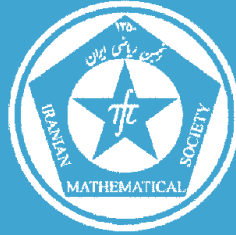


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THETA FUNCTIONS ON COVERS OF SYMPLECTIC GROUPS

S. FRIEDBERG* AND D. GINZBURG

To Freydoon Shahidi on his 70th birthday

ABSTRACT. We study the automorphic theta representation $\Theta_{2n}^{(r)}$ on the r -fold cover of the symplectic group Sp_{2n} . This representation is obtained from the residues of Eisenstein series on this group. If r is odd, $n \leq r < 2n$, then under a natural hypothesis on the theta representations, we show that $\Theta_{2n}^{(r)}$ may be used to construct a globally generic representation $\sigma_{2n-r+1}^{(2r)}$ on the $2r$ -fold cover of Sp_{2n-r+1} . Moreover, when $r = n$ the Whittaker functions of this representation attached to factorizable data are factorizable, and the unramified local factors may be computed in terms of n -th order Gauss sums. If $n = 3$ we prove these results, which in that case pertain to the six-fold cover of Sp_4 , unconditionally. We expect that in fact the representation constructed here, $\sigma_{2n-r+1}^{(2r)}$, is precisely $\Theta_{2n-r+1}^{(2r)}$; that is, we conjecture relations between theta representations on different covering groups.

Keywords: Symplectic group, metaplectic cover, theta representation, descent integral, unipotent orbit, generic representation, Whittaker function.

MSC(2010): Primary: 11F70; Secondary: 11F27, 11F55.

1. Introduction

The classical metaplectic group is a double cover of a symplectic group, and may be defined over a local field or the ring of adèles of a number field. This group arises in the study of theta functions, which may be constructed directly as sums over the global rational points of an isotropic subspace. This construction of theta functions appears to be special to the double cover case.

To generalize, let $r > 1$ be any integer, and let F be a number field with a full set of r -th roots of unity. Then there is an r -fold cover of the symplectic group $Sp_{2n}(\mathbb{A})$, where \mathbb{A} is the ring of adèles of F , denoted $Sp_{2n}^{(r)}(\mathbb{A})$. When $r > 2$

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one may construct theta functions for such a group, but only indirectly – theta functions are obtained as the multi-residues of the minimal parabolic (Borel) Eisenstein series on $Sp_{2n}^{(r)}(\mathbb{A})$ at the “right-most” (suitably normalized) pole. (This construction may be carried out when $r = 2$ as well.) As such, these theta functions are automorphic forms and realize an automorphic representation $\Theta_{2n}^{(r)}$, called the theta representation.

Since for a general cover the theta functions may only be defined via residues, it is considerably more challenging to determine basic information such as their Whittaker or other Fourier coefficients in this situation. Indeed, the Fourier coefficients of the theta function have not been determined for covers of SL_2 of any degree higher than 4. And when $n > 1$ very little is known. Moreover, except in the double cover case there are few examples of any automorphic functions on such a group, there is no information about when their Whittaker coefficients might be factorizable (one expects rarely), and it is not clear to what extent one might expect relations between theta functions or other automorphic forms on different covers of different groups.

In a first example of such information, the authors [7] have established relations between the Whittaker coefficients of certain automorphic functions on different covering groups in two specific cases, related to the conjectures of Patterson [17] and Chinta-Friedberg-Hoffstein [5] concerning the Fourier coefficients of theta functions on covers of GL_2 . The key to doing so was to adapt descent methods, originally used by Ginzburg-Rallis-Soudry [11] in the context of algebraic groups or double covers, to higher degree covering groups. In particular, the treatment of the adelic version of the conjecture of Chinta-Friedberg-Hoffstein relied on the study of the theta function on the three-fold cover of Sp_4 .

In this work we investigate theta functions and descent integrals on covers of symplectic groups of arbitrary rank. As we shall explain, for r odd a descent construction allows one to pass from the theta representation $\Theta_{2n}^{(r)}$ to a representation $\sigma_{2n-r+1}^{(2r)}$ realized inside the space of automorphic square-integrable functions on $Sp_{2n-r+1}^{(2r)}(F) \backslash Sp_{2n-r+1}^{(2r)}(\mathbb{A})$. We expect that this representation is in fact the theta representation $\Theta_{2n-r+1}^{(2r)}$ (Conjecture 4.2), and we show that $\sigma_{2n-r+1}^{(2r)}$ has non-zero projection to this representation. In Theorem 5.1 below we establish that if r is odd, $n \leq r < 2n$, then the representation $\sigma_{2n-r+1}^{(2r)}$ is globally generic, that is, has a non-zero Whittaker coefficient for some generic character. Moreover, if $r = n$, we show that the Whittaker coefficients of $\sigma_{n+1}^{(2n)}$ arising from factorizable inputs are factorizable. As a first new case, this is true of the Whittaker coefficients of the descent representation on the six-fold cover of Sp_4 . Then in Theorem 6.1 we show that the unramified local contributions to the Whittaker coefficients in the factorizable case may be expressed as sums of Whittaker coefficients of the theta representation on the n -fold cover of GL_n .

These coefficients are n -th order Gauss sums by the work of Kazhdan and Patterson [14]. Thus we exhibit a new class of automorphic forms on covering groups whose Whittaker coefficients are factorizable with algebraic factors at good places.

These results are proved on two conditions. One concerns the specific unipotent orbit attached to $\Theta_{2n}^{(r)}$ (Conjecture 2.2). This is an object of on-going study [4]; in the last section we give a sketch of the proof of this conjecture in the first new case, $n = r = 3$. A second concerns the precise characters within a given unipotent orbit which support a nonzero Fourier coefficient (Assumption 5.2). This is not needed in all cases, and in particular is not needed in the factorizable case $\sigma_{n+1}^{(2n)}$. Our results concerning Conjecture 2.2 are thus sufficient to establish unconditionally that the Whittaker function of the descent to the six-fold cover of Sp_4 is Eulerian for factorizable data, and to compute its unramified local factor in terms of cubic Gauss sums. We expect that this descent is in fact the theta function on the six-fold cover of Sp_4 .

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2. Definition and properties of the Theta representation

Let n and r be two natural numbers. Let F be a number field containing a full set μ_r of r -th roots of unity, and let \mathbb{A} be the ring of adeles of F . Let Sp_{2n} denote the symplectic group consisting of the $2n \times 2n$ matrices leaving invariant the standard symplectic form $\langle x, y \rangle = \sum_{i=1}^n (x_i y_{2n-i+1} - x_{n+i} y_{n-i+1})$. Denote by $Sp_{2n}^{(r)}(\mathbb{A})$ the metaplectic r -fold cover of the symplectic group $Sp_{2n}(\mathbb{A})$. This group consists of pairs (g, ζ) with $g \in Sp_{2n}(\mathbb{A})$ and $\zeta \in \mu_r$; it may be obtained in the standard way from covers of the local groups $Sp_{2n}(F_\nu)$ as ν ranges over the places of F , identifying the copies of μ_r . The multiplication in the local group is determined by a cocycle σ . If σ_{BLS} is the 2-cocycle of Banks-Levy-Sepanski [1] for $SL_{2n}(F_\nu)$ and w is the permutation matrix such that the conjugate of Sp_{2n} by w preserves the symplectic form $\langle x, y \rangle' = \sum_{i=1}^n x_i y_{n+i} - y_i x_{n+i}$, then we take $\sigma(g_1, g_2) = \sigma_{BLS}(wg_1w^{-1}, wg_2w^{-1})$. (This choice simplifies some of the calculations involving cocycles below.)

We shall be concerned with the theta representation defined on the group $Sp_{2n}^{(r)}(\mathbb{A})$, denoted $\Theta_{2n}^{(r)}$. This representation is defined as the space spanned by the residues of the Borel Eisenstein series on $Sp_{2n}^{(r)}(\mathbb{A})$, similarly to the definition for the general linear group in [14, p. 118]. Our first task is to give a brief account of this construction.

Let B_{2n} denote the standard Borel subgroup of Sp_{2n} , and let T_{2n} denote the maximal torus of Sp_{2n} which is a subgroup of B_{2n} . Working either locally at a completion of F or globally over \mathbb{A} , if H is any algebraic subgroup of Sp_{2n} , we let $H^{(r)}$ denote its inverse image in $Sp_{2n}^{(r)}$. We will call this inverse image parabolic if H has this property. Let $Z(T_{2n}^{(r)})$ denote the center of $T_{2n}^{(r)}$. Suppose first that r is odd. Given a character μ of T_{2n} , we may use it to define a genuine character, again denoted μ , of $Z(T_{2n}^{(r)})$. (The definition of a genuine representation depends on an embedding of the group of r -th roots of unity μ_r into \mathbb{C}^\times . We will fix this and omit it from the notation.) Extending it trivially to any maximal abelian subgroup of $T_{2n}^{(r)}$ and then inducing up, we obtain a representation of $Sp_{2n}^{(r)}$ which we denote by $\text{Ind}_{B_{2n}^{(r)}}^{Sp_{2n}^{(r)}} \mu$. This representation is determined uniquely by the choice of μ and this procedure may be carried out both locally and globally. Here we consider unnormalized induction, and shall include the modular character $\delta_{B_{2n}}^{1/2}$ when we require normalized induction.

Let s_i be complex variables, and let μ be the character of T_{2n} given by

$$\mu(\text{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1})) = |a_1|^{s_1} \cdots |a_n|^{s_n}.$$

If this construction is carried out over a local field, then $|a|$ denotes the normalized local absolute value, while if it is carried out over \mathbb{A} , then $|a|$ denotes the product of these over all places of F . One may form the induced representation $\text{Ind}_{B_{2n}^{(r)}(\mathbb{A})}^{Sp_{2n}^{(r)}(\mathbb{A})} \mu \delta_{B_{2n}}^{1/2}$, and for each vector in this space, one may construct the multi-variable Eisenstein series $E(h, s_1, \dots, s_n)$ defined on the group $Sp_{2n}^{(r)}(\mathbb{A})$. Computing the constant term as in [14, Proposition II.1.2], we deduce that the poles of the partial intertwining operator associated with the long Weyl element of Sp_{2n} are determined by

$$(2.1) \quad \frac{\prod_{i < j} [\zeta_S(r(s_i - s_j)) \zeta_S(r(s_i + s_j))] \prod_i \zeta_S(rs_i)}{\prod_{i < j} [\zeta_S(r(s_i - s_j) + 1) \zeta_S(r(s_i + s_j) + 1)] \prod_i \zeta_S(rs_i + 1)}.$$

Here S is a finite set of all places including the archimedean places and all finite places ν such that $|r|_\nu \neq 1$ and $\zeta_S(s)$ is the partial global zeta function.

The expression (2.1) has a multi-residue at the point

$$s_n = \frac{1}{r}, \quad r(s_i - s_{i+1}) = 1.$$

From this we deduce that the Eisenstein series $E(h, s_1, \dots, s_n)$ has a multi-residue at that point, and we denote the residue representation by $\Theta_{2n}^{(r)}$. If μ_0 denotes the character of T_{2n} defined by

$$\mu_0(\text{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1})) = |a_1|^{\frac{n}{r}} \cdots |a_{n-1}|^{\frac{2}{r}} |a_n|^{\frac{1}{r}},$$

then it follows that $\Theta_{2n}^{(r)}$ is a subquotient of the induced representation $\text{Ind}_{B_{2n}^{(r)}(\mathbb{A})}^{Sp_{2n}^{(r)}(\mathbb{A})} \mu_0 \delta_{B_{2n}}^{1/2}$. From this we deduce that $\Theta_{2n}^{(r)}$ is also a subrepresentation of the induced representation $\text{Ind}_{B_{2n}^{(r)}(\mathbb{A})}^{Sp_{2n}^{(r)}(\mathbb{A})} \chi_{Sp_{2n}^{(r)}, \Theta}$ where

$$\chi_{Sp_{2n}^{(r)}, \Theta}(\text{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1})) = |a_1|^{\frac{n(r-1)}{r}} \cdots |a_{n-1}|^{\frac{2(r-1)}{r}} |a_n|^{\frac{r-1}{r}}.$$

We turn to even degree coverings. We shall only be concerned with the covers of degree $2r$ where r is an odd integer. (This is the only case we consider here since the analogue of Conjecture 2.2 below is not yet formulated when r is even.) If r is odd, the definition of the theta representation for $Sp_{2m}^{(2r)}$ is similar. There is a small difference since the maximal parabolic subgroup of Sp_{2m} whose Levi part is GL_m splits under the double cover. Because of this, when computing the intertwining operator corresponding to the long Weyl element, one finds that its poles are determined by

$$\frac{\prod_{i < j} [\zeta_S(r(s_i - s_j)) \zeta_S(r(s_i + s_j))] \prod_i \zeta_S(2rs_i)}{\prod_{i < j} [\zeta_S(r(s_i - s_j) + 1) \zeta_S(r(s_i + s_j) + 1)] \prod_i \zeta_S(2rs_i + 1)}.$$

Accordingly we define $\Theta_{2m}^{(2r)}$ to be the multi-residue of the Eisenstein series at the point

$$s_m = \frac{1}{2r}, \quad r(s_i - s_{i+1}) = 1.$$

Then the representation $\Theta_{2m}^{(2r)}$ is a subquotient of the induced representation $\text{Ind}_{B_{2m}^{(r)}(\mathbb{A})}^{Sp_{2m}^{(2r)}(\mathbb{A})} \mu_e \delta_{B_{2m}}^{1/2}$ where μ_e denotes the character of T_{2m} defined by

$$\mu_e(\text{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1})) = |a_1|^{\frac{2m-1}{2r}} \cdots |a_{m-1}|^{\frac{3}{2r}} |a_m|^{\frac{1}{2r}}.$$

It follows that the representation $\Theta_{2m}^{(2r)}$ is also a subrepresentation of the induced representation $\text{Ind}_{B_{2m}^{(r)}(\mathbb{A})}^{Sp_{2m}^{(2r)}(\mathbb{A})} \chi_{Sp_{2m}^{(2r)}, \Theta}$ where

$$\chi_{Sp_{2m}^{(2r)}, \Theta}(\text{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1})) = |a_1|^{\frac{2mr - (2m-1)}{2r}} \cdots |a_{m-2}|^{\frac{6r-5}{2r}} |a_{m-1}|^{\frac{4r-3}{2r}} |a_m|^{\frac{2r-1}{2r}}.$$

We now develop some properties of the theta representations that follow from induction in stages, or, equivalently, by taking an $(n-1)$ -fold residue of $E(h, s_1, \dots, s_n)$ to obtain a maximal parabolic Eisenstein series attached to theta representations of lower rank groups. Let $1 \leq a \leq n$. Denote by $P_{2n,a}$ the maximal parabolic subgroup of Sp_{2n} whose Levi part is $GL_a \times Sp_{2(n-a)}$, and let $L_{2n,a}$ denote its unipotent radical. We write i for the inclusion of GL_a in $P_{2n,a}$ in this paragraph but we suppress i afterwards. From the block compatibility of the cocycle σ_{BLS} (Banks-Levy-Sepanski [1]) and a short computation it follows that if $g_1, g_2 \in GL_a$ over a local field then $\sigma(i(g_1), i(g_2)) =$

$\sigma_{GL_a}(g_1, g_2)^2(\det g_1, \det g_2)^{-1}$, where σ_{GL_a} is the metaplectic 2-cocycle of [1] for GL_a and (\cdot, \cdot) is the r -th order Hilbert symbol. Thus for both the r and $2r$ -fold covers, i determines an r -fold cover of GL_a . We write this cover $GL_a^{(r)}$ (which case we are in will always be clear from context).

Suppose first that r is odd. Let $\Theta_{GL_a}^{(r)}$ denote the theta representation of the group $GL_a^{(r)}(\mathbb{A})$, as constructed in [14] (or the corresponding character if $a = 1$). Let $E_a(g, s)$ denote the Eisenstein series of $Sp_{2n}^{(r)}(\mathbb{A})$ associated with the induced representation $\text{Ind}_{P_{2n,a}^{(r)}(\mathbb{A})}^{Sp_{2n}^{(r)}(\mathbb{A})}(\Theta_{GL_a}^{(r)} \otimes \Theta_{2(n-a)}^{(r)})\delta_{P_{2n,a}}^s$. (For the construction of such Eisenstein series more generally, see Brubaker and Friedberg [2].) Then it follows from induction in stages that the representation $\Theta_{2n}^{(r)}$ is a residue of $E_a(g, s)$ at the point $s = (r+1)/2r$. Of course, this can also be verified directly by studying the corresponding intertwining operators.

In the case of a cover of degree $2r$, r odd, the situation is roughly similar. Let $E_a(g, s)$ denote the Eisenstein series associated with the induced representation $\text{Ind}_{P_{2m,a}^{(2r)}(\mathbb{A})}^{Sp_{2m}^{(2r)}(\mathbb{A})}(\Theta_{GL_a}^{(r)} \otimes \Theta_{2(m-a)}^{(2r)})\delta_{P_{2m,a}}^s$. Then once again from induction in stages we deduce that $\Theta_{2m}^{(2r)}$ is the residue of $E_a(g, s)$ at the point $s = \frac{(r+1)(2m-a)+r}{2r(2m-a+1)}$.

From these observations, we will deduce the following proposition. Here and below, matrices are embedded in metaplectic groups by the trivial section without additional notation, and we call metaplectic elements diagonal or unipotent when their projections to the linear group are.

Proposition 2.1. *Suppose that r is odd. Let $\theta_{2n}^{(r)}$ be in the space of $\Theta_{2n}^{(r)}$. Then there exist functions $\theta_{GL_a}^{(r)} \in \Theta_{GL_a}^{(r)}$, $\theta_{2(n-a)}^{(r)} \in \Theta_{2(n-a)}^{(r)}$ such that for all diagonal g in $GL_a^{(r)}(\mathbb{A})$ which lies in the center of the Levi part of the parabolic group $P_{2n,a}^{(r)}(\mathbb{A})$ and for all unipotent $h \in Sp_{2(n-a)}^{(r)}(\mathbb{A})$, $v \in GL_a^{(r)}(\mathbb{A})$, one has*

$$(2.2) \quad \int_{L_{2n,a}(F) \backslash L_{2n,a}(\mathbb{A})} \theta_{2n}^{(r)}(u(gv, h)) du = \chi_{Sp_{2n}^{(r)}, \Theta}(g) \theta_{GL_a}^{(r)}(v) \theta_{2(n-a)}^{(r)}(h).$$

A similar identity holds in the even-degree cover case.

The requirement that h, v be unipotent could be dropped at the expense of a cocycle; being unipotent guarantees that it is 1.

Proof. Since $\Theta_{2n}^{(r)}$ is the residue of the Eisenstein series $E_a(\cdot, s)$, we first consider the constant term

$$(2.3) \quad \int_{L_{2n,a}(F) \backslash L_{2n,a}(\mathbb{A})} E_a(u(gv, h), s) du.$$

For $\operatorname{Re}(s)$ large we can unfold the Eisenstein series and deduce that (2.3) is equal to

$$(2.4) \quad \sum_{w \in P_{2n,a}(F) \backslash Sp_{2n}(F) / P_{2n,a}(F) L_{2n,a}^w(F) \backslash L_{2n,a}(\mathbb{A})} \int_{\gamma \in (w^{-1}P_{2n,a}(F)w \cap P_{2n,a}(F)) \backslash P_{2n,a}(F)} \sum f_s(w\gamma u(gv, h)) du.$$

Here $L_{2n,a}^w = w^{-1}L_{2n,a}w \cap L_{2n,a}^-$ where $L_{2n,a}^-$ is the conjugate of $L_{2n,a}$ by the long Weyl element. Also, we have $f_s \in \operatorname{Ind}_{P_{2n,a}^{(r)}(\mathbb{A})}^{Sp_{2n}^{(r)}(\mathbb{A})} (\Theta_{GL_a}^{(r)} \otimes \Theta_{2(n-a)}^{(r)}) \delta_{P_{2n,a}}^s$. Notice that all elements in $P_{2n,a}(F) \backslash Sp_{2n}(F) / P_{2n,a}(F)$ can be chosen to be Weyl elements. Similarly to [15, Section 1.2], one can check that for all Weyl elements in $P_{2n,a}(F) \backslash Sp_{2n}(F) / P_{2n,a}(F)$ which are not the long Weyl element, the inner summation is a certain Eisenstein series or product of such series. Moreover, one can also check that the Eisenstein series in the corresponding summand in (2.4) is holomorphic at $s = (r+1)/2r$. Hence, taking the residue in (2.3) and (2.4) at $s = (r+1)/2r$, we are left only with the long Weyl element, and for that element we obtain the identity

$$\int_{L_{2n,a}(F) \backslash L_{2n,a}(\mathbb{A})} \theta_{2n}^{(r)}(u(gv, h)) du = \operatorname{Res}_{s=(r+1)/2r} M_w f_s(gv, h)$$

where M_w is the intertwining operator attached to w . From this identity (2.2) follows. \square

Similar statements for other groups are established in [3, Proposition 3.4] and [7, Proposition 1].

We end this section with a general conjecture about the unipotent orbit attached to the representation $\Theta_{2n}^{(r)}$, and give a consequence of this conjecture. Recall that if π denotes an automorphic representation of a reductive group, then the set $\mathcal{O}(\pi)$ was defined in [9]. It is the largest unipotent orbit that supports a nonzero coefficient for this representation. The extension of the definition of this set to the metaplectic groups is clear. In this paper we are interested in the set $\mathcal{O}(\Theta_{2n}^{(r)})$. The conjecture regarding this set is given as follows. Assume that $r < 2n$. Write $2n = a_{n,r}r + b_{n,r}$ where $a_{n,r}$ and $b_{n,r}$ are both nonnegative numbers such that $0 \leq b_{n,r} \leq r-1$. We recall that a partition is a symplectic partition if any odd number in the partition occurs with even multiplicity. As defined in [6], given a partition λ of $2n$, we define the Sp collapse of λ to be the greatest symplectic partition which is smaller than λ . We have

Conjecture 2.2. Let r be an odd number. If $r < 2n$, then the set $\mathcal{O}(\Theta_{2n}^{(r)})$ consists of the partition which is the Sp collapse of the partition $(r^{a_{n,r}} b_{n,r})$. If $r > 2n$, then the representation $\Theta_{2n}^{(r)}$ is globally generic.

The conjecture, and the analogous local conjecture, are studied in [4]. We shall write Conjecture 2.2_{r,n} for this statement for a specific pair (r, n) . We remark that an extension to even degree covers has not been formulated at this time.

To give a consequence of this conjecture, let r be odd, let a be an integer with $1 \leq a \leq n - (r - 1)/2$, and let $b = n - (r - 1)/2 - a$. Let $U_{2n, n-b}$ be the unipotent radical of the standard parabolic subgroup of Sp_{2n} whose Levi part is $GL_1^{n-b} \times Sp_{2b}$. (In particular, $U_{2n, n}$ is the standard maximal unipotent subgroup of Sp_{2n} .) Let $U_{2n, n-b, 1}$ be the subgroup of $U_{2n, n-b}$ which consists of all matrices $u = (u_{i, j}) \in U_{2n, n-b}$ such that $u_{n-b, i} = 0$ for all $n - b + 1 \leq i \leq n$. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_{a-1})$ where for all i we have $\alpha_i \in \{0, 1\}$. Let ψ be a nontrivial character of $F \backslash \mathbb{A}$, which will be fixed throughout this paper. Let $\psi_{U_{2n, n-b, 1, \underline{\alpha}}}$ be the character of $U_{2n, n-b, 1}$ given by

$$\psi_{U_{2n, n-b, 1, \underline{\alpha}}}(u) = \psi \left(\sum_{i=1}^{a-1} \alpha_i u_{i, i+1} + \sum_{j=a}^{n-b} u_{j, j+1} \right).$$

Then we have the following result, which will be used later.

Lemma 2.3. *Suppose r is odd, $r < 2n$, and Conjecture 2.2_{r, n-i_0} holds for all $0 \leq i_0 \leq a - 1$. Then the integral*

$$(2.5) \quad \int_{U_{2n, n-b, 1}(F) \backslash U_{2n, n-b, 1}(\mathbb{A})} \theta_{2n}^{(r)}(ug) \psi_{U_{2n, n-b, 1, \underline{\alpha}}}(u) du$$

is zero for all choices of data, that is, for all $\theta_{2n}^{(r)} \in \Theta_{2n}^{(r)}$.

Proof. Consider first the case where $\alpha_i = 1$ for all i . Then the Fourier coefficient (2.5) is the Fourier coefficient which corresponds to the unipotent orbit $((2n - 2b)1^{2b})$. From Conjecture 2.2_{r, n} we have that $\mathcal{O}(\Theta_{2n}^{(r)})$ consists of the partition which is the Sp collapse of the partition $(r^{a_n, r} b_{n, r})$. Hence $((2n - 2b)1^{2b})$ is greater than or not related to $\mathcal{O}(\Theta_{2n}^{(r)})$. Indeed, this follows from the relation $2b = 2n - 2a - r + 1$ which implies that $((2n - 2b)1^{2b}) = ((r + 2a - 1)1^{2b})$. Thus, if $\alpha_i = 1$ for all i , then the integral (2.5) is zero for all choices of data.

Next assume that at least one of the scalars α_i is zero. Let $i_0 \leq a - 1$ be the largest index such that $\alpha_{i_0} = 0$. Then the integral (2.5) is equal to

$$(2.6) \quad \int_{H(F) \backslash H(\mathbb{A})} \int_{L_{2n, i_0}(F) \backslash L_{2n, i_0}(\mathbb{A})} \int_{U_{2(n-i_0), n-i_0-b, 1}(F) \backslash U_{2(n-i_0), n-i_0-b, 1}(\mathbb{A})} \theta_{2n}^{(r)}(luhg) \psi_{U_{2(n-i_0), n-i_0-b, 1}}(u) \psi_{H, \underline{\alpha}}(h) du dl dh.$$

Here H is a certain unipotent subgroup of GL_{i_0} which will not be important to us. Notice that the integration over $L_{2n, i_0}(F) \backslash L_{2n, i_0}(\mathbb{A})$ is the constant term along this unipotent group. Therefore, it follows from Proposition 2.1 above that

the integration along the quotient space $U_{2(n-i_0), n-i_0-b, 1}(F) \backslash U_{2(n-i_0), n-i_0-b, 1}(\mathbb{A})$ is the coefficient of the representation $\Theta_{2(n-i_0)}^{(r)}$ corresponding to the unipotent orbit $((2(n-i_0-b))1^{2b})$. Thus the vanishing of the integral (2.6) will follow if this unipotent orbit is greater than or not related to the orbit $\mathcal{O}(\Theta_{2(n-i_0)}^{(r)})$. From Conjecture 2.2 $_{r, n-i_0}$, this is the case if $2(n-i_0-b) > r$. Since this inequality follows from the relations $2(n-a-b) = r-1$ and $a > i_0$, the result is proved. \square

3. The descent construction

Recall that $U_{2m, k}$ denotes the unipotent radical of the standard parabolic subgroup of Sp_{2m} whose Levi part is $GL_1^k \times Sp_{2(m-k)}$. The quotient group $U_{2m, k-1} \backslash U_{2m, k}$ may be identified with the Heisenberg group in $2(m-k)+1$ variables, $\mathcal{H}_{2(m-k)+1}$. Indeed, there is a homomorphism $l : U_{2m, k} \mapsto \mathcal{H}_{2(m-k)+1}$ which is onto, and whose kernel is the group $U_{2m, k-1}$.

For the rest of this section we suppose that $r > 1$ is odd. Set $r' = (r-1)/2$. For $\alpha \in F^*$ let $\psi_{U, \alpha}$ be the character of $U_{r-1, r'}(F) \backslash U_{r-1, r'}(\mathbb{A})$ given by

$$\psi_{U, \alpha}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r'-1, r'} + \alpha u_{r', r'+1}) \quad u = (u_{i,j}).$$

From now on we shall suppose that Conjecture 2.2 $_{r, r'}$ holds, that is, that the representation $\Theta_{r-1}^{(r)}$ is globally generic. This is known if $r = 3$ [14]. Thus there is an $\alpha \in F^*$ and a choice of data, i.e. an automorphic function $\theta_{r-1}^{(r)}$ in the space of $\Theta_{r-1}^{(r)}$, such that the integral

$$(3.1) \quad \int_{U_{r-1, r'}(F) \backslash U_{r-1, r'}(\mathbb{A})} \theta_{r-1}^{(r)}(ug) \psi_{U, \alpha}(u) du$$

is not zero.

Remark 3.1. In fact we expect that for all $\alpha \in F^*$, the integral (3.1) will be nonzero for some choice of data.

Fix $\alpha \in F^*$ such that (3.1) is nonzero, and let ψ^α be the additive character $\psi^\alpha(x) = \psi(\alpha x)$. Let $\Theta_{2m}^{\psi^\alpha, \phi}$ denote the theta representation attached to the Weil representation and the additive character ψ^α and defined on the double cover of $Sp_{2m}(\mathbb{A})$. Here ϕ is a Schwartz function on \mathbb{A}^m . See for example [13, p. 8]. Since r is odd, using the isomorphism $\mu_{2r} \cong \mu_2 \times \mu_r$ we may map $g \in Sp_{2n-r+1}^{(2r)}(\mathbb{A})$ to its image in the double and r -fold covers; we will not introduce separate notation for this. To define the descent construction, consider the function of $g \in Sp_{2n-r+1}^{(2r)}(\mathbb{A})$ given by

$$(3.2) \quad f(g) = \int_{U_{2n, r'}(F) \backslash U_{2n, r'}(\mathbb{A})} \theta_{2n-r+1}^{\psi^\alpha, \phi}(l(u)g) \theta_{2n}^{(r)}(ug) \psi_{U_{2n, r'}}(u) du.$$

Here $\theta_{2n-r+1}^{\psi^\alpha, \phi}$ and $\theta_{2n}^{(r)}$ are vectors in the spaces of the representations $\Theta_{2n-r+1}^{\psi^\alpha, \phi}$ and $\Theta_{2n}^{(r)}$ (resp.), $\psi_{U_{2n,r'}}$ is the character of $U_{2n,r'}(\mathbb{A})$ given by $\psi_{U_{2n,r'}}(u_{i,j}) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r'-1,r'})$, and $g \in Sp_{2n-r+1}$ is embedded in Sp_{2n} as

$$g \mapsto \begin{pmatrix} I_{r'} & & \\ & g & \\ & & I_{r'} \end{pmatrix}.$$

(One could also consider descent integrals similar to (3.2) if r is even but we shall not do so here.) Then $f(g)$ is a genuine automorphic function defined on $Sp_{2n-r+1}^{(2r)}(\mathbb{A})$. Let $\sigma_{2n-r+1}^{(2r)}$ denote the representation of $Sp_{2n-r+1}^{(2r)}(\mathbb{A})$ generated by all the functions $f(g)$.

4. Computation of the constant term of the descent integral

Let V denote any standard unipotent subgroup of Sp_{2n-r+1} , and let ψ_V be a character of $V(F) \backslash V(\mathbb{A})$, possibly trivial. In this paper we will compute integrals of the type

$$(4.1) \quad \int_{V(F) \backslash V(\mathbb{A})} f(vg) \psi_V(v) dv,$$

where $f(g)$ is given by (3.2).

Using (3.2), we arrive at an iterated integral. We first unfold the theta function $\theta_{2n-r+1}^{\psi^\alpha, \phi}$. Collapsing summation and integration, and then using the formulas for the action of the Weil representation ω_{ψ^α} , the integral (4.1) is equal to

$$(4.2) \quad \int_{\mathbb{A}^{n-r'}} \int_{V(F) \backslash V(\mathbb{A})} \int_{U_{2n,r',1}(F) \backslash U_{2n,r',1}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(r)}(uvj(x)g) \psi_{U_{2n,r',1,\alpha}}(u) \psi_V(v) du dv dx.$$

Here $x = (x_1, \dots, x_{n-r'}) \in \mathbb{A}^{n-r'}$ is embedded in Sp_{2n} via the map $j(x) = I_{2n} + x_1 e_{r',r'+1}^* + x_2 e_{r',r'+2}^* + \cdots + x_{n-r'} e_{r',n}^*$, where $e_{i,j}$ denotes the (i,j) -th elementary matrix and $e_{i,j}^* = e_{i,j} - e_{2n-j+1,2n-i}$. Recall that the group $U_{2n,r',1}$ is the subgroup of $U_{2n,r'}$ consisting of all matrices $u = (u_{i,j}) \in U_{2n,r'}$ such that $u_{r',k} = 0$ for all $r'+1 \leq k \leq n$. The character $\psi_{U_{2n,r',1,\alpha}}(u)$ is defined as the product of $\psi_{U_{2n,r'}}(u)$ and the character $\psi_{U_{2n,r',1,\alpha}}^0$ of $U_{2n,r',1}$ given by $\psi_{U_{2n,r',1,\alpha}}^0(u) = \psi(\alpha u_{r',2n-r'+1})$.

At this point we consider the case when $V = L_{2n-r+1,a}$ where $1 \leq a \leq n-r'$ and ψ_V is the trivial character. Thus, V is the unipotent radical of the standard maximal parabolic subgroup of Sp_{2n-r+1} whose Levi part is $GL_a \times Sp_{2n-2a-r+1}$. We recall that $2b = 2n - 2a - r + 1$.

Let w_a denote the Weyl group element of Sp_{2n} defined by

$$w_a = \begin{pmatrix} & I_a & & & \\ I_{n-a-b} & & & & \\ & & I_{2b} & & \\ & & & I_{n-a-b} & \\ & & & & I_a \end{pmatrix}.$$

Since $\theta_{2n}^{(r)}$ is left-invariant under rational points, we may conjugate by w_a from left to right. Doing so, we see that the integral (4.2) is equal to

$$(4.3) \quad \int_{\mathbb{A}^{n-r'}} \int_{\text{Mat}_{(n-a-b-1) \times a}(F) \backslash \text{Mat}_{(n-a-b-1) \times a}(\mathbb{A})} \int_{V_0(F) \backslash V_0(\mathbb{A})} \int_{U_0(F) \backslash U_0(\mathbb{A})} \omega_{\psi, \alpha}(g) \phi(x) \theta_{2n}^{(r)}(u_0 v_0 k(y) w_a j(x) g) \psi_{V_0, \alpha}(v_0) du_0 dv_0 dy dx.$$

The notations are defined as follows. Recall that $L_{2n,a}$ denotes the unipotent radical of the standard maximal parabolic subgroup of Sp_{2n} whose Levi part is $GL_a \times Sp_{2(n-a)}$. The group U_0 is defined to be the subgroup of $L_{2n,a}$ consisting of all matrices in Sp_{2n} of the form

$$\begin{pmatrix} I_a & 0 & * & * & * \\ & I_{n-a-b} & 0 & 0 & * \\ & & I_{2b} & 0 & * \\ & & & I_{n-a-b} & 0 \\ & & & & I_a \end{pmatrix}.$$

The embedding of $\text{Mat}_{(n-a-b-1) \times a}$ into Sp_{2n} is given by

$$k(y) = \begin{pmatrix} I_a & & & & \\ y & I_{n-a-b} & & & \\ & & I_{2b} & & \\ & & & I_{n-a-b} & \\ & & & & y^* & I_a \end{pmatrix},$$

where y^* is chosen to make the matrix symplectic. Here the embedding of y is such that it consists of all matrices of size $(n-a-b) \times a$ such that its bottom row is zero. Finally V_0 is equal to the group $U_{2(n-a), n-a-b, 1}$, defined similarly to the definition of $U_{2n, a, 1}$ reviewed after (4.2) (in particular, the group V_0 is a subgroup of $Sp_{2(n-a)}$), and the character $\psi_{V_0, \alpha} := \psi_{U_{2(n-a), n-a-b, 1, \alpha}}$ as defined there. In coordinates, $\psi_{V_0, \alpha}(v) = \psi(v_{a+1, a+2} + v_{a+2, a+3} + \cdots + v_{n-b-1, n-b} + \alpha v_{n-b, n+b+1})$.

The next step is to carry out some root exchanges. The notion of *root exchange* is formulated abstractly in Ginzburg-Rallis-Soudry [13, Section 7.1]; a similar local statement may also be found in their paper [11, Section 2.2], and a global version also appears in the authors' paper [7, Section 2.2]. In our

We conclude that the integral (4.1) with $V = U_{2n-r+1,a}$ and $\psi_V = 1$ is equal to

$$(4.6) \quad \int_{\mathbb{A}^{n-r'}} \int_{\text{Mat}_{(n-a-b-1) \times a}(\mathbb{A})} \int_{V_0(F) \backslash V_0(\mathbb{A})} \int_{L_{2n,a}(F) \backslash L_{2n,a}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(r)}(uv_0 k(y) w_a j(x) g) \psi_{V_0, \alpha}(v_0) du dv_0 dy dx.$$

We shall prove momentarily that (4.6) is nonzero for some choice of data. Supposing this for the moment, let us analyze this integral when $g = \text{diag}(tI_a, I_{2n-r-2a+1}, t^{-1}I_a)$ is an element in the center of the Levi part of the maximal parabolic subgroup of Sp_{2n-r+1} whose Levi part is $GL_a \times Sp_{2n-r-2a+1}$, embedded in the r -fold metaplectic group via the trivial section. Suppose also that t is an r -th power. Inserting g into (4.6), we move it from right to left in the function $\theta_{2n}^{(r)}$. Moving it past $j(x)$ and making the corresponding variable change in x gives a factor of $|t|^{-a}$, as well as a factor of $\gamma(t^a)|t|^{a/2}$ from the action of the Weil representation. Then moving it past $k(y)$ we obtain a factor of $|t|^{-a(n-a-b-1)}$ from the corresponding change of variables in y . Clearly, g commutes with V_0 , and finally, applying Proposition 2.1 above we also get a factor of

$$\delta_{P_{2n,a}}^{\frac{r-1}{2r}}(\text{diag}(tI_a, I_{2(n-a)}, t^{-1}I_a)) = |t|^{a(r-1)(2n-a+1)/2r}$$

(here, recall that $P_{2n,a}$ is the maximal parabolic subgroup of Sp_{2n} whose unipotent radical is $L_{2n,a}$).

We conclude that, for $g = \text{diag}(tI_a, I_{2(n-a)}, t^{-1}I_a)$ as above and for $V = U_{2n-r+1,a}$, we have

$$\int_{V(F) \backslash V(\mathbb{A})} f(vg) dv = |t|^\beta \int_{V(F) \backslash V(\mathbb{A})} f(v) dv,$$

where

$$\beta = \frac{a(r-1)(2n-a+1)}{2r} - \frac{a}{2} - a(n-a-b-1).$$

It follows from Section 2 that $\chi_{Sp_{2n-r+1}, \Theta}^{(2r)}(g) = |t|^\beta$.

Finally, we prove that the integral (4.6) is not zero for some choice of data. This will also imply that the representation $\sigma_{2n-r+1}^{(2r)}$ is not zero. Suppose instead that (4.6) is zero for all choices of data. Arguing as in [7, Lemma 1], we may ignore the integration over the x and y variables, and deduce that the integral

$$\int_{V_0(F) \backslash V_0(\mathbb{A})} \int_{L_{2n,a}(F) \backslash L_{2n,a}(\mathbb{A})} \theta_{2n}^{(r)}(uv_0) \psi_{V_0, \alpha}(v_0) du dv_0$$

is zero for all choices of data. From Proposition 2.1 we deduce that the integral

$$(4.7) \quad \int_{V_0(F) \backslash V_0(\mathbb{A})} \theta_{2(n-a)}^{(r)}(v_0 h) \psi_{V_0, \alpha}(v_0) dv_0$$

is zero for all choice of data. Here $h \in Sp_{2(n-a)}^{(r)}(\mathbb{A})$.

To derive a contradiction, we will use the non-vanishing of the integral (3.1). Indeed, since $2b = 2n - 2a - r + 1$, the vanishing of (4.7) for all choices of data implies that the integral

$$(4.8) \quad \int_{L_{2b,b}(F) \backslash L_{2b,b}(\mathbb{A})} \int_{V_0(F) \backslash V_0(\mathbb{A})} \theta_{2b+r-1}^{(r)}(v_0 l) \psi_{V_0, \alpha}(v_0) dv_0 dl$$

is zero for all choices of data. We recall that $L_{2b,b}$ is the unipotent radical of the maximal parabolic subgroup of Sp_{2b} whose Levi part is GL_b . Arguing as in the computation of (4.2), we obtain that the vanishing of (4.8) implies the vanishing of the integral

$$(4.9) \quad \int_{U_{r-1,r'}(F) \backslash U_{r-1,r'}(\mathbb{A})} \int_{L_{2b+r-1,b}(F) \backslash L_{2b+r-1,b}(\mathbb{A})} \theta_{2b+r-1}^{(r)}(lv_0) \psi_{V_0, \alpha}(v_0) dl dv_0.$$

But the vanishing of (4.9) for all choice of data implies that (3.1) is zero for all choice of data, by Proposition 2.1. This is a contradiction.

Since the constant terms have been bounded and the descent is a compact integral of an automorphic form, the descent and its derivatives have uniformly moderate growth. Then from truncation as in the Corollary in [16, I.2.12], together with the above calculations, one obtains the following result.

Proposition 4.1. *Let r be odd, $r < 2n$, and suppose that Conjecture 2.2 _{r,i} holds for all i with $r' \leq i \leq n$. Then the representation $\sigma_{2n-r+1}^{(2r)}$ is not zero and it is a sub-representation of $L^2(Sp_{2n-r+1}^{(2r)}(F) \backslash Sp_{2n-r+1}^{(2r)}(\mathbb{A}))$. Moreover its projection to the residual spectrum is nonzero.*

One could also deduce this following [7, Propositions 4 and 5], by a sequence of expansions along abelian unipotent radicals. We omit the details, as in fact we expect that a stronger statement holds (compare [7, Conjecture 1]).

Conjecture 4.2. The representation $\sigma_{2n-r+1}^{(2r)}$ is equal to $\Theta_{2n-r+1}^{(2r)}$.

5. The Whittaker coefficient of the descent

In this section we study those cases in which the descent representation $\sigma_{2n-r+1}^{(2r)}$ is globally generic. Let us explain how these cases are predicted by

the dimension equation of the second author [10]. It follows from this equation together with (3.2) that the following equation should hold, where the dimension of a representation means its Gelfand-Kirillov dimension:

$$(5.1) \quad \dim \Theta_{2n-r+1}^{\psi^\alpha, \phi} + \dim \Theta_{2n}^{(r)} = \dim U_{2n, r'} + \dim \sigma_{2n-r+1}^{(2r)}.$$

Since $\Theta_{2n-r+1}^{\psi^\alpha, \phi}$ is the minimal representation of $Sp_{2n-r+1}^{(2)}(\mathbb{A})$, its dimension is $\frac{1}{2}(2n-r+1)$. An easy computation shows that $\dim U_{2n, r'} = \frac{1}{4}(r-1)(4n-r+1)$. If we want the representation $\sigma_{2n-r+1}^{(2r)}$ to be globally generic, its dimension should equal $\frac{1}{4}(2n-r+1)^2$. Using these values, (5.1) gives

$$(5.2) \quad \dim \Theta_{2n}^{(r)} = n^2 - n + \frac{1}{2}(r-1).$$

It is not hard to check that for a representation $\Theta_{2n}^{(r)}$ which satisfies both equation (5.2) and Conjecture 2.2_{r,n}, we must have $n \leq r < 2n$. If $n = 2k$, then for all $0 \leq i \leq k-1$ the dimension of the unipotent orbit $((4k-2i-2)(2i+2))$ is equal to $n^2 - n + \frac{1}{2}(r-1)$ with $r = 4k - 2i - 1$. If $n = 2k+1$, then for all $0 \leq i \leq k-1$ the dimension of the unipotent orbit $((4k-2i)(2i+2))$ is equal to $n^2 - n + \frac{1}{2}(r-1)$ with $r = 4k - 2i + 1$. In this case we also have the unipotent orbit $((2k+1)^2)$ whose dimension is $n^2 - n + \frac{1}{2}(r-1)$ with $r = 2k+1$. The above argument motivates the following result.

Theorem 5.1. *Suppose that r is odd, $n \leq r < 2n$. Suppose that Conjecture 2.2_{r,i} holds for $r' \leq i \leq n$. If $r \neq n, n+1$, suppose that Assumption 1 below holds as well. Then the representation $\sigma_{2n-r+1}^{(2r)}$ is globally generic. Moreover, in the special case where $r = n$, the Whittaker coefficient of $\sigma_{n+1}^{(2n)}$ attached to factorizable data $\theta_{2n}^{(n)} \in \Theta_{2n}^{(n)}$ and ϕ is factorizable.*

We shall see below that the hypotheses concerning Conjecture 2.2_{r,i} are satisfied when $r = n = 3$. Also, shortly after this paper was completed Prof. Fan Gao released a preprint [8] establishing that the local theta representation when $r = n$ has a unique Whittaker model. His result shows that Conjecture 4.2 is consistent with Theorem 5.1.

Proof. The proof is based on a computation of the integral (4.1) with V the maximal unipotent subgroup of Sp_{2n-r+1} and ψ_V the Whittaker character of V .

According to Conjecture 2.2_{r,n} the unipotent orbit attached to the representation $\Theta_{2n}^{(r)}$ is computed as follows. Let $r = 2n - 2l + 1$ for convenience, and write $2n = a_{n,r}(2n - 2l + 1) + b_{n,r}$. Then $a_{n,r} = 1$ and $b_{n,r} = 2l - 1$. Thus, the unipotent orbit attached to the representation $\Theta_{2n}^{(r)}$ is the Sp collapse of $((2n - 2l + 1)(2l - 1))$. This is $((2n - 2l)(2l))$ unless $n = 2l - 1$, in which case it is $((2l - 1)^2)$. This latter is the case $n = r$.

We consider first the case where the unipotent orbit attached to the representation $\Theta_{2n}^{(r)}$ is of the form $((2n - 2l)(2l))$. (In the above notations we have $l = i + 1$.) We first describe the set of Fourier coefficients attached to this unipotent orbit. Let P'_l denote the standard parabolic subgroup of Sp_{2n} whose Levi part is $GL_1^{n-2l} \times GL_2^l$. We embed this group in Sp_{2n} as the group of all matrices of the form $\text{diag}(a_1, \dots, a_{n-2l}, g_1, \dots, g_l, g_l^*, \dots, g_1^*, a_{2n-l}^{-1}, \dots, a_1^{-1})$. Here $a_i \in GL_1$ and $g_j \in GL_2$ and the starred entries are determined by the requirement that the matrix be in Sp_{2n} . Let U'_l denote the unipotent radical of P'_l . Then the Fourier coefficients of an automorphic function φ attached to the unipotent orbit $((2n - 2l)(2l))$ are given by

$$(5.3) \quad \int_{U'_l(F) \backslash U'_l(\mathbb{A})} \varphi(ug) \psi_{U'_l, a, b}(u) du$$

where $a, b \in F^*$ and the character $\psi_{U'_l, a, b}$ is defined by

$$(5.4) \quad \psi_{U'_l, a, b}(u) = \psi \left(\sum_{i=1}^{n-2l} u_{i, i+1} + \sum_{j=n-2l+1}^{n-2} u_{j, j+2} + au_{n-1, n+2} + bu_{n, n+1} \right).$$

Thus, to assert that the unipotent orbit $((2n - 2l)(2l))$ is attached to a certain representation, means first that there is a choice of a and b such that the Fourier coefficient (5.3) is not zero for some choice of data, and second that for all orbits which are greater than or not related to $((2n - 2l)(2l))$ all corresponding Fourier coefficients are zero. However, Conjecture 2.2 $_{r, n}$ does not specify for which values of a and b the integral (5.3) is nonzero for some choice of data. To prove the theorem, we will need a compatibility assumption which is stated as follows.

Assumption 5.2. Suppose that $r \neq n, n + 1$. Then there is a choice of vector φ in the space of $\Theta_{2n}^{(r)}$ such that the integral (5.3) with $a = \alpha$ is not zero.

This assumption, which will not be needed when the orbit is of the form (n^2) , is used to guarantee the non-vanishing of the Whittaker coefficient of the descent.

At this point we begin the computation of the Whittaker coefficient when $n \neq r$; the case $n = r$ will be treated separately. We compute the integral (4.1) with $V = U_{2n-r+1, n-r'}$, that is,

$$(5.5) \quad \int_{U_{2n-r+1, n-r'}(F) \backslash U_{2n-r+1, n-r'}(\mathbb{A})} f(ug) \psi_{U, b}(u) du.$$

As above $r' = (r - 1)/2$ and the character $\psi_{U, b}$, $b \in F^*$, is given by

$$\psi_{U, b} = \psi(u_{1,2} + u_{2,3} + \dots + u_{n-r'-1, n-r'} + bu_{n-r', n-r'+1}).$$

Similarly to the analysis of (4.1) and (4.2), the integral (5.5) is equal to

$$(5.6) \quad \int_{\mathbb{A}^{n-r'}} \int_{U_{2n-r+1, n-r'}(F) \backslash U_{2n-r+1, n-r'}(\mathbb{A})} \int_{U_{2n, r', 1}(F) \backslash U_{2n, r', 1}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(r)}(wvj(x)g) \psi_{U_{2n, r', 1, \alpha}}(u) \psi_{U, b}(v) du dv dx.$$

Define the following Weyl element w_0 of Sp_{2n} . First, let w denote the Weyl element of GL_n given by $w = \begin{pmatrix} I_{n-2l+1} & \\ & \nu \end{pmatrix}$ with $l = n - r'$ and $\nu = (\nu_{i,j}) \in GL_{2l-1}$ defined as follows. Let $\nu_{i,j} = 1$ for the pairs $(i, j) = (1, l); (3, l + 1); (5, l + 2); \dots; (2l - 1, 2l - 1)$ and $(i, j) = (2, 1); (4, 2); (6, 3) \dots (2l - 2, l - 1)$ and $\nu_{i,j} = 0$ otherwise. For example if $l = 2$ then $\nu = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then let $w_0 = \begin{pmatrix} w & \\ & w^* \end{pmatrix}$ where the star indicates that the matrix is in Sp_{2n} .

Since $\theta_{2n}^{(r)}$ is left-invariant under w_0 , inserting w_0 into $\theta_{2n}^{(r)}$ and moving it rightward, the integral (5.6) is equal to

$$(5.7) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(F) \backslash V_1(\mathbb{A})} \int_{U'_{l,1}(F) \backslash U'_{l,1}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(r)}(uvw_0j(x)g) \psi_{U'_{l, \alpha, b}}(u) du dv dx.$$

Here $U'_{l,1}$ is the subgroup consisting of $u = (u_{i,j}) \in U'_l$ such that $u_{2m, 2k+1} = 0$ for all $1 \leq m \leq l - 1$ and $m \leq k \leq l - 1$. Also, the group V_1 is the unipotent subgroup of Sp_{2n} consisting of all matrices of the form $I_{2n} + \sum_{i,j} r_{i,j} e_{i,j}^*$ where the sum is over all pairs $(i, j) = (2k + 1, 2m)$ with $1 \leq m \leq l - 1$ and $m - 1 \leq k \leq l - 2$.

Let Z_l denote the unipotent subgroup of U'_l consisting of all matrices of the form $I_{2n} + \sum_{i,j} r_{i,j} e_{i,j}^*$ where the sum is over all pairs $(i, j) = (2m, 2k + 1)$ with $1 \leq m \leq l - 1$ and $m \leq k \leq l - 1$. Then Z_l is a subgroup of U'_l which satisfies the condition that $U'_l = Z_l \cdot U'_{l,1}$. We perform a root exchange in the integral (5.7). Expanding the integral (5.7) along the quotient $Z_l(F) \backslash Z_l(\mathbb{A})$ and exchanging roots with the group V_1 , we deduce that this integral is equal to

$$(5.8) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{U'_l(F) \backslash U'_l(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(r)}(uvw_0j(x)g) \psi_{U'_{l, \alpha, b}}(u) du dv dx.$$

Using [7, Lemma 1], we deduce that the integral (5.8) is not zero for some choice of data if and only if the integral (5.3) with $a = 1$ is not zero for some choice of data. When $r \neq n + 1$, i.e. $n \neq 2l$, we use Assumption 5.2 to deduce the nonvanishing of integral (5.8), or equivalently, the nonvanishing of a Whittaker coefficient of $\sigma_{2n-r+1}^{(r)}$.

When $r = n + 1$, so $n = 2l$, we may deduce the nonvanishing of the integral (5.8) without requiring an analogue of Assumption 5.2. Indeed, this situation is essentially the same as the cases studied in Ginzburg-Rallis-Soudry [11, 12]. In those works, the residue representation is defined on the linear group $Sp_{2n}(\mathbb{A})$ rather than a covering group. In our case, we are concerned with the $r = (n + 1)$ -fold cover, and the representation we use to define the descent is $\Theta_{2n}^{(n+1)}$. However, the crucial ingredient for the computations of the Whittaker coefficients is the unipotent orbit attached to the residue representations. In both cases it is the orbit (n^2) . Since the computations for the proof of the theorem involve only manipulations of unipotent groups, the cover does not enter. Since the notations in those references are different, for the convenience of the reader we sketch the argument.

Starting with the integral (5.8), we choose $b = -\alpha/4$. Let $\gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 \end{pmatrix}$, and define $\gamma_0 = \text{diag}(\gamma, \dots, \gamma, \gamma^*, \dots, \gamma^*) \in Sp_{2n}(F)$. This is analogous to the matrix a defined in [11, (4.8)]. Since $\theta_{2n}^{(n+1)}$ is invariant under γ_0 , inserting it in the integral (5.8) and moving it rightward we obtain

$$(5.9) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{U'_1(F) \backslash U'_1(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n+1)}(u \gamma_0 v w_0 j(x) g) \psi_{U'_1, \alpha}(u) du dv dx$$

where $\psi_{U'_1, \alpha}$ is given by

$$\psi_{U'_1, \alpha}(u) = \psi(u_{1,3} + u_{2,4} + u_{3,5} + u_{4,6} + \dots + u_{n-3, n-1} + u_{n-2, n} + \alpha u_{n-1, n+1}).$$

Define the following Weyl element w'_0 of Sp_{2n} . If $w'_0 = (w'_{0, i, j})$ then $w'_{0, i, 2i-1} = 1$ for all $1 \leq i \leq n$. All other entries are $0, \pm 1$ and are determined uniquely such that $w'_0 \in Sp_{2n}$. This Weyl element was denoted by ν in [11, p. 881]. Inserting w'_0 in the integral (5.9), we obtain

$$(5.10) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{Y(F) \backslash Y(\mathbb{A})} \int_{U_{2n, n, 0}(F) \backslash U_{2n, n, 0}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n+1)}(u y w'_0 \gamma_0 v w_0 j(x) g) \psi_{U_{2n, n, \alpha}}(u) du dy dv dx.$$

Here Y is the unipotent group consisting of all matrices in Sp_{2n} of the form $\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}$ such that $y_{i, j} = 0$ if $i \geq j$. The group $U_{2n, n, 0}$ is the subgroup of $U_{2n, n}$ such that $u_{i, j} = 0$ for all $1 \leq i \leq n$ and $n + 1 \leq j \leq n + i$. Finally, the character $\psi_{U_{2n, n, \alpha}}$ is the character of $U_{2n, n}$ defined by $\psi_{U_{2n, n, \alpha}}(u) = \psi(u_{1,2} + u_{2,3} + \dots + u_{n-2, n-1} + \alpha u_{n-1, n})$.

At this point we carry out a sequence of root exchanges. This process is described in detail in [11, Section 5] so we will omit the details here. Carrying out this process, we deduce that the integral (5.8) is equal to

$$(5.11) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{Y(\mathbb{A})} \int_{U_{2n,n}(F) \backslash U_{2n,n}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n+1)}(uyw'_0 \gamma_0 v w_0 j(x)g) \psi_{U_{2n,n},\alpha}(u) du dy dv dx.$$

Then once again [7, Lemma 1], implies that integral (5.11) is not zero for some choice of data if and only if the inner integration along the quotient $U_{2n,n}(F) \backslash U_{2n,n}(\mathbb{A})$ is not zero for some choice of data. The nonvanishing of this integral follows from Proposition 2.1 with $a = n$ and the global genericity of the representation $\Theta_{GL_n}^{(n+1)}$ [14].

To complete the proof of the theorem we need to consider the case when n is odd and $r = n$. In this case we have $l = (n + 1)/2$. To use formulas from the previous case, we change the notation, and now let P'_l be the standard parabolic subgroup of Sp_{2n} whose Levi part is $GL_1 \times GL_2^{(n-1)/2}$ and U'_l be its unipotent radical. (Above we would have written P'_{l-1}, U'_{l-1} .) We again start with the integral (5.5), define w_0 as above, and after conjugation we obtain the integral (5.7), with the groups $U'_{l,1}$ and V_1 changed as follows. Now $U'_{l,1}$ is the subgroup of U'_l consisting of $u = (u_{i,j}) \in U'_l$ such that $u_{2m-1,2k} = 0$ for all $1 \leq m \leq l-1$ and $m \leq k \leq l-1$, and V_1 is the unipotent subgroup of Sp_{2n} consisting of all matrices of the form $I_{2n} + \sum_{i,j} r_{i,j} e_{i,j}^*$ where the sum ranges over all pairs $(i, j) = (2k, 2m-1)$ where $1 \leq m \leq l-1$ and $m-1 \leq k \leq l-2$.

Let Z_l now be the subgroup of U'_l consisting of all matrices of the form $I_{2n} + \sum_{i,j} r_{i,j} e_{i,j}^*$ where the sum is over all pairs $(i, j) = (2m-1, 2k)$ where $1 \leq m \leq l-1$ and $m \leq k \leq l-1$ together with the condition that $r_{1,2} = 0$. Notice that in this case it is not true that $U'_l = Z_l \cdot U'_{l,1}$. Performing root exchange similarly to our treatment of the integral (5.7) we obtain

$$\int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{U'_{l,0}(F) \backslash U'_{l,0}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n)}(uvw_0 j(x)g) \psi_{U'_{l,\alpha,b}}(u) du dv dx.$$

Here, $U'_{l,0}$ is the subgroup of U'_l consisting of all matrices $u = (u_{i,j}) \in U'_l$ such that $u_{1,2} = 0$, and the character $\psi_{U'_{l,1,b}}$ is given by

$$\psi_{U'_{l,1,b}}(u) = \psi \left(\sum_{i=1}^{n-2} u_{i,i+2} + \alpha u_{n-1,n+2} + b u_{n,n+1} \right).$$

Expand the above integral along the group $z(y) = I_{2n} + y e_{1,2}^*$. We obtain

$$\int_{\mathbb{A}^{n-r'} V_1(\mathbb{A})} \int_{\beta \in F} \sum_{F \backslash \mathbb{A} U'_{l,0}(F) \backslash U'_{l,0}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n)}(z(y)uvw_0 j(x)g) \psi_{U'_{l,\alpha,b}}(u) \psi(\beta y) du dy dv dx.$$

Assume that $b = -\alpha/4$. Then it is not hard to check that if $\beta \neq -\alpha/2$ then the two inner integrations give a coefficient which is associated to the unipotent orbit $((n+1)(n-1))$. Since n is odd this is a symplectic partition. However, it follows from Conjecture 2.2 $_{n,n}$ that the unipotent orbit associated with $\Theta_{2n}^{(n)}$

is (n^2) . Therefore, the only nonzero contribution to the above integral is from $\beta = -\alpha/2$. Let $\gamma = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ & 1 \end{pmatrix}$ and let $\gamma_0 = \text{diag}(\gamma, \dots, \gamma, \gamma^*, \dots, \gamma^*) \in Sp_{2n}(F)$. Using the invariance of $\theta_{2n}^{(n)}$ by this element and moving it rightward, we obtain the integral

$$(5.12) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{U'_i(F) \backslash U'_i(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n)}(u\gamma v w_0 j(x)g) \psi_{U'_i, \alpha}(u) du dv dx,$$

where $\psi_{U'_i, \alpha}$ is given by

$$\psi_{U'_i, \alpha}(u) = \psi(u_{1,2} + u_{2,4} + u_{3,5} + \dots + u_{n-3, n-1} + u_{n-2, n} + \alpha u_{n-1, n+1}).$$

Next we introduce the Weyl element $w_0^* \in Sp_{2n}$ defined by

$$w_0^* = \begin{pmatrix} 1 & & \\ & w'_0 & \\ & & 1 \end{pmatrix}.$$

(Here w'_0 was defined following (5.9).) Using invariance by w_0^* and moving this element rightward, (5.12) is equal to

$$\int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{Y_0(F) \backslash Y_0(\mathbb{A})} \int_{U_{2n, n, 0}(F) \backslash U_{2n, n, 0}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n)}(u y_0 w_0^* \gamma_0 v w_0 j(x)g) \\ \times \psi_{U_{2n, n, \alpha}}(u) du dy_0 dv dx.$$

Here $U_{2n, n, 0}$ is now the subgroup of $U_{2n, n}$ of $u = (u_{i, j}) \in U_{2n, n}$ such that $u_{i, j} = 0$ for $2 \leq i \leq n$ and $n+1 \leq j \leq n+i-1$, and Y_0 is the group of all matrices of the form

$$y_0 = \begin{pmatrix} 1 & & \\ & y' & \\ & & 1 \end{pmatrix}$$

such that $y' \in Y$ (where Y , as well as the character $\psi_{U_{2n, n}}$, was defined following (5.10)).

Carrying out root exchanges similarly to our treatment of (5.10), we obtain the integral

$$(5.13) \quad \int_{\mathbb{A}^{n-r'}} \int_{V_1(\mathbb{A})} \int_{Y_0(\mathbb{A})} \int_{U_{2n, n}(F) \backslash U_{2n, n}(\mathbb{A})} \omega_{\psi^\alpha}(g) \phi(x) \theta_{2n}^{(n)}(u y_0 w_0^* \gamma_0 v w_0 j(x)g) \psi_{U_{2n, n, \alpha}}(u) du dy_0 dv dx.$$

It is not hard to check that the integral (5.13) is not zero for some choice of data. To complete the proof of the theorem, observe that Proposition 2.1 allows us to express the integral of $\psi_{U_{2n, n, \alpha}}$ times any right translate of $\theta_{2n}^{(n)}$ over $U_{2n, n}(F) \backslash U_{2n, n}(\mathbb{A})$ in terms of the Whittaker function of the representation $\Theta_{GL_n}^{(n)}$. Since the Whittaker functional of this representation is unique up to scalars (see Kazhdan-Patterson [14, Theorem II.2.1]), we conclude that

the integral (5.13) is factorizable provided the data $\theta_{2n}^{(n)} \in \Theta_{2n}^{(n)}$ and ϕ are factorizable. \square

6. Some local computations

We have just seen that for factorizable data the Whittaker coefficient of $\sigma_{n+1}^{(2n)}$, n odd, can be expressed as a product of local Whittaker functionals. We now analyze the corresponding local integral in the unramified case. Consider a place ν where all data is unramified. In this Section, F denotes the corresponding local field, and in all integrals we understand that we are taking the F -valued points of the algebraic groups that are shown. We continue to use the convention that all matrices are embedded in the corresponding metaplectic groups via the trivial section, unless otherwise noted.

In this section, let $T_{2n}^{(n)}$ denote the inverse image in the local n -fold metaplectic group of the standard maximal torus of $Sp_{2n}(F)$, and let $Z(T_{2n}^{(n)})$ denote its center. Let p be a generator of the maximal ideal in the ring of integers \mathcal{O} of F . Since n is odd, -1 is an n -th power and hence $(p, p) = 1$. This implies that $(p^{k_1}, p^{k_2}) = 1$ for all integers k_1, k_2 . The subgroup of $T_{2n}^{(n)}$ generated by $Z(T_{2n}^{(n)})$ and by all matrices $\text{diag}(p^{k_1}, \dots, p^{k_n}, p^{-k_n}, \dots, p^{-k_1})$ is a maximal abelian subgroup of $T_{2n}^{(n)}$. Denote this subgroup by $T_{2n,0}^{(n)}$. For a diagonal matrix $t = \text{diag}(a, a^*) \in Sp_{2n}(F)$ we shall denote $t_0 = a \in GL_n(F)$. Also, recall that since we are in the unramified case the maximal compact subgroup $Sp_{2n}(\mathcal{O})$ of $Sp_{2n}(F)$ splits under the n -fold cover; fix an injection ι of $Sp_{2n}(\mathcal{O})$ into $Sp_{2n}^{(n)}(F)$.

Let $f_W^{(n)}$ be the function on $Sp_{2n}(F)$ defined as follows. First, it is right-invariant under $\iota(Sp_{2n}(\mathcal{O}))$. Second, $f_W^{(n)}(ug) = \psi_{U_{2n,n},\alpha}(u)f_W^{(n)}(g)$ for all $u \in U_{2n,n}(F)$. (The character $\psi_{U_{2n,n},\alpha}$ was defined before (5.11) above.) Recall that $\alpha \in F^*$ was defined in (3.1). For almost all places α is a unit. Hence, after a suitable conjugation by a torus element we may replace $\psi_{U_{2n,n},\alpha}$ by $\psi_{U_{2n,n},1}$ which we write simply $\psi_{U_{2n,n}}$. In doing so we make use of the right $\iota(Sp_{2n}(\mathcal{O}))$ -invariance of f_W . So from now on we set $\alpha = 1$ without loss. Finally, for all $(t, \zeta) \in T_{2n,0}^{(n)}$ ($t = \text{diag}(a, a^*) \in Sp_{2n}(F)$, $\zeta \in \mu_n$) we have $f_W^{(n)}((t, \zeta)) = \zeta \delta_Q^{\frac{n-1}{2n}}(t) W_{GL_n}^{(n)}((t_0, 1))$. Here $W_{GL_n}^{(n)}$ is the normalized Whittaker function for the local theta representation of $GL_n^{(n)}$ that is unramified with respect to the subgroup $GL_n(\mathcal{O})$ embedded in $GL_n^{(n)}(F)$ in a way that is compatible with ι , and $Q = P_{2n,n}$ is the maximal parabolic subgroup of Sp_{2n} whose Levi part is GL_n . (Recall from [14] that the space of Whittaker functionals for the theta function of $GL_n^{(n)}(F)$ is one dimensional.)

Consider the integral

$$(6.1) \quad W_{Sp_{n+1}}^{(2n)}(g) = \int_{F^{n-r'}} \int_{V_1} \int_{Y_0} \omega_\psi(g) \phi(x) f_W^{(n)}(y_0 w_0^* \gamma_0 v w_0 j(x) g) dy_0 dv dx,$$

where the function ϕ is the normalized unramified Schwartz function.

It may be desirable to compute $W_{Sp_{n+1}}^{(2n)}(g)$ for an arbitrary torus element; however, a general formula would be quite complicated. We will concentrate on a special case (though for $n = 3$ it is general) where the computations, though not trivial, are simpler. Let n_1, n_2 be non-negative integers, $a = p^{n_1}$ and $b = p^{n_2}$, and let

$$g = \text{diag}(ab, b, I_{n-3}, b^{-1}, a^{-1}b^{-1}).$$

Substituting this element in (6.1), changing variables in x and using the invariant properties of the functions involved, we obtain

$$\gamma(a) |ab^2|^{-1/2} \int_{V_1} \int_{Y_0} f_W^{(n)}(y_0 w_0^* \gamma_0 v w_0 g) dy_0 dv.$$

Here $\gamma(a)$ is the Weil factor and the factor $\gamma(a) |ab^2|^{-1/2}$ is obtained from the change of variables in x together with the properties of the Weil representation. We can ignore the integration over x since ϕ is the unramified Schwartz function.

The matrices γ_0 and γ were defined before (5.12) above. We have the factorization $\gamma = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 \end{pmatrix}$, and this induces a factorization of γ_0 : $\gamma_0 = \gamma'_0 w'_0 \gamma''_0$. Changing variables in V_1 we may ignore the matrix γ''_0 . Moving γ'_0 to the left and using the left transformation of $f_W^{(n)}$ by matrices in $U_{2n,n}$, we obtain the integral

$$\gamma(a) |ab^2|^{-1/2} \int_{V_1} \int_{Y_0} f_W^{(n)}(y_0 w_0^* w'_0 v w_0 g) \psi_{Y_0}(y_0) dy_0 dv.$$

Here we recall that n is odd and that Y_0 consists of all matrices of the form

$$y_0 = \begin{pmatrix} 1 & & & \\ & I_{n-1} & & \\ & y & I_{n-1} & \\ & & & 1 \end{pmatrix} \quad \text{with } y \in \text{Mat}_{(n-1) \times (n-1)}, y_{i,j} = 0 \text{ for all } i \geq j;$$

we have $\psi_{Y_0}(y_0) = \psi(y_{\frac{n-1}{2}, \frac{n+1}{2}})$.

Next we move the matrix g to the left and the Weyl element $w_0^* w'_0 w_0$ to the right. Doing so, we obtain the integral

$$(6.2) \quad \gamma(a) |ab^2|^{-\frac{n-2}{2}} \int_{V_2} \int_{Y_0} f_W^{(n)}(t y_0 v_2) \psi_{Y_0}(y_0) dy_0 dv_2$$

with $t = \text{diag}(ab, b, I_{2n-4}, b^{-1}, a^{-1}b^{-1})$. The group V_2 consists of all matrices of the form $\begin{pmatrix} I_n & \\ & I_n \end{pmatrix}$ where $v = \sum_{i,j} r_{i,j}(e_{i,j} + e_{n-j+1, n-i+1})$ and the sum is over all pairs (i, j) such that $1 \leq i \leq \frac{n-3}{2}$ and $1 \leq j \leq i+1$. The extra factor of $|ab^2|^{-\frac{n-3}{2}}$ is obtained from the change of variables in V_1 .

Integral (6.2) is the analogous to the integral that appears on the right hand side of Ginzburg-Rallis-Soudry [12, p. 255], Eqn. (3.7). Indeed, the matrices y_0 and v_2 can be viewed as matrices of Sp_{2n-2} embedded in Sp_{2n} in the obvious way. This comparison enables us to proceed as in [12, pp. 256–260], and conclude that integral (6.2) is a sum of two terms. First we have

$$\gamma(a) |ab^2|^{-\frac{n-2}{2}} f_W^{(n)}(t) = \gamma(a) |ab^2|^{\frac{2n-1}{2n}} W_{GL_n}^{(n)}(t_0).$$

This corresponds to the term which is denoted as Case 1 in [12, p. 260]. The second term we obtain is the one corresponding to Case 2 in that reference. It is equal to

$$(6.3) \quad \gamma(a) |ab^2|^{-\frac{n-2}{2}} \int_{|h|>1} f_W^{(n)}(ty(h)) \psi(h) |h|^{\frac{(n-1)(n-3)}{2}} dh,$$

with $y(h) = \text{diag}(I_2, h^{-1}I_{n-2}, hI_{n-2}, I_2)$. Write $h = p^{-m}\epsilon$ where ϵ is a unit. Then (6.3) is equal to

$$\gamma(a) |ab^2|^{-\frac{n-2}{2}} \sum_{m=1}^{\infty} q^{m(1+\frac{(n-1)(n-3)}{2})} \int_{|\epsilon|=1} f_W^{(n)}(ty(p^{-m}\epsilon)) \psi(p^{-m}\epsilon) d\epsilon.$$

The factorization $y(p^{-m}\epsilon) = y(p^{-m})y(\epsilon)$ contributes, via the cocycle defining the metaplectic group, the factor $(\epsilon, p)_n^{m(n-2)}$. Hence the above integral is equal to

$$(6.4) \quad \gamma(a) |ab^2|^{-\frac{n-2}{2}} \sum_{m=1}^{\infty} q^{m(1+\frac{(n-1)(n-3)}{2})} f_W^{(n)}(ty(p^{-m})) \int_{|\epsilon|=1} (\epsilon, p)_n^{m(n-2)} \psi(p^{-m}\epsilon) d\epsilon.$$

The inner integral is zero unless $m = 1$, and when $m = 1$ it is equal to $q^{-1/2} G_{n-2}^{(n)}(p)$ where the last term is a normalized n -th order Gauss sum, as in [7, Section 6]. Substituting this into (6.4) we obtain

$$\gamma(a) |ab^2|^{-\frac{n-2}{2}} q^{-\frac{(n-2)(2n-1)}{2n}} W_{GL_n}^{(n)}(t_0 y(p)_0).$$

Thus we have established the following result, relating the Whittaker coefficients of the descent on a suitable cover of the symplectic group to the Whittaker coefficients of the theta function on a cover of the general linear group.

Theorem 6.1. *Let $n \geq 3$ be an odd integer. Let $a = p^{n_1}$, $b = p^{n_2}$, with $n_1, n_2 \geq 0$. Let*

$$t = \text{diag}(ab, b, I_{n-3}, b^{-1}, a^{-1}b^{-1}) \in Sp_{n+1}(F),$$

$$t_0 = \text{diag}(ab, b, I_{n-2}) \in GL_n(F), \quad t_1 = \text{diag}(ab, b, pI_{n-2}) \in GL_n(F).$$

Suppose that Conjecture 2.2_{n,i} holds for $(n-1)/2 \leq i \leq n$. Then we have the identity

$$(6.5) \quad W_{Sp_{n+1}}^{(2n)}(t) = \gamma(a) |ab^2|^{\frac{2n-1}{2n}} \left(W_{GL_n}^{(n)}(t_0) + q^{-\frac{(n-2)(2n-1)}{2n}} W_{GL_n}^{(n)}(t_1) \right).$$

7. A case of Conjecture 2.2

Suppose $n = 3$. Then Conjecture 1_{3,i} is known for $i = 1$ ([14, Corollary II.2.6]) and $i = 2$ ([7, Proposition 3]). In this section we sketch the proof of Conjecture 1_{3,3}.

Proposition 7.1. *Conjecture 2.2_{3,3} holds. That is, $\mathcal{O}(\Theta_6^{(3)}) = (3^2)$.*

As a consequence, the hypotheses of Theorem 6.1 are satisfied for $n = 3$, and we conclude unconditionally the Whittaker function for the descent on the 6-fold cover of Sp_4 is Eulerian, and given at unramified places by formula (6.5).

Proof. To prove the Proposition we need to prove that the representation $\Theta_6^{(3)}$ has no nonzero Fourier coefficients corresponding to the unipotent orbits (6) and (42). Once we do so, an argument similar to the one given in Ginzburg [9, Theorem 3.1], will imply that this representation also has no nonzero Fourier coefficients corresponding to the orbit (41²). We also need to show that $\Theta_6^{(3)}$ has a nonzero Fourier coefficient corresponding to the unipotent orbit (3²). This last follows from Proposition 2.1. So the critical step is the vanishing of Fourier coefficients attached to the two unipotent orbits (6) and (42).

For convenience, we drop the prior notation for unipotent subgroups and characters, and will introduce the ones needed anew.

To prove this claim for $\Theta_6^{(3)}$ we make use of the fact that this representation can be obtained as a residue of an Eisenstein series induced from a suitable theta function of a smaller rank group. This is explained in Section 2 above. First we consider the unipotent orbit (6). Since $\Theta_6^{(3)}$ is a residue of $E_{\Theta_4}^{(3)}(g, s)$, in order to prove that $\Theta_6^{(3)}$ is not globally generic it is enough to prove that $E_{\Theta_4}^{(3)}(g, s)$ is not globally generic. Moreover, let U denote the standard maximal unipotent of Sp_6 , and let ψ_U denote a Whittaker character of U ; without loss we take $\psi_U(u) = \psi(u_{1,2} + u_{2,3} + au_{3,4})$ with $a \in F^*$. Then a standard unfolding argument shows that the integral

$$\int_{U(F) \backslash U(\mathbb{A})} E_{\Theta_4}^{(3)}(ug, s) \psi_U(u) du$$

is not zero for some choice of data if and only if the representation $\Theta_4^{(3)}$ is globally generic. However, it follows from Friedberg-Ginzburg [7, Lemma 2], that this representation is not globally generic. Hence, for $\text{Re}(s)$ large, the above integral is zero for all choices of data, and by continuation it follows that $\Theta_6^{(3)}$ is not globally generic.

Next we consider the Fourier coefficients corresponding to the unipotent orbit (42). Let $V \subseteq U$ denote the subgroup $V = \{u = (u_{i,j}) \in U : u_{2,3} = u_{4,5} = 0\}$. This is the unipotent radical of the parabolic subgroup of Sp_6 whose Levi part is $L = GL_1 \times GL_2$. The embedding of L into Sp_6 is given by $L = \{\text{diag}(g_1, g_2, g_2^*, g_1^{-1}) : g_1 \in GL_1; g_2 \in GL_2\}$. Let $\alpha_1, \alpha_2 \in F$ and $\alpha_3, \alpha_4 \in F^*$. Consider the Fourier coefficient defined by

$$(7.1) \quad \int_{V(F) \backslash V(\mathbb{A})} \varphi(vg) \psi_{V, \alpha_i}(v) dv,$$

where $\psi_{V, \alpha_i}(v) = \psi(\alpha_1 v_{1,2} + \alpha_2 v_{1,3} + \alpha_3 v_{3,4} + \alpha_4 v_{2,5})$. This Fourier coefficient is associated with the unipotent orbit (42) provided the connected component of the stabilizer of ψ_{V, α_i} inside $L(F)$ is trivial (see for example Ginzburg [9]). If $\alpha_3 \alpha_4 = -\epsilon^2$ for some $\epsilon \in F^*$, then one can find a suitable $\gamma_L \in L(F)$ and $\alpha \in F^*$ such that the integral (7.1) is equal to

$$(7.2) \quad \int_{V(F) \backslash V(\mathbb{A})} \varphi(v\gamma_L g) \tilde{\psi}_{V, \alpha}(v) dv,$$

where $\tilde{\psi}_{V, \alpha}(v) = \psi(\alpha v_{1,2} + v_{1,3} + v_{2,4})$. We will say that the Fourier coefficient (7.1) is of split type if $\alpha_3 \alpha_4 = -\epsilon^2$ for some $\epsilon \in F^*$, and non-split otherwise.

Since at half of the places ν we have the condition $\alpha_3 \alpha_4 = -\epsilon^2$ for some $\epsilon \in F_\nu^*$, to complete the proof it is enough to prove that for every finite local unramified place, the Jacquet module $J_{V, \tilde{\psi}_{V, \alpha}}(\Theta_6^{(3)})$ is zero for all characters $\tilde{\psi}_{V, \alpha}$ with $\alpha \in F^*$, where $\Theta_6^{(3)}$ now denotes the local theta representation at ν . To do so we consider the Jacquet module associated with the unipotent orbit (3²). Let R denote the subgroup of U defined by $R = \{u \in U : u_{1,2} = u_{3,4} = u_{5,6} = 0\}$. Then R is the unipotent radical of the maximal parabolic subgroup of Sp_6 whose Levi part is $GL_2 \times SL_2$. Let ψ_R denote the character $\psi_R(r) = \psi(r_{1,3} + r_{2,4})$ of R (that is, of $R(F_\nu)$; from now on we take the F_ν points of the algebraic groups in question without changing the notation).

Next, we compute $J_{RN, \psi_R \psi_\alpha}(\Theta_6^{(3)})$, where N is the unipotent radical of SL_2 embedded in Sp_6 as the group of all matrices

$$n(x) = \begin{pmatrix} 1 & x & & & & \\ & 1 & & & & \\ & & 1 & x & & \\ & & & 1 & & \\ & & & & 1 & -x \\ & & & & & 1 \end{pmatrix},$$

and ψ_α is the character of N defined by $\psi_\alpha(n(x)) = \psi(\alpha x)$, with $\alpha \in F_\nu$. Using the one-parameter subgroups $\{z(k) = I_6 + ke_{3,4}\}$ and $\{y(m) = I_6 + me_{2,3} - me_{4,5}\}$, we perform root exchanges using the Lemma in [11, Section 2]. This gives the isomorphism of vector spaces (and as modules for a suitable unipotent group)

$$(7.3) \quad J_{RN, \psi_R \psi_\alpha}(\Theta_6^{(3)}) \cong J_{V, \tilde{\psi}_{V, \alpha}}(\Theta_6^{(3)}).$$

Thus, to prove Proposition 7.1, it is enough to prove that for all $\alpha \neq 0$, the Jacquet module $J_{RN, \psi_R \psi_\alpha}(\Theta_6^{(3)})$ is zero.

It is not hard to check that the stabilizer of ψ_R inside $GL_2 \times SL_2$ is SL_2 embedded diagonally. Note that the restriction of the 3-fold cover to SL_2 , embedded diagonally, is a trivial cover. Thus, $J_{R, \psi_R}(\Theta_6^{(3)})$, if not zero, defines an unramified representation of SL_2 . Let σ be an irreducible unramified quotient of $J_{R, \psi_R}(\Theta_6^{(3)})$. Then there is a nonzero mapping

$$J_{R, \psi_R}(\Theta_6^{(3)}) \rightarrow \sigma \rightarrow \text{Ind}_B^{SL_2} \chi \delta_B^{1/2}.$$

By Frobenius reciprocity, this implies that the space

$$\text{Hom}_{GL_1}(J_{RN, \psi_R}(\Theta_6^{(3)}), \chi)$$

is not zero. We shall show below that GL_1 acts trivially on $J_{RN, \psi_R}(\Theta_6^{(3)})$. Knowing this, it follows that GL_1 must also act trivially on its quotient χ . It follows that $\chi = 1$, and so $\sigma = 1$. Since the Jacquet module is always of finite length, we then conclude that all Jacquet modules $J_{RN, \psi_R \psi_\alpha}(\Theta_6^{(3)})$ with $\alpha \neq 0$ are zero. Thus once we have established that GL_1 acts trivially on $J_{RN, \psi_R}(\Theta_6^{(3)})$, the Proposition will be proved.

To do so, we use the isomorphism (7.3) above with $\alpha = 0$. If $\alpha = 0$, then the Jacquet module $J_{RN, \psi_R \psi_0}(\Theta_6^{(3)}) = J_{RN, \psi_R}(\Theta_6^{(3)}) \cong J_{V, \psi_V}(\Theta_6^{(3)})$, if not zero, defines an unramified character of GL_1 , which is embedded in Sp_6 as the group of all matrices $\{t(a)\}$, with $t(a) = \text{diag}(a, a^{-1}, a, a^{-1}, a, a^{-1})$. Here $\psi_V = \tilde{\psi}_{V, \alpha}$ with $\alpha = 0$. For $1 \leq i \leq 3$, let w_i denote the three simple reflections of the

Weyl group of Sp_6 . In matrices we have

$$w_1 = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}; \quad w_2 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}; \quad w_3 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Let $Y = \{y_1(m_1) = I_6 + m_1 e_{4,3}\}$, and let U_1 be the subgroup of the maximal unipotent subgroup U of Sp_6 consisting of all matrices $u = (u_{i,j}) \in U$ such that $u_{2,4} = u_{3,5} = u_{3,4} = 0$. Let $\psi_{U_1}(u_1) = \psi(u_1(1,2) + u_1(2,3))$. Then $U_1 Y = (w_3 w_2) V (w_3 w_2)^{-1}$. Hence, $J_{V, \psi_V}(\Theta_6^{(3)})$ is isomorphic to $J_{U_1, \psi_{U_1}}(\Theta_6^{(3)})$. Let $X = \{I_6 + k e_{2,4} + k e_{3,5}\}$, and perform a root exchange of this unipotent subgroup with the group Y . Then, since $\Theta_6^{(3)}$ is not (locally) generic, we obtain that $J_{U_1, \psi_{U_1}}(\Theta_6^{(3)})$ is isomorphic to $J_{U, \psi'_U}(\Theta_6^{(3)})$, with $\psi'_U(u) = \psi(u_{1,2} + u_{2,3})$. This Jacquet module is not zero. Indeed, using an argument similar to [3, Theorem 2.3] the nonvanishing of this Jacquet module follows from the genericity of the theta representation of $GL_3^{(3)}$. From this we deduce that the Jacquet module $J_{RN, \psi_R}(\Theta_6^{(3)})$ is not zero. Moreover, the Jacquet module $J_{U, \psi'_U}(\Theta_6^{(3)})$ acts by the character $|a|^4$ under the torus $\text{diag}(aI_3, a^{-1}I_3) \in Sp_6$. Taking into an account the various root exchanges, we deduce that the group $GL_1 = \{t(a)\}$ acts trivially on the Jacquet module $J_{RN, \psi_R}(\Theta_6^{(3)})$, as claimed. This completes the proof of the Proposition. \square

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