmatrix}$$

we see that

$$\begin{aligned} t = 1 + pe_ie_j &= \prod_k (1+pe_k)^{-1/2 P_{ijk}}, \\ e_j &= t\sum_k Q_{ijk} e_k \\ e_i &= t\sum_k Q_{jik} e_k. \end{aligned}$$

The exponents $P_{ijk} \in \mathbb{Z}_p$ are defined by

$$\log(1+pe_ie_j) = -\frac{1}{2}\sum_k P_{ijk} \log(1+pe_k).$$

Such an expression exists since exp and $\log(1+X)$ are inverse diffeomorphisms between $p\mathcal{O}_L$ and $1+p\mathcal{O}_L$. For $p > 2$ and $x \in p\mathcal{O}_L$,

$$\log(1+x) \equiv x \quad [p^2].$$

Indeed it suffices to show that $v_p(\frac{x^n}{n}) \geq 2 \quad (n \geq 2)$. This is clear for $n = 2$; for $n \geq 3$,

$$v_p(\frac{x^n}{n}) \geq n - \log_p n \geq n - \log_3 n \geq 2 \quad (n \geq 3),$$

with equality only if $n = p = 3$. Dividing the previous equality by p, we see that

$$e_ie_j \equiv -\frac{1}{2}\sum_k P_{ijk} e_k \quad [p]. \tag{3.4}$$

Moreover,

$$\begin{aligned} (1+pe_ie_j)^{-1} e_j &= \sum_k Q_{ijk} e_k \\ (1+pe_ie_j)^{-1} e_i &= \sum_k Q_{jik} e_k \end{aligned} \tag{3.5}$$

and the coefficients verify

$$\begin{aligned} Q_{ijj} &\equiv 1[p] \\ Q_{ijk} &\equiv 0[p] \quad (k \neq j) \end{aligned} \tag{3.6}$$

We now have the relation:

$$(1+X_i)(1+Y_j) = \prod_{\alpha,\beta,\gamma} (1+Y_\alpha)^{Q_{ij\alpha}} (1+H_\beta)^{P_{ij\beta}} (1+X_\gamma)^{Q_{ji\gamma}}. \tag{C}$$

Now, as in Section 2.4, we consider in the non–commutative algebra $\mathcal{A} = \mathbb{Z}_p\{\{Y_i, H_j, X_k\}\}$ the ideal $\mathcal{R}$ generated by the relations (A), (B) and (C) and

(Comm$_0$) *The variables* Y_i, H_j, X_k *(separately) commute.*

We have again

Theorem 3.1. *The natural homomorphism*

$$\mathcal{A}/\mathcal{R} \longrightarrow \Lambda$$

is an isomorphism.

If we consider, as in Section 2.4, $\bar{\mathcal{A}}$ and Ω, we have likewise:

Theorem 3.2. *The natural homomorphism*

$$\bar{\mathcal{A}}/\bar{\mathcal{R}} \longrightarrow \Omega$$

is an isomorphism.

3.2. We sketch the proof, which is very similar to that given for $\mathbb{Q}_p$. We choose the coefficient α of Section 2.1 equal to $\frac{1}{2}$. Again, Λ is filtered by $\frac{1}{2}\mathbb{N}$, $gr^\nu\Lambda$ having a basis given by the independent elements

$$p^t Y^\ell H^m X^n\,,\ t + \frac{1}{2}|\ell| + |m| + \frac{1}{2}|n| = \nu.$$

The elements $Y^\ell H^m X^n$ ($\frac{1}{2}|\ell| + |m| + \frac{1}{2}|n| = \nu$) give a basis of $gr^\nu\Omega$.

The foregoing argument shows that the natural maps $\mathcal{A} \to \Lambda$, $\bar{\mathcal{A}} \to \Omega$ are well–defined. We give weights (1,2,1) to the elements Y_i, H_j, X_k and replace in Ω the filtration by $\frac{1}{2}\mathbb{N}$ by the filtration by $n = 2\nu \in \mathbb{N}$. We have the analogue of Lemma 2.12; first we have

Lemma 3.3. *In the algebra* $\bar{\mathcal{A}}$

(i) $(1+X_k)^{q_{ijk}} \equiv 1 \pmod{Fil^4}, \quad k \neq j$

$(1+X_k)^{q_{ijk}} \equiv 1 + X_k \pmod{Fil^4}, \quad k = j.$

The same relations are verified by the Y_k.

(ii) *Same relations, with* q_{ijk} *replaced by* Q_{ijk}.

(iii) $\prod_k (1+H_k)^{P_{ijk}} \equiv 1 + \sum P_{ijk} H_k \pmod{Fil^4}.$

Relation (iii) is obvious; (i) and (ii) follow from (3.3) and (3.6), since $p \geq 5$.

Lemma 3.4. *In the algebra* $\bar{\mathcal{B}} = \bar{\mathcal{A}}/\bar{\mathcal{R}}$,

(i) $X_j H_i - H_i X_j \equiv 0 \pmod{Fil^4},$

(ii) $H_i Y_i - Y_j H_i \equiv 0 \pmod{Fil^4},$

(iii) $X_j Y_j - Y_i X_i \equiv \sum_k P_{ijk} H_k \pmod{Fil^3}.$

Part (i) follows from (A) and (i) of the previous Lemma; similarly (ii) follows from (B) and (i). Now from (C) we have, using (ii) of that Lemma:

$$\begin{aligned}(1+X_i)(1+Y_i) &\equiv (1+Y_i)\prod(1+H_\beta)^{P_{ij\beta}}(1+X_i) \pmod{Fil^4}\\ &\equiv 1 + Y_j + X_i + Y_j X_i + \sum \bar{P}_{ijk} H_k \pmod{Fil^3}.\end{aligned}$$

By Lemma 3.4, we see first that $\dim gr^1\bar{\mathcal{B}} = 2r$ (linear terms in the X_i, Y_i), then that $\dim gr^2\bar{\mathcal{B}} = \frac{2r(2r+1)}{2} + r$ (symmetric polynomials of degree 2 in the X_i, Y_j and linear forms in the H_k). Finally, the argument given for the proof of Lemma 2.11 shows that the dimension of $gr^n\bar{\mathcal{B}}$ is smaller than the dimension of the space of symmetric polynomials of degree n in the Y_i H_i, X_k (with the correct degrees.) This proves Theorem 3.2, and Theorem 3.1 follows.

Changing notations, we now have:

Corollary 3.5. *The Iwasawa algebra* Λ_G *of* G *is the quotient of* $\mathcal{A}= \mathbb{Z}_p\{\{Y_i, H_i, X_i, Z_i\}\}$ *by the ideal* $\mathcal{R}$ *generated by relations* (A), (B), (C), (Comm$_0$), *and*
(Comm$_1$) *The variables* Z_i *commute, and commute with* Y_i, H_i, X_i.

3.3. We now show that, at least formally (in the colloquial sense) there exists a map $\Lambda_{G_r} \to \Lambda_{G_1}$ where G_r is the group associated to L and G_1 the group associated to $\mathbb{Q}_p$. Denote by $\mathcal{A}_r$ the algebra of non–commutative series in $4r$ variables of Corollary 3.5, and $\mathcal{A}$ the algebra of non–commutative series in 4 variables. We would like to define a map b: $\mathcal{A}_r \to \mathcal{A}$ sending the ideal $\mathcal{R}_r$ of relations to $\mathcal{R}$. We work with the Iwasawa algebras Λ_r, Λ of the group S.

Recall the relations defining $\mathcal{A}$:

(a) $(1+X)(1+H) = (1+H)(1+X)^q$,
(b) $(1+H)(1+Y) = (1+Y)^q(1+H)$,
(c) $(1+X)(1+Y) = (1+Y)^Q(1+H)^{-2}(1+X)^Q$.

where $q = 1+p$, $Q = (1+p)^{-1}$
and $\mathcal{A}_r$:

$$\text{(A)}\quad (1+X_j)(1+H_i) = (1+H_i)\prod_{k=1}^{r}(1+X_k)^{q_{ijk}},$$

$$\text{(B)}\quad (1+H_i)(1+Y_j) = \prod_{k=1}^{r}(1+Y_k)^{q_{ijk}}(1+H_i),$$

$$\text{(C)}\quad (1+X_i)(1+Y_j) = \prod_{\alpha,\beta,\gamma}(1+Y_\alpha)^{Q_{ij\alpha}}(1+H_\beta)^{P_{ij\beta}}(1+X_\gamma)^{Q_{ji\gamma}}),$$

where

$$\begin{aligned} (1+pe_i)e_j &= \sum_k q_{ijk}e_k, \\ (1+pe_ie_j)^{-1}e_j &= \sum_k Q_{ijk}e_k, \\ \log(1+pe_ie_j) &= -\tfrac{1}{2}\sum_k P_{ijk}\log(1+pe_k). \end{aligned}$$

The map b is defined if we give the images of $1+X_i$, $1+H_i$, $1+Y_i$. We first try to satisfy the relations (a,b).

The set $\mathcal{A}_1^\times$ of elements of the form $1+F$, where F belongs to the prime ideal (Y,H,X) of $\mathcal{A}$, is a multiplicative subgroup of $\mathcal{A}^\times$; it is a compact pro–p–group. In particular g^x is well–defined if $x \in \mathbb{Z}_p$ and g belongs to $\mathcal{A}_1^\times$.

It is natural to consider a map b given by

$$\begin{aligned} 1+H_i &\mapsto (1+H)^{\alpha_i} \\ 1+X_j &\mapsto (1+X)^{\beta_j} \\ 1+Y_j &\mapsto (1+Y)^{\beta_j}. \end{aligned}$$

If $\mathbf{h} = 1+H$, $\mathbf{x} = 1+X$, the relation (a) can be written

$$\mathbf{h}^{-1}\mathbf{x}\mathbf{h} = \mathbf{x}^q$$

which implies, for α, $\beta \in \mathbb{Z}_p$:

$$\mathbf{h}^{-\alpha}\mathbf{x}^{\beta}\mathbf{h}^{\alpha} = \mathbf{x}^{\beta q^{\alpha}}. \tag{3.7}$$

Note that q^α is well–defined since $q = 1+p$.

The relation (a) will follow from (A) if

$$(1+X)^{\beta_j}(1+H)^{\alpha_i} = (1+H)^{\alpha_i}(1+X)^{\sum\limits_k q_{ijk}\beta_k}$$

which is true, according to (3.7), if

$$\beta_j q^{\alpha_i} = \sum_k q_{ijk}\beta_k. \tag{3.8}$$

Comparing with the expression of the exponents in relations (A), (B) and (C), we see that this will be verified if

$$\beta_j = e_j\,,\ q^{\alpha_i} = 1+pe_i\,, \tag{3.9}$$

or equivalently

$$\beta_j = e_j\,,\ \alpha_i = \frac{\log(1+pe_i)}{\log(1+p)}\,. \tag{3.10}$$

Furthermore, it can be checked that there is no other solution to the equations (3.8) for α_i, $\beta_j \in \mathcal{O}_L$. In particular, there is no solution with exponents in $\mathbb{Z}_p$. This implies that the images of our generators $1+H_i$, etc. will have coefficients in L. However, even if we extend coefficients, their images are not in $\Lambda \otimes L$.

The reductions $\bar{e_j}$ of e_j in k_L form a basis of k_L over $\mathbb{F}_p$. We may assume that $\bar{e}_1 = 1$, and then $|e_j - r| = 1$ $(j > 1)$ for any $r \in \mathbb{Z}$. The formal series

$$(1+X)^{e_j} = \sum_{m=0}^{\alpha} \binom{e_i}{m} X^m = \sum a_m X^m \tag{3.11}$$

then verifies, for the canonical valuation on $\mathcal{O}_L$:

$$v_p(a_m) = -v_p(m!) = -\frac{1}{p-1}(m - \operatorname{Schiff} m)$$

(Lazard's notation, [11, III. 1.1.2.5]) where Schiff m is the sum of the digits of m in base p. In particular the coefficients tend to infinity (in absolute value). We will return to the growth of the images at the end of this section.

We neglect this for now, and (computing formally) consider the other relations. By the usual symmetry, (b) is automatically satisfied. We must consider relation (c). Applying our map to (C), we must investigate in $\mathcal{A}$ the relation

$$(1+X)^{\beta_i}(1+Y)^{\beta_j} = (1+Y)^{\Sigma\beta_k Q_{ijk}}(1+H)^{\Sigma\alpha_k P_{ijk}}\cdot(1+X)^{\Sigma\beta_k Q_{jik}}, \tag{3.12}$$

where β_i, α_i are given by (3.10). This is to be compared with (c).

Unlike the two previous cases, then is no formal fashion to deduce (3.12) from (c). We must return to the way (c) was obtained. It was equivalent to the relation (in S):

$$\mathbf{xy} = \begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ p & 1\end{pmatrix} = \begin{pmatrix}1 & \\ Qp & 1\end{pmatrix}\begin{pmatrix}(1+p)^{-1/2} & \\ & (1+p)^{-1/2}\end{pmatrix}^{-2}\begin{pmatrix}1 & Q\\ & 1\end{pmatrix}.$$

In the additive groups U and N, it is natural to assume that $\mathbf{x}^{\beta_i} = \mathbf{x}^{e_i} = \begin{pmatrix}1 & e_i\\ & 1\end{pmatrix}$, and $\mathbf{y}^{e_j} = \begin{pmatrix}1 & \\ pe_j & 1\end{pmatrix}$ (elements now of the L–points of the groups). Now

$$\mathbf{x}^{e_i}\mathbf{y}^{e_j} = \begin{pmatrix}1 & e_i\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ pe_j & 1\end{pmatrix} = \begin{pmatrix}1+pe_ie_j & e_i\\ pe_j & 1\end{pmatrix}.$$

We are reduced to the computation of Section 3.1, and this yields relation (C) which, after applying the map b, implies (3.12).

This argument is admittedly formal, even more than the previous ones. It shows, however, that we will have to use the extension of scalars to make sense of the map b between (some algebras replacing) Λ_r and Λ.

3.4. We end this section by noting that the apparent problem posed by the growth of the series (3.11) naturally presents its own solution. Consider simply the additive group N (over $\mathbb{Z}_p$). Clearly the series (3.11) does not belong to the Iwasawa algebra, dual to the continuous functions.

However we see that its coefficients a_m always satisfy

$$v_p(a_m) \geq -\frac{1}{p-1}(m - \mathrm{Schiff}\, m) = -v_p(m!).$$

On the other hand, an analytic function on $\mathbb{Z}_p$ (seen as a rigid–analytic space, or, in Lazard's terminology, a saturated pro-p group) admits a Mahler expansion

$$f(x) = \sum_m c_m \binom{x}{m}$$

with $v(c_m) - v(m!) \to \infty$. (Amice's theorem, cf [11, III. 1.3.8].) This implies that the Iwasawa series (3.11) is convergent on such functions. Thus we are naturally led to replace the Iwasawa algebra by the dual of the space of globally analytic functions (possibly with coefficients in L).

4. **The holomorphic base change map**

4.1. In this section we will make sense of the "formal" base change map of Section 3.3 by restricting it to the algebra of global distributions. We will in fact work rationally; in the previous sections this would amount to replacing an Iwasawa algebra Λ by $\Lambda \otimes \mathbb{Q}_p$.

We first check that our groups S, S_r are naturally group objects in the category of (affinoid) rigid–analytic spaces (in fact, all isomorphic to B^N, B being the closed ball of radius 1.)

Consider for simplicity the group S of $\mathbb{Z}_p$–points. It is in bijection with $\mathbb{Z}_p^3$ by the variables y, x and $-2\frac{\log(t)}{\log(1+p)}$. We will denote this last variable by z in the next computations. Let, as before Lemma 2.2,

$$g = \begin{pmatrix} t & tx \\ tpy & tpxy + t^{-1} \end{pmatrix}$$

and let

$$\gamma = \begin{pmatrix} \tau & \tau\xi \\ \tau p\eta & \tau p\xi\eta + \tau^{-1} \end{pmatrix}.$$

Then the product $g\gamma$, equal to

$$\begin{pmatrix} t\tau + pt\tau x\eta & t\tau\xi + t\tau px\xi\eta + t\tau^{-1}x \\ t\tau py + t\tau p^2xy\eta + \tau t^{-1}p\eta & * \end{pmatrix}$$

has parameters given by

$$\begin{aligned} T &= t\tau(1 + px\eta), \\ TY &= t\tau(y + pxy\eta + t^{-2}\eta), \\ TX &= t\tau(\xi + px\xi\eta + \tau^{-2}x). \end{aligned}$$

With parameters z, x, y, we see that the coordinates of the product are then in the Tate algebra in the six variables. Note that $t = \exp(-\frac{1}{2}\ z\ \log(1+p))$ is, an a function of z, in the Tate algebra in one variable, and is invertible. Then T an invertible element in the Tate algebra $\mathcal{T}_6(\mathbb{Q}_p)$ of the product. A similar argument applies to the map $g \mapsto g^{-1}$. Thus we can see S as a rigid analytic group, isomorphic as a space to the closed 3–ball B^3. (This was already proved by Lazard, cf. [11, III. 3.3.2].)

The same formulas, and the same argument, apply to $S_r \cong B^3(L)$, a rigid analytic space of dimension 3 over L.

4.2. Now let $\mathcal{A}_1 \cong \mathcal{T}_3(\mathbb{Q}_p)$ be the space of globally defined analytic functions on S.[2] We assume that the coefficients are in L. If $f \in \mathcal{A}_1$, f defines naturally

[2]We apologise for the conflict of notations with the previous paragraphs. The spaces of non-commutative polynomials will no longer occur in this section. Also (already in section 4.1) X, Y, Z, T denote objects different from the formal variables of Section 2,3.

a function in $\mathcal{T}_3(L)$ (with the same coefficients). We denote it by $b_0^*(f)$. There is a comultiplication $m^*\colon \mathcal{A}_1 \to \mathcal{A}_1 \widehat{\otimes} \mathcal{A}_1$,

$$f(g) \mapsto f(g_1 g_2).$$

It is obviously compatible with the map $b_0^*\colon \mathcal{A}_1 \to \mathcal{A}'_r = \mathcal{T}_3(L)$ which we have defined. (E.g., use that $\mathbb{Z}_p$ is dense in $\mathcal{O}_L$ for the canonical topology.)

A function in $\mathcal{A}_1$ can be paired with an element of the Iwasawa algebra of S (but we obtain a small space of the space of test functions). Similarly, a function in $\mathcal{A}'_r$ can be integrated against a distribution in the Iwasawa algebra of S_r.

Now the Iwahori subgroup S_r, a rigid–analytic space of dimension 3 over L, defines by restriction of scalars a rigid-analytic space $Res(S_r)$ of dimension $3r$ over $\mathbb{Q}_p$, but it then has a larger space $\mathcal{A}_r$ of analytic functions. See [3] for the restriction of scalars in the formal case; by the results in [13] this gives the restriction of scalars in the rigid-analytic case. All that we will need is that for functions the corresponding map $\mathcal{A}'_r \mapsto \mathcal{A}_r$ is given by the explicit formula (4.1) below. Thus we get a map $b^*\mathcal{A}_1 \to \mathcal{A}_r$; the image is comprised of the functions (in $\mathcal{A}_r$) which are "L–holomorphic"; again, this is compatible with the comultiplication. Note that all these spaces are Banach spaces, with natural norms.

The construction of $Res(S_r)$ yields the following formula for functions, (cf. [3]).[3] Let $\mathcal{O}_L = \bigoplus \mathbb{Z}_p e_i$, so $v = \sum v_i e_i$ ($v \in \mathcal{O}_L, v_i \in \mathbb{Z}_p$). We can apply this to the variables x, y, z. If

$$f = \sum a(m,n,q) x^m z^n y^q \in \mathcal{A}_1,$$

the corresponding function on $Res(S_r)$ is

$$F = \sum a(m,n,q) (\sum x_i e_i)^m (\sum z_i e_i)^n (\sum y_i e_i)^q \tag{4.1}$$

$$= \sum_{m_i, n_i, q_i} \prod x_i^{m_i} \prod z_i^{n_i} \prod y_i^{q_i} a(\sum m_i, \sum n_i, \sum q_i) \binom{m}{m_i} \binom{n}{n_i} \binom{q}{q_i} \underline{e}^m \underline{e}^n \underline{e}^q,$$

where $\underline{e}^m = \prod e_i^{m_i}$, etc., obviously in the Tate algebra in the $3r$ variables. The map $f \mapsto F$ is continuous if the Tate algebras are provided with the sup norm; it is compatible with the comultiplication.

Consider now the spaces of (global) distributions $\mathcal{D}$, $\mathcal{D}_r$ on the rigid-analytic spaces S and $Res(S_r)$ (with coefficients in L.) These are simply the dual spaces of $\mathcal{A}_1$ and $\mathcal{A}_r$.

By duality we obtain a map

$$b\colon \mathcal{D}_r \longrightarrow \mathcal{D}.$$

---

[3] We will review the restriction of scalars more precisely in part II of this paper.

Since the convolution can be defined by means of the coproduct on the spaces $\mathcal{A}$, we see that b is a homomorphism of algebras for the convolution.

In fact we can say more. Return to the formulas in Section 4.1 giving the product map in coordinates, with

$$t = \exp(-\frac{1}{2}\ z\ \ \log(1+p)) = \exp(-\frac{1}{2}\ z\pi),\ z = -2\frac{\log t}{\log(1+p)},$$

where $\pi = \log(1+p)$.

The local parameter Z associated to T is then, with obvious notation,

$$z + \zeta - 2\frac{1}{\log(1+p)}\log(1+px\eta),$$

a convergent series in (z, ζ, x, η) with integral coefficients. Furthermore, $(1+px\eta)^{-1}$ is such a series, as is t^{-2}:

$$t^{-2} = \exp(z\pi) = \sum a_n z^n$$

with

$$v(a_n) = \frac{p-2}{p-1}n + \mathrm{Schiff}(n) > 0$$

for $n \geq 1$.Therefore the coordinates Z, X, Y of the product belong to the Tate algebra in the six variables with *integral* coefficients.[4]

Let $\delta\colon \mathcal{A} = \mathcal{A}_S \to \mathcal{A}_S \widehat{\otimes} \mathcal{A}_S$ be the comultiplication. Then convolution in $\mathcal{D}$ is defined by

$$\langle S * T, f\rangle = \langle S \otimes T, \delta(f)\rangle,$$

while the norm in $\mathcal{D}$ is given by

$$\|T\| = Sup_{f\in\mathcal{A}_0}\langle T, f\rangle.$$

where $\mathcal{A}_0$ is the unit ball in $\mathcal{A}$. For $f \in \mathcal{A}_0$, the previous computation show that $\delta(f)$ is a series with integral coefficients. By [2, Section 2.1, Proposition 1], we then have $|\langle S \otimes T, \delta(f)\rangle| \leq 1$ if $\|S\| \leq 1$ and $\|T\| \leq 1$. Thus

$$\|S * T\| \leq \|S\|.\|T\|.$$

This obviously applies, with the same proof, to $Res(S_r)$. Since the restriction of scalars preserves integral functions, we finally have the following result. We write $\mathcal{D}(L), \mathcal{D}_r(L)$ to emphasise the fact that the coefficients are in L.

Theorem 4.1. (i) *The map b defines a morphism of convolution algebras $\mathcal{D}_r(L) \to \mathcal{D}(L)$.*
(ii) *$\mathcal{D}(L), \mathcal{D}_r(L)$ are Banach algebras for convolution, the product verifying in fact $\|S * T\| \leq \|S\|.\|T\|$.*
(iii) $\|b(f)\| \leq \|f\|$ $(f \in \mathcal{D}_r(L)$.

[4] Note that in this rigid-analytic computation we do not have to assume that $p > 3$. We did not check which of the other arguments apply.

By adding the center (a direct factor for $p \geq 3$), Theorem 4.1 obviously extends to the full Iwahori subgroups G, G_r.

4.3. It is now easy to see that the map $b : \mathcal{D}_r \to \mathcal{D}$ is the map that we had formally defined in Section 3.4. Consider for instance the effect of b on $1 + X_i$:

$$1 + X_i \mapsto (1 + X)^{e_i}.$$

The distribution $1 + X_i$ belongs to $\mathcal{D}_r$, and is simply the evaluation of F at the point $(x_i = 1, 0, \ldots, 0)$ of the ball $\mathbb{Z}_p^{3r}$. If we compose with the map $f \mapsto F$ (cf. (4.1)), we obtain the map $f \mapsto f(e_i, 0, 0)$ where f is seen as an analytic function on the L-adic group. This distribution does not belong to the Iwasawa algebra: it has no meaning on the continuous functions. However, $(1 + X)^v$ is the map $f \mapsto f(v, 0, 0)$ if $v \in \mathbb{Z}_p$; for $v \in \mathcal{O}_L$, the power $(1 + X)^{e_i}$ which appeared in the computations of Section 3.4 is then naturally identified with $f \mapsto f(e_i, 0, 0)$. The same argument applies to the other variables.

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