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STRONG EXPONENT BOUNDS FOR THE LOCAL RANKIN-SELBERG CONVOLUTION

C.J. BUSHNELL* AND G. HENNIART

Abstract. Let F be a non-Archimedean locally compact field. Let σ and τ be finite-dimensional representations of the Weil-Deligne group of F. We give strong upper and lower bounds for the Artin and Swan exponents of $\sigma\otimes\tau$ in terms of those of σ and τ. We give a different lower bound in terms of $\sigma\otimes\check\sigma$ and $\tau\otimes\check\tau$. Using the Langlands correspondence, we obtain the bounds for Rankin-Selberg exponents.

Keywords: Local Langlands correspondence, Weil-Deligne groups and representations, tensor products, Artin exponent, Swan exponent, Rankin-Selberg exponent.
MSC(2010): Primary: 22E50; Secondary: 11S37.

1. Introduction

1.1. Let F be a non-Archimedean, locally compact field. For integers $m,n\geqslant 1$ let π, ρ be irreducible, smooth, complex representations of the general linear groups $\mathrm{GL}_m(F)$, $\mathrm{GL}_n(F)$ respectively. If s is a complex variable and ψ a non-trivial smooth character of F, we consider the L-function $\mathrm{L}(\pi\times\rho,\mathrm{s})$ and the local constant $\varepsilon(\pi\times\rho,s,\psi)$ of [17] or [21, 22]. If q is the cardinality of the residue field of F, the local constant takes the form

$$\varepsilon(\pi\times\rho,s,\psi)=\varepsilon(\pi\times\rho,0,\psi)\,q^{-s(\mathrm{Ar}(\pi\times\rho)+mnc(\psi))}.$$

Here, $c(\psi)$ is an integer depending only on ψ. The integer $\mathrm{Ar}(\pi\times\rho)$ depends only on the pair (π,ρ). Here we call it the *Rankin-Selberg exponent* of (π,ρ).

If we take $n=1$ and let ρ be the trivial character 1 of $F^\times\cong\mathrm{GL}_1(F)$, then $\varepsilon(\pi\times 1,s,\psi)$ is the Godement-Jacquet local constant $\varepsilon(\pi,s,\psi)$ [10], and $\mathrm{Ar}(\pi\times 1)$ is denoted simply $\mathrm{Ar}(\pi)$. The aim of this paper is to give strong, universal estimates for $\mathrm{Ar}(\pi\times\rho)$ in terms of $\mathrm{Ar}(\pi)$ and $\mathrm{Ar}(\rho)$. We give a second lower bound in terms of exponents of the pairs $(\pi,\check\pi)$, $(\rho,\check\rho)$. These results are Corollaries A–C below.

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1.2. We fix a separable algebraic closure $\bar{F}/F$ of the field F, and form the Weil group $\mathcal{W}_F = \mathcal{W}_{\bar{F}/F}$. Let $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ be the set of equivalence classes of finite-dimensional representations of the Weil-Deligne group defined by $\mathcal{W}_F$ (see 1.1). With π and ρ as before, the Langlands correspondence [11, 15, 18, 19] associates to π, ρ representations ${}^L\pi, {}^L\rho \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$. These have dimension m, n respectively.

For $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, of dimension d, let $\varepsilon(\sigma, s, \psi)$ be the Langlands-Deligne local constant [3, 23] of σ. Again, $\varepsilon(\sigma, s, \psi) = \varepsilon(\sigma, 0, \psi)\, q^{-s(\mathrm{Ar}(\sigma)+dc(\psi))}$ and the integer $\mathrm{Ar}(\sigma)$ is the *Artin exponent* of σ. A defining property of the Langlands correspondence [14, 16] is that

$$\varepsilon(\pi\times\rho, s, \psi) = \varepsilon({}^L\pi\otimes{}^L\rho, s, \psi).$$

Consequently, $\mathrm{Ar}(\pi\times\rho) = \mathrm{Ar}({}^L\pi\otimes{}^L\rho)$ and $\mathrm{Ar}(\pi) = \mathrm{Ar}({}^L\pi)$. We may therefore tackle the Rankin-Selberg exponent via the Artin exponent of tensor products of representations of the Weil-Deligne group.

1.3. We state our results for representations of the Weil-Deligne group. If $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, $\sigma \neq 0$, write

$$\eta(\sigma) = \mathrm{Ar}(\sigma)/\dim\sigma, \quad \sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}},\ \sigma\neq 0.$$

Convention. When σ is the zero representation, $\eta(\sigma)$ is undefined. So, use of the symbol $\eta(\sigma)$ here will *always* entail the implicit assumption $\sigma \neq 0$.

Say that σ is *η-minimal* if $\mathrm{Ar}(\sigma) \leqslant \mathrm{Ar}(\chi\otimes\sigma)$, for any character χ of $\mathcal{W}_F$.

Theorem A. *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is η-minimal, then*

$$\eta(\sigma\otimes\tau) \geqslant \tfrac{1}{2}\max\{\eta(\sigma), \eta(\tau)\},$$

for all $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$.

A trivial example shows that some hypothesis of minimality is required for a result of this kind: for fixed $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ and a character χ of $\mathcal{W}_F$, one has $\eta((\chi\otimes\sigma)\otimes(\chi^{-1}\otimes\tau)) = \eta(\sigma\otimes\tau)$. For suitable choice of χ, one has $\eta(\chi\otimes\sigma) = \eta(\chi^{-1}\otimes\tau) = \eta(\chi)$ and this may be taken as large as desired.

Further examples show that the constant $\frac{1}{2}$ is best possible: there are many pairs of *irreducible* representations (σ, τ), with σ being η-minimal, for which $2\eta(\sigma\otimes\tau) = \eta(\sigma) = \eta(\tau)$. However, by restricting the class of representation one can get better constants: see the examples in 3.3 and 4.5.

There is a second, rather different, lower bound. This avoids the necessity for a minimality condition by using the operation $\sigma \mapsto \check{\sigma}$ of contragredience on $\widehat{\mathcal{W}}_F^{\mathcal{D}}$.

Theorem B. *If $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, then*

$$\eta(\sigma\otimes\check{\tau}) \geqslant \tfrac{1}{2}\big(\eta(\sigma\otimes\check{\sigma}) + \eta(\tau\otimes\check{\tau})\big).$$

If σ and τ are indecomposable, then $\eta(\sigma\otimes\check{\tau}) \geqslant \max\{\eta(\sigma\otimes\check{\sigma}), \eta(\tau\otimes\check{\tau})\}$.

The easy example $\sigma = \tau$ shows that the constant $\frac{1}{2}$ is again best possible. With regard to upper bounds, we prove:

Theorem C. *Let $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ have dimensions m, n, respectively. The Artin exponent $\mathrm{Ar}(\sigma \otimes \tau)$ satisfies*

$$\mathrm{Ar}(\sigma \otimes \tau) \leqslant n\mathrm{Ar}(\sigma) + m\mathrm{Ar}(\tau) - \min\{\mathrm{Ar}(\sigma), \mathrm{Ar}(\tau)\}.$$

If $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ are irreducible, then

$$\eta(\sigma \otimes \tau) \leqslant \max\{\eta(\sigma), \eta(\tau)\}.$$

Both aspects of the result are best possible.

1.4. Let π, ρ be irreducible, smooth, complex representations of $\mathrm{GL}_m(F)$, $\mathrm{GL}_n(F)$, respectively. Set $\eta(\pi \times \rho) = \mathrm{Ar}(\pi \times \rho)/mn$ and $\eta(\pi) = \mathrm{Ar}(\pi)/m$, with the same convention regarding zero representations. Say that π is *η-minimal* if $\eta(\pi) \leqslant \eta(\chi\pi)$ for all characters χ of $F^\times$. The Langlands correspondence respects contragredience and twisting with characters, so we have the following consequences of Theorems A–C.

Corollary A. *Let π, ρ be irreducible representations of the groups $\mathrm{GL}_m(F)$, $\mathrm{GL}_n(F)$, respectively. If π is η-minimal, then*

$$\eta(\pi \times \rho) \geqslant \tfrac{1}{2}\max\{\eta(\pi), \eta(\rho)\}.$$

Corollary B. *If π, ρ are irreducible representations of the groups $\mathrm{GL}_m(F)$, $\mathrm{GL}_n(F)$, respectively, then*

$$\eta(\pi \times \check\rho) \geqslant \tfrac{1}{2}\big(\eta(\pi \times \check\pi) + \eta(\rho \times \check\rho)\big).$$

If π and ρ are essentially square-integrable, then

$$\eta(\pi \times \check\rho) \geqslant \max\{\eta(\pi \times \check\pi), \eta(\rho \times \check\rho)\}.$$

Corollary C. *Let π, ρ be irreducible representations of the groups $\mathrm{GL}_m(F)$, $\mathrm{GL}_n(F)$, respectively. The Rankin-Selberg exponent satisfies*

$$\mathrm{Ar}(\pi \times \rho) \leqslant n\mathrm{Ar}(\pi) + m\mathrm{Ar}(\rho) - \min\{\mathrm{Ar}(\pi), \mathrm{Ar}(\rho)\}.$$

If the representations π and ρ are cuspidal, then

$$\eta(\pi \times \rho) \leqslant \max\{\eta(\pi), \eta(\rho)\}.$$

In all of these statements, the representations π, ρ are assumed *smooth.* Corollary C may also be found in [1], where it receives a different proof.

Beyond remarking that the representation π is essentially square-integrable (resp. cuspidal) if and only if ${}^L\pi \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable (resp. irreducible), there is nothing more to be said about these corollaries.

1.5. We return to the Galois side. Let $\widehat{\mathcal{W}}_F^{\mathrm{ss}}$ be the set of equivalence classes of finite-dimensional, smooth, semisimple representations of $\mathcal{W}_F$.

There is a parallel, but distinct, family of estimates governing the *Swan exponent* $\mathrm{Sw}(\sigma)$, $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, in place of the Artin exponent. We include them here since, in applications, the Swan exponent often occurs more naturally than the Artin exponent and it can be bothersome to switch between the two languages. The exponent $\mathrm{Sw}(\sigma)$ depends only on the restriction of σ to $\mathcal{W}_F$, so nothing is lost by treating Sw as a function on $\widehat{\mathcal{W}}_F^{\mathrm{ss}}$.

If $\sigma \neq 0$, we set $\varsigma(\sigma) = \mathrm{Sw}(\sigma)/\dim\sigma$. Again, use of the symbol $\varsigma(\sigma)$ entails the implicit assumption $\sigma \neq 0$.

Say that σ is ς-*minimal* if $\varsigma(\sigma) \leqslant \varsigma(\chi\otimes\sigma)$ for all characters χ of $\mathcal{W}_F$. (Note that the concepts of η-minimality and ς-minimality are distinct.) We then have the following results.

Theorem AS. *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$ is ς-minimal, then*

$$\varsigma(\sigma\otimes\tau) \geqslant \tfrac{1}{2}\max\{\varsigma(\sigma), \varsigma(\tau)\},$$

for all $\tau \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$.

Theorem BS. *If $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$, then*

$$\varsigma(\sigma\otimes\check{\tau}) \geqslant \tfrac{1}{2}\big(\varsigma(\sigma\otimes\check{\sigma}) + \varsigma(\tau\otimes\check{\tau})\big).$$

If σ and τ are irreducible, then $\varsigma(\sigma\otimes\check{\tau}) \geqslant \max\{\varsigma(\sigma\otimes\check{\sigma}), \varsigma(\tau\otimes\check{\tau})\}$.

Theorem CS. *Let $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$ have dimensions m, n, respectively. The Swan exponent $\mathrm{Sw}(\sigma\otimes\tau)$ satisfies*

$$\mathrm{Sw}(\sigma\otimes\tau) \leqslant n\,\mathrm{Sw}(\sigma) + m\,\mathrm{Sw}(\tau) - \min\{\mathrm{Sw}(\sigma), \mathrm{Sw}(\tau)\}.$$

If $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ are irreducible, then $\varsigma(\sigma\otimes\tau) \leqslant \max\{\varsigma(\sigma), \varsigma(\tau)\}$.

1.6. We review some background material in Section 2. The proof of Theorem A starts in Section 3, where we deal with irreducible representations. At present, these can only be treated via parallel properties for irreducible *cuspidal* representations of general linear groups and then using the Langlands correspondence. The method relies on the explicit formula for $\mathrm{Ar}(\pi\times\rho)$ in [6], combining the classification theory of [7], [8, 9] with the interpretation [22] of the Rankin-Selberg exponent as a relative Plancherel measure. This is where the factor $\frac{1}{2}$ of Theorem A first appears and reveals itself as best possible. The main part of the proofs of Theorems A and AS is in Section 4. The arguments are all conducted on the Galois side. They are essentially elementary although, in places, they feel intricate.

Theorems B and BS are treated in Section 5. The proofs start from relatively simple properties of tensor products of irreducible representations observed in [12, 5] but are equally intricate. For the pairs A/AS, B/BS of parallel theorems, the proofs start together. We then concentrate on the more involved case of the Artin exponent. That done, the argument for the Swan exponent follows a shorter version of the same route, obtained by a simple change of vocabulary. We indicate the process briefly at the ends of the relevant sections. The results are not so easy to deduce from each other, and nothing seems to be gained from constructing an artificial framework in which they can be treated together. The proofs of Theorems C and CS are short, and combined in Section 6.

2. **Representations of the Weil-Deligne group**

We retain the notations $\mathcal{W}_F$, $\widehat{\mathcal{W}}_F^{\mathrm{ss}}$ and $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ of the introduction. Let $\widehat{\mathcal{W}}_F^{\mathrm{irr}}$ be the set of isomorphism classes of *irreducible* smooth representations of $\mathcal{W}_F$. Starting from the discussions in [3] and [23], we recall some basic features of representations $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$. We define the Artin exponent in terms of the Langlands-Deligne local constant and collect a number of facts and simple results for use in later sections.

2.1. Let q be the cardinality of the residue class field of F. Let $x \mapsto \|x\|$ denote the unique character of $\mathcal{W}_F$ that is trivial on the inertia subgroup of $\mathcal{W}_F$ and takes the value q^{-1} on geometric Frobenius elements.

For our purposes, *a representation σ of the Weil-Deligne group of F* is a pair $(\sigma_{\mathcal{W}}, \mathfrak{n})$ consisting of a finite-dimensional, smooth, semisimple representation $\sigma_{\mathcal{W}} : \mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(V)$ and a nilpotent endomorphism $\mathfrak{n}$ of the vector space V such that

$$\sigma_{\mathcal{W}}(g)\,\mathfrak{n} = \|g\|\,\mathfrak{n}\,\sigma_{\mathcal{W}}(g), \quad g \in \mathcal{W}_F.$$

We denote by $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ the set of isomorphism classes of such representations. For $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, we rarely use the notation $\sigma_{\mathcal{W}}$ but speak instead of the restriction of σ to $\mathcal{W}_F$. In the same spirit, a representation $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$ defines an element $(\sigma, 0)$ of $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ that we continue to denote by σ.

The set $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ admits a notion of direct sum,

$$(\sigma, \mathfrak{m}) \oplus (\tau, \mathfrak{n}) = (\sigma \oplus \tau, \mathfrak{m} \oplus \mathfrak{n}).$$

We say $(\sigma, \mathfrak{n})$ is *indecomposable* if it cannot be expressed in this way as a direct sum in which both factors are non-trivial. Surely any $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ may be expressed as a direct sum of indecomposable elements of $\widehat{\mathcal{W}}_F^{\mathcal{D}}$. Such a decomposition is unique up to permutation of the isomorphism classes of indecomposable factors.

To define the tensor product $(\sigma, \mathfrak{m}) \otimes (\tau, \mathfrak{n})$, let σ act on a vector space V and τ on W. One sets

$$(\sigma, \mathfrak{m}) \otimes (\tau, \mathfrak{n}) = (\sigma \otimes \tau, \mathfrak{m} \otimes 1_W + 1_V \otimes \mathfrak{n}).$$

2.2. We recall the standard first example of an element of $\widehat{\mathcal{W}}_F^{\mathcal{D}}$. Let $n \geqslant 1$ be an integer and let $\mathrm{sp}_n \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$ denote the direct sum of the characters $x \mapsto \|x\|^i$, for $0 \leqslant i \leqslant n-1$. We view sp_n as acting on $V = \mathbb{C}^n$. The space V admits a *regular* nilpotent endomorphism $\mathfrak{n}$ such that $\mathrm{Sp}_n(1) = (\mathrm{sp}_n, \mathfrak{n})$ is a representation of the Weil-Deligne group. The isomorphism class of $\mathrm{Sp}_n(1)$ is independent of the choice of $\mathfrak{n}$.

More generally, let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$. We define

$$\mathrm{Sp}_n(\sigma) = \sigma \otimes \mathrm{Sp}_n(1).$$

An exercise [23, (4.1.5)] yields:

Fact 2.1. *A representation $\Sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable if and only if $\Sigma = \mathrm{Sp}_n(\sigma)$, for an integer $n \geqslant 1$ and a representation $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$. Moreover, $\mathrm{Sp}_n(\sigma) \cong \mathrm{Sp}_{n'}(\sigma')$ if and only if $n = n'$ and $\sigma \cong \sigma'$.*

2.3. We recall the definition of the Artin exponent $\mathrm{Ar}(\sigma)$ and the Swan exponent $\mathrm{Sw}(\sigma)$, for $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$.

Let ψ be a non-trivial smooth character of F and s a complex variable. If $\dim \sigma = n$, the Langlands-Deligne local constant $\varepsilon(\sigma, s, \psi)$ takes the form

$$\varepsilon(\sigma, s, \psi) = \varepsilon(\sigma, 0, \psi)\, q^{-(\mathrm{Ar}(\sigma)+nc(\psi))s}.$$

The constant $\varepsilon(\sigma, 0, \psi)$ is non-zero. The exponent $\mathrm{Ar}(\sigma)$ is a non-negative integer depending only on σ and $c(\psi)$ is an integer depending only on ψ. The function $\sigma \mapsto \mathrm{Ar}(\sigma)$ is additive with respect to direct sums. In simple cases, it is given as follows.

Fact 2.2.
(1) *If χ is an unramified character of $\mathcal{W}_F$, then $\mathrm{Ar}(\mathrm{Sp}_r(\chi)) = r-1$.*
(2) *Let $(\sigma, \mathfrak{n}) \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$. If σ is a direct sum of unramified characters of $\mathcal{W}_F$, then $\mathrm{Ar}(\sigma, \mathfrak{n})$ equals the rank of the linear operator $\mathfrak{n}$.*
(3) *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$ is not an unramified character, then $\mathrm{Ar}(\mathrm{Sp}_r(\sigma)) = r\mathrm{Ar}(\sigma)$.*

These are the key instances of a general formula [23, (4.1.6)] (but note that, in the terminology of [23], all $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ are Φ-semisimple).

We define the *Swan exponent* $\mathrm{Sw}(\sigma)$ of $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$: if $\sigma = (\sigma_{\mathcal{W}}, \mathfrak{n})$, then $\mathrm{Sw}(\sigma) = \mathrm{Sw}(\sigma_{\mathcal{W}})$. On $\widehat{\mathcal{W}}_F^{\mathrm{ss}}$, the function $\sigma \mapsto \mathrm{Sw}(\sigma)$ is additive with respect to direct sums. If χ is an unramified character of $\mathcal{W}_F$, then $\mathrm{Sw}(\chi) = \mathrm{Ar}(\chi) = 0$. If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$ is not an unramified character, then $\mathrm{Sw}(\sigma) = \mathrm{Ar}(\sigma) - \dim \sigma$.

As in the introduction, it is helpful to have normalized exponents. For $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, $\sigma \neq 0$, set

$$\eta(\sigma) = \mathrm{Ar}(\sigma)/\dim \sigma, \qquad \varsigma(\sigma) = \mathrm{Sw}(\sigma)/\dim \sigma.$$

The use of either of these symbols carries the presumption that σ is not zero.

The ς-invariant has a helpful property. For a real number $x \geqslant 0$, let $\mathcal{W}_F^x$ be the corresponding ramification subgroup of $\mathcal{W}_F$ (see [20, Section IV.3]). From [13, Théorème 3.5] , we have:

Fact 2.3. *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$, then $\varsigma(\sigma) = \inf\big\{x \geqslant 0 : \mathcal{W}_F^x \subset \operatorname{Ker}\sigma\big\}$. If $\varsigma(\sigma) > 0$, then the restriction $\sigma\big|\mathcal{W}_F^{\varsigma(\sigma)}$ does not contain the trivial character.*

We make repeated use of the following observation.

Lemma 2.4. *If $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$, then*

$$\begin{aligned} \varsigma(\sigma\otimes\tau) &\leqslant \max\{\varsigma(\sigma), \varsigma(\tau)\}, \quad \textit{and} \\ \eta(\sigma\otimes\tau) &\leqslant \max\{\eta(\sigma), \eta(\tau)\}. \end{aligned}$$

Equality holds in the first instance if $\varsigma(\tau) \neq \varsigma(\sigma)$, in the second if $\eta(\tau) \neq \eta(\sigma)$.

Proof. The inequality concerning ς follows from [5, 3.1 Corollary]. Suppose that $s = \varsigma(\sigma) > \varsigma(\tau)$. For $x \geqslant 0$, let $\mathcal{W}_F^{x+}$ denote the closure of the group $\bigcup_{y>x} \mathcal{W}_F^y$. By Fact 2.3, each of the representations σ, τ, $\sigma\otimes\tau$ is trivial on $\mathcal{W}_F^{s+}$ while τ is trivial on $\mathcal{W}_F^s$. The restriction of σ to $\mathcal{W}_F^s$ does not contain the trivial character. It follows that any irreducible component ξ of $\sigma\otimes\tau$ satisfies $\varsigma(\xi) = s$, whence $\mathrm{Sw}(\xi) = s\dim\xi$ and $\varsigma(\sigma\otimes\tau) = s$.

The assertions concerning η now follow from the definitions via a simple calculation. ☐

Note. We have proved the parts of Theorems C and CS relating to irreducible representations. We do not return to those results until Section 5.

2.4. We consider tensor products of indecomposable elements of $\widehat{\mathcal{W}}_F^{\mathcal{D}}$ that are *unramified* on restriction to $\mathcal{W}_F$.

Proposition 2.5. *Let m, n be positive integers. If χ, ξ are unramified characters of $\mathcal{W}_F$, then*

$$\begin{aligned} \mathrm{Ar}(\mathrm{Sp}_m(\chi)\otimes\mathrm{Sp}_n(\xi)) &= mn - \min\{m,n\}, \\ \eta(\mathrm{Sp}_m(\chi)\otimes\mathrm{Sp}_n(\xi)) &= \max\{\eta(\mathrm{Sp}_m(\chi)), \eta(\mathrm{Sp}_n(\xi))\}. \end{aligned}$$

Proof. The two assertions are visibly equivalent so we prove the first. There are positive integers r_i and unramified characters χ_i of $\mathcal{W}_F$, $1 \leqslant i \leqslant l$, so that

$$\mathrm{Sp}_m(\chi)\otimes\mathrm{Sp}_n(\xi) = \bigoplus_{i=1}^{l} \mathrm{Sp}_{r_i}(\chi_i).$$

In particular, $\sum_{i=1}^{l} r_i = mn$. Using the definition of η and the additivity of the exponent Ar, we get

$$\begin{aligned}\eta\big(\bigoplus_{i=1}^{l} \mathrm{Sp}_{r_i}(\chi_i)\big) &= \sum_{i=1}^{l} r_i \eta\big(\mathrm{Sp}_{r_i}(\chi_i)\big)/mn \\ &= \sum_{i=1}^{l} (r_i - 1)/mn = 1 - l/mn.\end{aligned}$$

We therefore need to compute l.

Write $\mathrm{Sp}_m(\chi) = (\sigma, \mathfrak{m})$, where $\mathfrak{m}$ is a regular nilpotent endomorphism of $\mathbb{C}^m$. Likewise, write $\mathrm{Sp}_n(\xi) = (\tau, \mathfrak{n})$, so that $\mathrm{Sp}_m(\chi) \otimes \mathrm{Sp}_n(\xi) = (\sigma \otimes \tau, \mathfrak{l})$, where $\mathfrak{l} = \mathfrak{m} \otimes 1 + 1 \otimes \mathfrak{n}$. In this form, the integer $mn - l$ is the rank of the nilpotent operator $\mathfrak{l}$ (Fact 2.2(2)). It is therefore enough to recall:

Lemma 2.6. *Let $\mathfrak{m}$ (resp. $\mathfrak{n}$) be a regular nilpotent endomorphism of the vector space $V = \mathbb{C}^m$ (resp. $W = \mathbb{C}^n$). The operator $\mathfrak{l} = \mathfrak{m} \otimes 1_W + 1_V \otimes \mathfrak{n}$ has rank $mn - \min\{m, n\}$.*

The proof of the lemma is a straightforward exercise which completes the proof of the proposition. □

3. Irreducible representations

We prove Theorems A and AS for *irreducible* representations of $\mathcal{W}_F$, taking an indirect approach. We state and prove analogous results for irreducible *cuspidal* representations of general linear groups $\mathrm{GL}_n(F)$ and then use the Langlands correspondence.

3.1. We need some definitions. Let π be an irreducible *cuspidal* representation of $\mathrm{GL}_n(F)$, for an integer $n \geqslant 1$. If χ is a character of $F^\times$, then $\chi\pi$ denotes the representation $g \mapsto \chi(\det g)\pi(g)$, $g \in \mathrm{GL}_n(F)$.

We recalled in the introduction the definition of the Artin exponent $\mathrm{Ar}(\pi)$ of π. We also use the notation $\eta(\pi) = \mathrm{Ar}(\pi)/n$.

The *Swan exponent* $\mathrm{Sw}(\pi)$ of π is defined by $\mathrm{Sw}(\pi) = \mathrm{Ar}(\pi) - n$ *except* in the case where $n = 1$ and π is an unramified character of $F^\times = \mathrm{GL}_1(F)$. In that case, $\mathrm{Sw}(\pi) = 0$. In all cases, $\mathrm{Sw}(\pi) \geqslant 0$. We also use the notation $\varsigma(\pi) = \mathrm{Sw}(\pi)/n$.

If $\sigma = {}^L\pi \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$ is the irreducible representation of $\mathcal{W}_F$ attached to π by the Langlands correspondence, then $\mathrm{Ar}(\sigma) = \mathrm{Ar}(\pi)$ and $\eta(\sigma) = \eta(\pi)$. The definitions ensure that $\mathrm{Sw}(\pi) = \mathrm{Sw}(\sigma)$ and $\varsigma(\pi) = \varsigma(\sigma)$.

We make a similar modification to the Rankin-Selberg exponent $\mathrm{Ar}(\pi \times \rho)$ defined in the introduction.

Definition 3.1. Let ρ (resp. π) be an irreducible cuspidal representation of $\mathrm{GL}_m(F)$ (resp. $\mathrm{GL}_n(F)$). Let d be the number of unramified characters χ of

$F^\times$ such that $\chi\rho \cong \check{\pi}$. Set

$$\mathrm{Sw}(\pi\times\rho) = \mathrm{Ar}(\pi\times\rho) - mn + d,$$
$$\varsigma(\pi\times\rho) = \mathrm{Sw}(\pi\times\rho)/mn.$$

Note that if, in this definition, we have $d \neq 0$, then $m = n$ and d divides n. As a consequence of the definition and corresponding properties of the Artin exponent, we have:

Fact 3.2. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(F)$.*

(1) *If $\sigma = {}^L\pi \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$, then $\mathrm{Sw}(\pi) = \mathrm{Sw}(\sigma)$ and $\varsigma(\pi) = \varsigma(\sigma)$.*
(2) *If ρ is an irreducible cuspidal representation of $\mathrm{GL}_m(F)$ and ${}^L\rho = \tau$, then $\mathrm{Sw}(\pi\times\rho) = \mathrm{Sw}(\sigma\otimes\tau)$ and $\varsigma(\pi\times\rho) = \varsigma(\sigma\otimes\tau)$.*

3.2. We remark on some upper bounds.

Proposition 3.3. *For $i = 1, 2$, let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(F)$.*

(1) *We have $\varsigma(\pi_1\times\pi_2) \leqslant \max\{\varsigma(\pi_1), \varsigma(\pi_2)\}$, with equality in the case $\varsigma(\pi_1) \neq \varsigma(\pi_2)$.*
(2) *We have $\eta(\pi_1\times\pi_2) \leqslant \max\{\eta(\pi_1), \eta(\pi_2)\}$, with equality in the case $\eta(\pi_1) \neq \eta(\pi_2)$.*

Proof. This follows from Lemma 2.4 via the Langlands correspondence. □

3.3. We consider the more substantial problem of lower bounds.

Proposition 3.4. *For $i = 1, 2$, let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(F)$. If π_1 is ς-minimal then $\varsigma(\pi_1\times\pi_2) \geqslant \frac{1}{2}\max\{\varsigma(\pi_1), \varsigma(\pi_2)\}$.*

Proof. It is easier to work with $\check{\pi}_2$ in place of π_2. By [5, 5.4 Corollary],

$$\varsigma(\pi_1\times\check{\pi}_1) \leqslant \max\{\varsigma(\pi_1\times\check{\pi}_2), \varsigma(\pi_2\times\check{\pi}_1)\}.$$

Since $\varsigma(\pi_1\times\check{\pi}_2) = \varsigma(\pi_2\times\check{\pi}_1)$, we get $\varsigma(\pi_1\times\check{\pi}_1) \leqslant \varsigma(\pi_1\times\check{\pi}_2)$. We are therefore reduced to treating the following special case.

Lemma 3.5. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(F)$, for some $n \geqslant 1$. If π is ς-minimal, then $\varsigma(\pi\times\check{\pi}) \geqslant \varsigma(\pi)/2$.*

Proof. We first show that if $\varsigma(\pi) = 0$ then also $\varsigma(\pi\times\check{\pi}) = 0$. In Definition 3.1, we have $m = n$ while $n = d$ by [7, (6.2.5)], so the assertion follows from [6, 6.5 Theorem (i)]. We therefore assume $\varsigma(\pi) > 0$. Thus π contains a simple character θ, attached to a simple stratum $[\mathfrak{a}, m, 0, \beta]$ in the matrix algebra $\mathrm{M_n(F)}$. The algebra $F[\beta]$ is a field, of degree d_β, say, over F and ramification index e_β. Indeed, θ is "m-simple" (cf. [4], especially Corollary 1), so e_β equals the F-period of the hereditary order $\mathfrak{a}$ and $\varsigma(\pi) = m/e_\beta$. The element β determines a certain non-negative integer $\mathfrak{c}(\beta)$, as in [6, Section 6.4], such that $\varsigma(\pi\times\check{\pi}) = \mathfrak{c}(\beta)/d_\beta^2$ ([6, 6.5 Theorem (i)]).

We next choose a simple stratum $[\mathfrak{a}, m, m-1, \alpha]$ equivalent to $[\mathfrak{a}, m, m-1, \beta]$. Let the field extension $F[\alpha]/F$ have degree d_α and ramification index e_α. It follows from [2, 3.1 Proposition] that $\mathfrak{c}(\alpha)/d_\alpha^2 \leqslant \mathfrak{c}(\beta)/d_\beta^2$. The element α is minimal over F (in the sense of [7, 1.4.14] and, since π is ς-minimal, $d_\alpha > 1$. To calculate $\mathfrak{c}(\alpha)$, we take a simple stratum $[\mathfrak{a}', m', 0, \alpha]$ in the matrix algebra $\mathrm{End}_F(F[\alpha]) \cong \mathrm{M}_{\mathrm{d}_\alpha}(\mathrm{F})$. The integer m' is me_α/e_β (cf. [7, (1.2.4)]) and, by [5, 4.1 Proposition], $\mathfrak{c}(\alpha) = m' d_\alpha(d_\alpha - 1))/e_\alpha$. Therefore,

$$\varsigma(\pi \times \check{\pi}) \geqslant \mathfrak{c}(\alpha)/d_\alpha^2 = (1 - d_\alpha^{-1})\, m/e_\beta \geqslant m/2e_\beta = \tfrac{1}{2}\,\varsigma(\pi),$$

as required. □

This completes the proof of the proposition. □

Example 3.6. Let π be an irreducible, cuspidal representation of $\mathrm{GL}_2(F)$. Suppose that π is ς-minimal and $\varsigma(\pi) > 0$. In the proof of the last lemma, we get $\alpha = \beta$ and $d_\alpha = 2$. This implies $\varsigma(\pi \times \check{\pi}) = \frac{1}{2}\varsigma(\pi)$. The constant $\frac{1}{2}$ in the proposition is therefore best possible as applied to arbitrary representations.

Example 3.7. One can improve the constant by restricting the class of representations under consideration. For example, if $\ell \geqslant 3$ is a prime number and if ρ is an irreducible, ς-minimal, cuspidal representation of $\mathrm{GL}_\ell(F)$ with $\varsigma(\rho) > 0$, the same argument gives $\varsigma(\rho \times \check{\rho}) = (1 - \ell^{-1})\varsigma(\rho) > \frac{1}{2}\varsigma(\rho)$.

We translate in terms of Artin exponents. Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(F)$. If π is η-minimal, it is then ς-minimal. (The converse does not hold: the case $n = 1$ and $\mathrm{Ar}(\pi) = 1$ provides an example).

Corollary 3.8. *For $i = 1, 2$, let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(F)$. If π_1 is η-minimal then $\eta(\pi_1 \times \pi_2) \geqslant \frac{1}{2} \max\{\eta(\pi_1), \eta(\pi_2)\}$.*

Proof. If either π_i is an unramified character of $F^\times$, there is nothing to prove so we assume otherwise. Suppose next that π_2 is not an unramified twist of $\check{\pi}_1$. Thus

$$\begin{aligned}\eta(\pi_1 \times \pi_2) = \varsigma(\pi_1 \times \pi_2) + 1 &\geqslant \tfrac{1}{2} \max\{\varsigma(\pi_1), \varsigma(\pi_2)\} + 1 \\ &= \tfrac{1}{2} \max\{\eta(\pi_1), \eta(\pi_2)\} + \tfrac{1}{2}.\end{aligned}$$

Finally, suppose π_2 is an unramified twist of $\check{\pi}_1$. Thus $\varsigma(\pi_1 \times \pi_2) = \varsigma(\pi_1 \times \check{\pi}_1)$. The lemma then gives $\varsigma(\pi_1 \times \pi_2) \geqslant \frac{1}{2}\varsigma(\pi_1) = \frac{1}{2}\varsigma(\pi_2)$. In this case, $n_1 = n_2$ and, since π_1 is not an unramified character of $F^\times$, we have $n_1 > 1$. If $d(\pi_1)$ is the number of unramified characters χ for which $\chi\pi_1 \cong \pi_1$, we have $d(\pi_1)/n_1^2 \leqslant 1/n_1 \leqslant \frac{1}{2}$. So,

$$\begin{aligned}\eta(\pi_1 \times \pi_2) &= \varsigma(\pi_1 \times \pi_2) + 1 - d(\pi_1)/n_1^2 \\ &\geqslant \varsigma(\pi_1 \times \pi_2) + \tfrac{1}{2} \\ &\geqslant \tfrac{1}{2} \max\{\eta(\pi_1), \eta(\pi_2)\},\end{aligned}$$

as required. □

Example 3.9. In Example 3.6, we may choose π so that $d(\pi) = 2$. We then get $\eta(\pi \times \check{\pi}) = \frac{1}{2}\eta(\pi)$. The constant $\frac{1}{2}$ is thus best possible for Artin exponents as well.

4. **First lower bound**

We prove Theorem A, then deal with Theorem AS at the end of the section.

4.1. We make a simple reduction.

Proposition 4.1. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ be η-minimal. If the inequality*

$$\eta(\sigma \otimes \tau) \geqslant \tfrac{1}{2} \max\{\eta(\sigma), \eta(\tau)\}$$

holds when $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable, then it holds for all $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$.

Proof. Let $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ and write $\tau = \bigoplus_{j\in J} \tau_j$, where each τ_j is an indecomposable element of $\widehat{\mathcal{W}}_F^{\mathcal{D}}$. Put $\alpha_j = \dim \tau_j / \dim \tau$, so that $\sum_{j\in J} \alpha_j = 1$. By hypothesis, $2\eta(\sigma \otimes \tau_j) \geqslant \eta(\sigma)$, so

$$2\eta(\sigma \otimes \tau) = \sum_{j\in J} 2\alpha_j \eta(\sigma \otimes \tau_j) \geqslant \sum_{j\in J} \alpha_j \eta(\sigma) = \eta(\sigma).$$

The hypothesis also gives $2\eta(\sigma \otimes \tau_j) \geqslant \eta(\tau_j)$ so

$$2\eta(\sigma \otimes \tau) \geqslant \sum_{j\in J} \alpha_j \eta(\tau_j) = \eta(\tau),$$

as required. □

We therefore have to prove:

Theorem 4.2. *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is η-minimal and $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable, then*

$$\eta(\sigma \otimes \tau) \geqslant \tfrac{1}{2} \max\{\eta(\sigma), \eta(\tau)\}.$$

This will take us to the end of Section 4.4.

4.2. Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$. Say that σ is *η-homogeneous* if there exists $a \in \mathbb{R}$ such that $\eta(\tau) = a$, for every irreducible factor τ of σ on $\mathcal{W}_F$. When this holds, we write $\ell_0(\sigma) = a$.

Example 4.3. If τ is irreducible and $\sigma = \mathrm{Sp}_r(\tau)$, then σ is η-homogeneous with $\ell_0(\sigma) = \eta(\tau)$. Moreover, $\ell_0(\sigma) = \eta(\sigma)$ if $\ell_0(\sigma) \neq 0$. If $\ell_0(\sigma) = 0$, then τ is an unramified character and $\eta(\sigma) = 1 - r^{-1}$.

Proposition 4.4. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ be η-homogeneous and write $\sigma = \bigoplus_{j\in J} \mathrm{Sp}_{r_j}(\tau_j)$, where the τ_j are irreducible and $r_j \geqslant 1$.*

(1) *The representation σ is η-minimal if and only if each τ_j is η-minimal.*

(2) *If σ is η-minimal, then*

$$\eta(\chi \otimes \sigma) = \max\{\eta(\sigma), \eta(\chi)\},$$

for all characters χ of $\mathcal{W}_F$.

Proof. Set $a = \ell_0(\sigma)$. If $a = 0$, then σ is η-minimal, each $\mathrm{Sp}_{r_j}(\tau_j)$ is η-minimal and (2) is implied by Fact 2.2. We assume henceforth that $a > 0$.

Take first the case where σ is *indecomposable,* say $\sigma = \mathrm{Sp}_r(\tau)$. If $\dim\tau = 1$, it is clear that σ is η-minimal if and only if $\eta(\tau) = 0$, contrary to hypothesis. Therefore $\dim\tau > 1$, so $a = \eta(\sigma) = \eta(\tau)$. Also, if χ is a character of $\mathcal{W}_F$, then $\eta(\chi\otimes\sigma) = \eta(\chi\otimes\tau)$. In particular, $\eta(\chi\otimes\sigma) \geqslant \eta(\sigma)$ if and only if $\eta(\chi\otimes\tau) \geqslant \eta(\tau)$ and (1) follows. For (2), suppose τ to be η-minimal and set $\eta(\chi) = c$. If $c > a$, then $\eta(\chi \otimes \tau) = c > a = \eta(\sigma)$, using Lemma 2.4. If, however, $c < a$, we get $\eta(\chi \otimes \sigma) = a = \eta(\sigma)$. Suppose finally that $c = a$. As τ is η-minimal, $\eta(\chi \otimes \tau) = \eta(\tau)$ and (2) is done in this case. We have also shown that, if σ is *not* minimal, there exists χ with $\eta(\chi) = a$ and $\eta(\chi \otimes \sigma) < \eta(\sigma)$.

For the general case, we set $\sigma = \bigoplus_{j\in J}\sigma_j$, where σ_j is indecomposable. Put $\alpha_j = \dim\sigma_j/\dim\sigma$, so that

$$\sum_j \alpha_j = 1 \quad \text{and} \quad \eta(\chi\otimes\sigma) = \sum_j \alpha_j\eta(\chi\otimes\sigma_j).$$

Suppose σ_j is not η-minimal, for some $j \in J$. We have just shown that there exists χ with $\eta(\chi) = a$ and $\eta(\chi\otimes\sigma_j) < \eta(\sigma_j)$. On the other hand, $\eta(\chi\otimes\sigma_k) \leqslant \eta(\sigma_k)$ for $k \neq j$, so

$$\eta(\chi\otimes\sigma) = \sum_{i\in J}\alpha_i\eta(\chi\otimes\sigma_i) < \sum_{i\in J}\alpha_i\eta(\sigma_j) = \eta(\sigma),$$

whence σ is not η-minimal.

Assume, therefore, that every σ_j is η-minimal. Let $c = \eta(\chi)$. If $c \geqslant a$, the discussion of the indecomposable case gives $\eta(\chi \otimes \sigma_j) = c$, $j \in J$, so $\eta(\chi\otimes\sigma) = c \geqslant \eta(\sigma)$. If, however, $c < a$, we get $\eta(\chi\otimes\sigma) = a = \eta(\sigma)$. Thus σ is η-minimal and we have also proven (2). $\square$

4.3. We use Proposition 4.4 to prove a special case of Theorem 4.2.

Proposition 4.5. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ be η-minimal and η-homogeneous. If $\tau \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ is irreducible, then*

$$\eta(\sigma\otimes\tau) \geqslant \tfrac{1}{2}\max\{\eta(\sigma), \eta(\tau)\}.$$

Proof. Combining Proposition 4.4 with the decomposition technique used in Section 4.1, we reduce to the case where σ is indecomposable and η-minimal. Let $a = \ell_0(\sigma)$.

Consider first the case where $a = 0$, that is, $\sigma = \mathrm{Sp}_r(\chi)$ with χ an unramified character of $\mathcal{W}_F$ and $r \geqslant 1$. Thus $\eta(\sigma) = (r-1)/r$. On the other hand,

$\sigma\otimes\tau = \mathrm{Sp}_r(\chi\otimes\tau)$ and so

$$\eta(\sigma\otimes\tau) = \begin{cases} \eta(\tau) & \text{if } \eta(\tau)\neq 0,\\ \eta(\sigma) & \text{if } \eta(\tau)=0. \end{cases}$$

In the first case, we have $\eta(\tau)\geqslant 1>\eta(\sigma)$ while, in the second, $\eta(\tau)\leqslant\eta(\sigma)$. The result therefore holds when $a=0$.

From now on, we assume $a>0$. We write σ as $\mathrm{Sp}_r(\rho)$, where ρ is irreducible and η-minimal. We have $\eta(\rho)=\eta(\sigma)=a$ and $\dim\rho>1$. Assume initially that there is no unramified character χ of $\mathcal{W}_F$ such that $\chi\otimes\rho\cong\check\tau$. This means that no irreducible component of $\rho\otimes\tau$ is unramified, so $\mathrm{Ar}(\sigma\otimes\tau)=r\mathrm{Ar}(\rho\otimes\tau)$ and

$$2\eta(\sigma\otimes\tau) = 2\eta(\rho\otimes\tau)\geqslant \max\{\eta(\rho),\eta(\tau)\},$$

by Corollary 3.8. Since $\eta(\sigma)=\eta(\rho)$, we are done in this case.

For the remaining case, we may assume $\check\tau\cong\rho$: in particular, $\eta(\tau)=a$. Let d be the number of unramified characters χ for which $\chi\otimes\rho\cong\rho$. Thus d divides $m=\dim\rho>1$. To estimate $\mathrm{Ar}(\sigma\otimes\tau)=\mathrm{Ar}(\mathrm{Sp}_r(1)\otimes\rho\otimes\check\rho)$, we write

$$\rho\otimes\check\rho = \rho'\oplus\chi_1\oplus\cdots\oplus\chi_d,$$

where the χ_i are unramified characters of $\mathcal{W}_F$ and every irreducible component of ρ' has strictly positive exponent. Thus

$$\eta(\rho\otimes\check\rho) = (1-d/m^2)\eta(\rho').$$

Also, $\eta(\rho\otimes\check\rho)\geqslant\frac12\eta(\rho)$ by Corollary 3.8. Taking this into account, we have

$$\begin{aligned}\eta(\sigma\otimes\tau) &= \eta\big((\mathrm{Sp}_r(1)\otimes\rho')\oplus\sum_{i=1}^d \mathrm{Sp}_r(\chi_i)\big)\\ &= (1-d/m^2)\,\eta(\mathrm{Sp}_r(1)\otimes\rho') + d(r-1)/rm^2\\ &= (1-d/m^2)\,\eta(\rho') + d(r-1)/rm^2\\ &\geqslant (1-d/m^2)\,\eta(\rho')\\ &\geqslant \tfrac12\eta(\rho) = \tfrac12\eta(\tau).\end{aligned}$$

Since, in this case, $\eta(\rho)=\eta(\sigma)$ the proof is complete. □

We may now deal with Theorem 4.2 in the case where σ is η-homogeneous.

Corollary 4.6. *Let $\sigma\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ be η-minimal and η-homogeneous. If $\tau\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable, then $\eta(\sigma\otimes\tau)\geqslant\frac12\max\{\eta(\sigma),\eta(\tau)\}$.*

Proof. As in the proof of the proposition, it is enough to treat the case where σ is indecomposable and η-minimal. Thus $\sigma=\mathrm{Sp}_r(\sigma')$ and $\tau=\mathrm{Sp}_s(\tau')$, for integers $r,s\geqslant 1$ and irreducible representations σ', τ'. We have

$$\sigma\otimes\tau = \big(\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(1)\otimes\sigma'\big)\otimes\tau'.$$

The representation $\sigma'' = \mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(1)\otimes\sigma'$ is η-minimal and η-homogeneous with $\ell_0(\sigma'') = \ell_0(\sigma) = \eta(\sigma')$, so the proposition gives

$$\eta(\sigma\otimes\tau) = \eta(\sigma''\otimes\tau') \geqslant \tfrac{1}{2}\max\{\eta(\sigma''),\eta(\tau')\}.$$

It is therefore enough to show that

$$(*)\qquad \max\{\eta(\sigma''),\eta(\tau')\} \geqslant \max\{\eta(\sigma),\eta(\tau)\}.$$

To do this, we write the tensor product $\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(1)$ as a sum of indecomposable representations: there are unramified characters χ_i and positive integers r_i, $1\leqslant i\leqslant l$, such that

$$\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(1) = \bigoplus_{i=1}^{l}\mathrm{Sp}_{r_i}(\chi_i).$$

We have $\sum_{i=1}^{l} r_i = rs$ and, by Proposition 2.5, $l = \min\{r,s\}$. Accordingly,

$$\sigma'' = \bigoplus_{i=1}^{l}\mathrm{Sp}_{r_i}(\chi_i\otimes\sigma').$$

If σ' is not unramified, then $\eta(\sigma'') = \eta(\sigma') = \eta(\sigma)$. Likewise, if τ' is not unramified then $\eta(\tau') = \eta(\tau)$. So, if neither σ' nor τ' is unramified, we get $\max\{\eta(\sigma''),\eta(\tau')\} = \max\{\eta(\sigma),\eta(\tau)\}$, proving $(*)$, whence the proposition, in this case.

Suppose next that σ' is unramified. By Proposition 2.5, we have

$$\eta(\sigma'') = \sum_i (r_i{-}1)/rs = 1 - \tfrac{l}{rs},$$

while $\eta(\sigma) = (r{-}1)/r \leqslant \eta(\sigma'')$. If τ' is not unramified, then $\eta(\tau') = \eta(\tau)$ and we are done. If τ' is unramified, then $\eta(\tau') = 0$ and $\eta(\tau) = (s{-}1)/s \leqslant \eta(\sigma'')$, since $l\leqslant r$. This proves $(*)$ and the proposition in this case.

There remains the case of σ' not unramified, τ' unramified. Following the reduction at the beginning of the proof, the hypotheses apply equally to the pair (τ,σ) in place of (σ,τ), so there is nothing more to do. □

4.4. We enter the final stage of the proof of Theorem 4.2. We proceed in two steps.

Proposition 4.7. *If $\sigma\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ is η-minimal then $\eta(\sigma\otimes\tau)\geqslant\frac{1}{2}\eta(\sigma)$, for all indecomposable $\tau\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$.*

Proof. Write $\sigma = \xi_1\oplus\xi_2\oplus\cdots\oplus\xi_r$, where the ξ_i are indecomposable. Setting $\dim\xi_i/\dim\sigma = \alpha_i$, we have

$$\eta(\sigma\otimes\tau) = \sum_{i=1}^{r}\alpha_i\eta(\xi_i\otimes\tau).$$

There is a character χ such that $\chi\otimes\tau$ is η-minimal. Since τ is indecomposable, $\chi\otimes\tau$ is η-homogeneous and Corollary 3.8 gives

$$\eta(\xi_i\otimes\tau) = \eta\big((\chi^{-1}\otimes\xi_i)\otimes(\chi\otimes\tau)\big) \geqslant \tfrac{1}{2}\,\eta(\chi^{-1}\otimes\xi_i).$$

Consequently,

$$\begin{aligned}\eta(\sigma\otimes\tau) &= \sum_{i=1}^{r}\alpha_i\eta(\xi_i\otimes\tau)\\ &\geqslant \tfrac{1}{2}\sum_{i=1}^{r}\alpha_i\eta(\chi^{-1}\otimes\xi_i)\\ &= \tfrac{1}{2}\eta(\chi^{-1}\otimes\sigma).\end{aligned}$$

As σ is η-minimal, so $\eta(\chi^{-1}\otimes\sigma)\geqslant\eta(\sigma)$ and the result follows. □

For Theorem A, it remains only to prove:

Proposition 4.8. *If $\sigma\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ is η-minimal and $\tau\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ is indecomposable, then $\eta(\sigma\otimes\tau)\geqslant\frac{1}{2}\,\eta(\tau)$.*

Proof. The representation τ is η-homogeneous. If τ is η-minimal, the result follows from Corollary 4.6 and Proposition 4.1. We therefore assume τ is not η-minimal.

To proceed further, we need to extend 3.2 Proposition. Write

$$\sigma = \bigoplus_{i\in I}\mathrm{Sp}_{r_i}(\xi_i),$$

for irreducible representations ξ_i and integers $r_i\geqslant 1$. Let $c=\max\eta(\xi_i)$. If $c=0$, all ξ_i are unramified and the proposition follows from Corollary 4.6. We assume henceforward that $c\geqslant 1$.

Define $\sigma_{\max}$ as the sum of all factors $\mathrm{Sp}_{r_i}(\xi_i)$ for which $\eta(\xi_i)=c$, and σ' as the sum of the others.

Lemma 4.9. *The representation σ' is either zero or η-minimal.*

Proof. Assume $\sigma'\neq 0$. Let $d_{\max}=\dim\sigma_{\max}$ and $d'=\dim\sigma'$. Set $d=d_{\max}+d'=\dim\sigma$. Let ϕ be a character of $\mathcal{W}_F$ and write $s=\eta(\phi)$. We compare the expressions

$$\begin{aligned}d\eta(\sigma) &= d_{\max}\eta(\sigma_{\max})+d'\eta(\sigma'),\\ d\eta(\phi\otimes\sigma) &= d_{\max}\eta(\phi\otimes\sigma_{\max})+d'\eta(\phi\otimes\sigma').\end{aligned}$$

Suppose that $s<c$. Every irreducible component ξ of $\sigma_{\max}$ has $\eta(\xi)=c\geqslant 1$. By Lemma 2.4, $\eta(\phi\otimes\xi)=c$, whence $\eta(\phi\otimes\sigma_{\max})=c=\eta(\sigma_{\max})$. As $\eta(\phi\otimes\sigma)\geqslant\eta(\sigma)$, so $\eta(\phi\otimes\sigma')\geqslant\eta(\sigma')$.

If, on the other hand, $s\geqslant c$, a similar argument gives $\eta(\phi\otimes\sigma')=s\geqslant c$. We show that $c>\eta(\sigma')$. To do this, we may assume σ' to be indecomposable, say $\sigma'=\mathrm{Sp}_r(\xi)$ with ξ irreducible. If ξ is not unramified, then $\eta(\sigma')=\eta(\xi)<c$,

by the definition of c. If ξ is unramified, then $\eta(\sigma') = 1-\frac{1}{r} < 1 \leqslant c$. Overall, $\eta(\phi \otimes \sigma') = s \geqslant c > \eta(\sigma')$ in this case also. Thus $\eta(\phi \otimes \sigma') \geqslant \eta(\sigma')$ for all ϕ, as required. $\square$

If the representation σ' is zero, then $\sigma = \sigma_{\max}$ is η-homogeneous and η-minimal. The proposition in this case is given by Corollary 3.8. We assume therefore that $\sigma' \neq 0$. Certainly $\sigma_{\max} \neq 0$ so, using induction on the Jordan-Hölder length of σ, we may assume

$$\eta(\sigma' \otimes \tau) \geqslant \tfrac{1}{2}\, \eta(\tau).$$

The indecomposable representation τ is not η-minimal (by assumption), and so is not unramified on $\mathcal{W}_F$. If we write $s = \eta(\tau)$, then $s \geqslant 1$. Let χ be a character such that $\chi \otimes \tau$ is η-minimal. It follows from Lemma 2.4 that $\eta(\chi) = s$.

Examining cases, suppose first that $s > c$. We show that $\eta(\sigma \otimes \tau) = s > s/2$, implying the result in this situation. Let ξ (resp. θ) be an irreducible composition factor of σ (resp. τ). Thus $\varsigma(\theta) = s-1 > 0$ and $\varsigma(\xi) = c-1$. By Fact 2.3, an irreducible component μ of $\xi \otimes \theta$ satisfies $\varsigma(\mu) = s-1$. Thus $\eta(\mu) = s = \eta(\sigma \otimes \tau)$, as asserted.

If, on the other hand, $s < c$, the same argument gives $\eta(\sigma_{\max} \otimes \tau) = c$. By inductive hypothesis, $\eta(\sigma' \otimes \tau) \geqslant s/2$. Writing $\alpha = d_{\max}/d$, $\beta = d'/d$, we have

$$\begin{aligned} \eta(\sigma \otimes \tau) &= \alpha\eta(\sigma_{\max} \otimes \tau) + \beta\eta(\sigma' \otimes \tau) \\ &\geqslant \alpha c + \tfrac{1}{2}\beta s \geqslant \tfrac{1}{2} s. \end{aligned}$$

It remains to treat the case $s = c$. If ξ is an irreducible factor of $\sigma' \otimes \tau$, Fact 2.3 implies $\eta(\xi) = s$. Thus $\eta(\sigma' \otimes \tau) = s = c$ and

$$\eta(\sigma \otimes \tau) = \alpha\eta(\sigma_{\max} \otimes \tau) + \beta s.$$

However, $\eta(\sigma_{\max} \otimes \tau) = \eta\big((\chi^{-1} \otimes \sigma_{\max}) \otimes (\chi \otimes \tau)\big)$ while, by the very first case of this proof, we have

$$\eta\big((\chi^{-1} \otimes \sigma_{\max}) \otimes (\chi \otimes \tau)\big) \geqslant \tfrac{1}{2}\, \eta(\chi^{-1} \otimes \sigma_{\max}).$$

If ξ is an irreducible factor of σ', then $\eta(\xi) < s = c$ and $\eta(\chi^{-1} \otimes \xi) = s$. Since σ is η-minimal,

$$\begin{aligned} \eta(\chi^{-1} \otimes \sigma) &= \alpha\eta(\chi^{-1} \otimes \sigma_{\max}) + \beta\eta(\chi^{-1} \otimes \sigma') \\ &= \alpha\eta(\chi^{-1} \otimes \sigma_{\max}) + \beta s \\ &\geqslant \eta(\sigma) = \alpha s + \beta\eta(\sigma'). \end{aligned}$$

That is,

$$\alpha\eta(\chi^{-1} \otimes \sigma_{\max}) \geqslant (\alpha-\beta)s + \beta\eta(\sigma')$$

and, overall,

$$\begin{aligned}\eta(\sigma\otimes\tau) &= \alpha\eta\big((\chi^{-1}\otimes\sigma_{\max})\otimes(\chi\otimes\tau)\big)+\beta s\\ &\geqslant \tfrac12\alpha\eta(\chi^{-1}\otimes\sigma_{\max})+\beta s\\ &\geqslant \tfrac12(\alpha{-}\beta)s+\tfrac12\beta\eta(\sigma')+\beta s\\ &\geqslant \tfrac12(\alpha{+}\beta)s\\ &= \tfrac12 s=\tfrac12\eta(\tau).\end{aligned}$$

That is, $\eta(\sigma\otimes\tau)\geqslant\frac12\eta(\tau)$ as required. □

This completes the proofs of Theorem 4.2 and Theorem A. □

4.5. We digress to highlight a special case. Say that $\sigma\in\widehat{\mathcal W}{}_F^{\mathcal D}$ is *unramified* if its restriction to $\mathcal W_F$ is a sum of unramified characters. Any such σ is both η-minimal and η-homogeneous.

Example 4.10. If $\sigma\in\widehat{\mathcal W}{}_F^{\mathcal D}$ is unramified, then $\eta(\sigma\otimes\tau)\geqslant\max\{\eta(\sigma),\eta(\tau)\}$, for all $\tau\in\widehat{\mathcal W}{}_F^{\mathcal D}$.

To justify this, one applies the argument of Section 3.1 twice to reduce to the case where both σ and τ are indecomposable. The proof of Proposition 4.5 gives the result when τ is irreducible. In the proof of Corollary 4.6, we still get $\eta(\sigma\otimes\tau)=\eta(\sigma''\otimes\tau')$, so $\eta(\sigma\otimes\tau)\geqslant\max\{\eta(\sigma''),\eta(\tau')\}$. In the same proof, we have shown that $\max\{\eta(\sigma''),\eta(\tau')\}\geqslant\max\{\eta(\sigma),\eta(\tau)\}$, whence the assertion.

Starting again from the first case of the proof of Proposition 4.5, one may equally conclude:

Example 4.11. Let $\sigma\in\widehat{\mathcal W}{}_F^{\mathcal D}$ be unramified. If $\tau\in\widehat{\mathcal W}{}_F^{\mathcal D}$ has no unramified direct factor, then $\eta(\sigma\otimes\tau)=\eta(\tau)$.

For a tensor product of unramified representations, one may derive an explicit formula from Lemma 2.6.

4.6. We prove Theorem AS: if $\sigma,\tau\in\widehat{\mathcal W}{}_F^{\mathrm{ss}}$ and if σ is ς-minimal, then $\varsigma(\sigma\otimes\tau)\geqslant\frac12\max\{\varsigma(\sigma),\varsigma(\tau)\}$.

If σ and τ are irreducible, this follows from Proposition 3.4. An argument identical to Proposition 4.1 shows it is enough to prove the theorem under the additional hypothesis that τ is irreducible.

Say that $\sigma\in\widehat{\mathcal W}{}_F^{\mathrm{ss}}$ is ς-homogeneous if, for some a, we have $\varsigma(\sigma')=a$ for all irreducible components σ' of σ. With this definition, the analogue of Proposition 4.4 holds with the same proof. In light of the case already done, where σ and τ are irreducible, the analogue of Proposition 4.5 is immediate here and the Corollary is redundant. The propositions of Section 4.4 hold, with identical proofs, and the theorem is proved. □

5. **Symmetric lower bound**

We prove Theorem B and deal with Theorem BS at the end of the section. We first accumulate some preliminary results concerning irreducible or indecomposable representations.

5.1. We start with what amounts to a special case of the theorem.

Proposition 5.1. *If σ, τ are irreducible representations of $\mathcal{W}_F$, then*

$$\varsigma(\sigma\otimes\check\tau) \geqslant \max\{\varsigma(\sigma\otimes\check\sigma), \varsigma(\tau\otimes\check\tau)\}, \quad \textit{and}$$
$$\eta(\sigma\otimes\check\tau) \geqslant \max\{\eta(\sigma\otimes\check\sigma), \eta(\tau\otimes\check\tau)\}.$$

Proof. For the first assertion, we follow [12] but use the notation and layout of [5, Sections 2.5, 3.1] . The set $\widehat{\mathcal{W}}_F^{\mathrm{irr}}$ carries a canonical pairing Δ with non-negative real values [5, (2.5.3)]. It has the property $\Delta(\sigma,\sigma) \leqslant \Delta(\sigma,\tau)$, for all $\tau\in\widehat{\mathcal{W}}_F^{\mathrm{irr}}$. As in [5, Section 3.1], there is a continuous, strictly increasing function $\varSigma_\sigma$ such that $\varSigma_\sigma(\Delta(\sigma,\tau)) = \varsigma(\sigma\otimes\check\tau)$. Therefore $\varsigma(\sigma\otimes\check\sigma)\leqslant\varsigma(\sigma\otimes\check\tau)$, as desired.

In the second assertion, suppose first that $\sigma\not\cong\chi\otimes\tau$, for any unramified character χ of $\mathcal{W}_F$. It follows that $\eta(\sigma\otimes\check\tau) = \varsigma(\sigma\otimes\check\tau){+}1$. The first assertion then gives $\eta(\sigma\otimes\check\tau) \geqslant \max\{\varsigma(\sigma\otimes\check\sigma){+}1, \varsigma(\tau\otimes\check\tau){+}1\}$. However,

$$\eta(\sigma\otimes\check\sigma) = \varsigma(\sigma\otimes\check\sigma) + 1 - d_\sigma/m^2,$$

where d_σ is the number of unramified characters χ such that $\chi\otimes\sigma\cong\sigma$ and $m=\dim\sigma$. Likewise for τ, and the result follows.

If, on the other hand, there is an unramified character ϕ such that $\tau\cong\phi\otimes\sigma$, we get $\eta(\sigma\otimes\check\tau) = \eta(\sigma\otimes\check\sigma) = \eta(\tau\otimes\check\tau)$, and there is nothing to do. ☐

5.2. The exponent has a striking ultrametric property.

Proposition 5.2. *If $\sigma,\tau,\rho\in\widehat{\mathcal{W}}_F^{\mathrm{irr}}$, then*

$$\varsigma(\sigma\otimes\check\tau) \leqslant \max\{\varsigma(\sigma\otimes\check\rho), \varsigma(\rho\otimes\check\tau)\},$$
$$\eta(\sigma\otimes\check\tau) \leqslant \max\{\eta(\sigma\otimes\check\rho), \eta(\rho\otimes\check\tau)\}.$$

Proof. The first inequality is [5, 3.1 Corollary]. To deduce the second, let $d_{\sigma\tau}$ be the number of unramified characters χ for which $\tau\cong\chi\otimes\sigma$, and similarly for the other pairs. Let $m=\dim\sigma$, $n=\dim\tau$ and $l=\dim\rho$. Thus

$$\eta(\sigma\otimes\check\tau) = \varsigma(\sigma\otimes\check\tau){+}1 - d_{\sigma\tau}/mn,$$

and similarly for the others. The first part of the proposition yields

$$\eta(\sigma\otimes\check\tau) \leqslant \max\{\eta(\sigma\otimes\check\rho) + d_{\sigma\rho}/ml, \eta(\rho\otimes\check\tau) + d_{\rho\tau}/nl\} - d_{\sigma\tau}/mn.$$

This gives the result if $d_{\sigma\tau} = d_{\sigma\rho} = d_{\rho\tau} = 0$. If $d_{\sigma\tau}\neq 0$, then $m=n$ and $d_{\sigma\rho} = d_{\rho\tau}$. Also, $\eta(\sigma\otimes\check\tau) = \eta(\sigma\otimes\check\sigma)$ and $\eta(\rho\otimes\check\tau) = \eta(\rho\otimes\check\sigma)$. The desired inequality thus reduces to $\eta(\sigma\otimes\check\sigma)\leqslant\eta(\sigma\otimes\check\rho)$, which follows from

Proposition 4.1. Similarly, if $d_{\sigma\rho} \neq 0$, we have to check that $\eta(\sigma\otimes\check\tau) \leqslant \max\{\eta(\sigma\otimes\check\sigma), \eta(\sigma\otimes\check\tau)\}$, and this is immediate. $\square$

5.3. We generalize Propositions 5.1 and 5.2 to indecomposable representations. To do this, we need some explicit formulas.

Let σ, τ be irreducible representations of $\mathcal{W}_F$ of dimension m, n respectively. Let d_σ be the number of unramified characters χ such that $\sigma \cong \chi\otimes\sigma$. Define d_τ similarly, and let $d_{\sigma\tau}$ be the number of unramified characters χ such that $\sigma\cong\chi\otimes\tau$. Let $r \geqslant s \geqslant 1$ be integers, and set $\Sigma = \mathrm{Sp}_r(\sigma)$, $\mathsf{T} = \mathrm{Sp}_s(\tau)$.

Lemma 5.3. *With the notation above, we have*

$$\begin{aligned}\eta(\Sigma\otimes\check{\mathsf{T}}) &= \eta(\sigma\otimes\check\tau) + d_{\sigma\tau}(1-r^{-1})/mn,\\ \eta(\Sigma\otimes\check\Sigma) &= \eta(\sigma\otimes\check\sigma) + d_\sigma(1-r^{-1})/m^2,\\ \eta(\mathsf{T}\otimes\check{\mathsf{T}}) &= \eta(\tau\otimes\check\tau) + d_\tau(1-s^{-1})/n^2.\end{aligned}$$

Proof. The second and third relations are instances of the first, so we need only prove that one.

We write $\sigma\otimes\check\tau = \rho\oplus\chi_1\oplus\chi_2\oplus\cdots\oplus\chi_d$, where every component of ρ is not unramified and χ_j is an unramified character, $1\leqslant j\leqslant d = d_{\sigma\tau}$. Thus

$$\eta(\sigma\otimes\check\tau) = (mn{-}d)\eta(\rho)/mn.$$

We also have $\check{\mathsf{T}} = \mathrm{Sp}_s(\zeta)$, for some unramified character ζ, so

$$\Sigma\otimes\check{\mathsf{T}} = \mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\sigma\otimes\check\tau.$$

Set $\mathsf{R} = \mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\rho$, so that $\eta(\mathsf{R}) = \eta(\rho)$. Expanding, we get

$$\eta(\Sigma\otimes\check{\mathsf{T}}) = (mn{-}d)\eta(\rho)/mn + d\eta(\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\varSigma_j\chi_j)/mn.$$

Recalling that $r\geqslant s$, Proposition 2.5 gives

$$\eta(\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\phi) = 1{-}r^{-1},$$

for any unramified character ϕ. Therefore,

$$\eta(\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\varSigma_j\chi_j) = d^{-1}\sum_j \eta(\mathrm{Sp}_r(1)\otimes\mathrm{Sp}_s(\zeta)\otimes\chi_j) = 1{-}r^{-1}$$

and, altogether,

$$\begin{aligned}\eta(\Sigma\otimes\check{\mathsf{T}}) &= (mn{-}d)\eta(\rho)/mn + d(1{-}r^{-1})/mn\\ &= \eta(\sigma\otimes\check\tau) + d(1{-}r^{-1})/mn,\end{aligned}$$

as required. $\square$

Proposition 5.4. *If $\Sigma, \mathsf{T}\in\widehat{\mathcal{W}}_F^{\mathcal{D}}$ are indecomposable, then*

$$\eta(\Sigma\otimes\check{\mathsf{T}}) \geqslant \max\{\eta(\Sigma\otimes\check\Sigma), \eta(\mathsf{T}\otimes\check{\mathsf{T}})\}.$$

Proof. We write $\Sigma = \mathrm{Sp}_r(\sigma)$ and $\mathsf{T} = \mathrm{Sp}_s(\tau)$, for $\sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$. Using the notation of the lemma, suppose first that $d_{\sigma\tau} = 0$. Using Proposition 5.1 and the formulas from the lemma, we get

$$\begin{aligned}\eta(\Sigma \otimes \check{\mathsf{T}}) = \eta(\sigma \otimes \check{\tau}) = \varsigma(\sigma \otimes \check{\tau}) + 1 \\ \geqslant \varsigma(\sigma \otimes \check{\sigma}) + 1 = \eta(\sigma \otimes \check{\sigma}) + d_\sigma/m^2 \geqslant \eta(\Sigma \otimes \check{\Sigma}),\end{aligned}$$

and likewise $\eta(\Sigma \otimes \check{\mathsf{T}}) \geqslant \eta(\mathsf{T} \otimes \check{\mathsf{T}})$.

Suppose therefore that $d_{\sigma\tau} \neq 0$. There is then an unramified character χ for which $\tau \cong \chi \otimes \sigma$. In particular, $m = n$ and $d_{\sigma\tau} = d_\sigma = d_\tau = d$, say. From the formulas above, we get $\eta(\Sigma \otimes \check{\mathsf{T}}) = \eta(\Sigma \otimes \check{\Sigma}) \geqslant \eta(\mathsf{T} \otimes \check{\mathsf{T}})$, which is enough. □

Proposition 5.5. *If $\Sigma, \mathsf{T}, \mathsf{R} \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ are indecomposable then*

$$\eta(\Sigma \otimes \check{\mathsf{T}}) \leqslant \max\{\eta(\Sigma \otimes \check{\mathsf{R}}), \eta(\mathsf{R} \otimes \check{\mathsf{T}})\}.$$

Proof. There are representations $\sigma, \tau, \rho \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$ and integers r, s, t such that $\Sigma = \mathrm{Sp}_r(\sigma)$, $\mathsf{T} = \mathrm{Sp}_s(\tau)$ and $\mathsf{R} = \mathrm{Sp}_t(\rho)$. Let $\dim\sigma = m$, $\dim\tau = n$ and $\dim\rho = l$. Define integers $d_{\sigma\tau}$, d_σ etc., as before.

Take first the case where σ is not an unramified twist of τ. That is, $d_{\sigma\tau} = 0$ and $\eta(\Sigma \otimes \check{\mathsf{T}}) = \eta(\sigma \otimes \check{\tau})$. If, for example, $d_{\sigma\rho} \neq 0$ then $l = m$ and

$$\eta(\Sigma \otimes \check{\mathsf{R}}) = \eta(\sigma \otimes \check{\rho}) + d_{\sigma\rho}(1 - q^{-1})/m^2 \geqslant \eta(\sigma \otimes \check{\rho}),$$

where $q = \max\{r, t\}$. If, on the other hand, $d_{\sigma\rho} = 0$, we get the conclusion $\eta(\Sigma \otimes \check{\mathsf{R}}) = \eta(\sigma \otimes \check{\rho})$. Similarly for the pair (ρ, τ), so the desired inequality now follows from Proposition 5.2.

We therefore assume $d_{\sigma\tau} \neq 0$. Our assumption $r \geqslant s$ implies $\eta(\Sigma \otimes \check{\mathsf{T}}) = \eta(\Sigma \otimes \check{\Sigma})$, while $\eta(\Sigma \otimes \check{\Sigma}) \leqslant \eta(\Sigma \otimes \check{\mathsf{R}})$, by Proposition 5.4. □

5.4. We now prove the main statement of Theorem B, that is:

Theorem 5.6. *If $\sigma_1, \sigma_2 \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$, then*

$$\eta(\sigma_1 \otimes \check{\sigma}_2) \geqslant \tfrac{1}{2}\big(\eta(\sigma_1 \otimes \check{\sigma}_1) + \eta(\sigma_2 \otimes \check{\sigma}_2)\big).$$

Proof. We proceed by induction on $r_1 r_2$, where r_i is the number of isomorphism classes of indecomposable direct factors of σ_i. The case $r_1 r_2 = 1$ follows from Proposition 5.4, so we assume $r_1 r_2 \geqslant 2$.

If k is a positive integer, we may replace σ_1 by $k\sigma_1 = \sigma_1 \oplus \sigma_1 \oplus \cdots \oplus \sigma_1$ (k copies) without changing r_1 or the formula to be proved. Likewise for σ_2. We may therefore assume that $\dim\sigma_1 = \dim\sigma_2$. Next, we choose an indecomposable direct factor τ_i of σ_i, $i = 1, 2$, so as to minimize $\eta(\tau_1 \otimes \check{\tau}_2)$.

Lemma 5.7. *Using the preceding notation, there are positive integers k, a, b such that*

$$k\sigma_1 = \rho_1 \oplus a\tau_1, \quad k\sigma_2 = \rho_2 \oplus b\tau_2,$$

where

(a) $\dim\rho_1 = \dim\rho_2$ *and either*
(b) ρ_1 *has no direct factor equivalent to* τ_1 *or*
(c) ρ_2 *has no direct factor equivalent to* τ_2.

Proof. Let m_i be the multiplicity of τ_i in σ_i and write $d_i = \dim\tau_i$. By symmetry, we may assume $d_1m_1 \geqslant d_2m_2$. Thus

$$d_1\sigma_2 = \rho_2 \oplus d_1m_2\tau_2,$$

for a subspace ρ_2 with no factor τ_2. Likewise,

$$d_1\sigma_1 = \rho_1' \oplus d_1m_1\tau_1,$$

for a subspace ρ_1' with no factor τ_1. We have

$$\dim\rho_2 - \dim\rho_1' = d_1^2m_1 - d_1d_2m_2.$$

This integer is divisible by d_1 and is non-negative. So, $d_1\sigma_1$ admits a decomposition $d_1\sigma_1 = \rho_1 \oplus a\tau_1$ in which $\dim\rho_1 = \dim\rho_2$ and a is a positive integer. The result follows with $k = d_1$ and $b = d_1m_2$. □ □

We may replace (σ_1, σ_2) by $(k\sigma_1, k\sigma_2)$ without changing anything. To simplify notation, we assume that Lemma 5.7 holds with $k = 1$. The hypothesis $r_1r_2 > 1$ implies that one of the spaces ρ_i is non-zero, so both are. Extending notation in the obvious way, we have $r(\rho_1)r(\rho_2) < r_1r_2$ so, by inductive hypothesis,

$$2\,\eta(\rho_1 \otimes \check\rho_2) \geqslant \eta(\rho_1 \otimes \check\rho_1) + \eta(\rho_2 \otimes \check\rho_2).$$

Put

$$\alpha = \frac{\dim\rho_1}{\dim\sigma_1} = \frac{\dim\rho_2}{\dim\sigma_2}, \quad \beta = 1-\alpha.$$

Applying the definition of η to the relations $\sigma_1 = \rho_1 \oplus a\tau_1$, $\sigma_2 = \rho_2 \oplus b\tau_2$, we get

$$\begin{aligned}
\eta(\sigma_1 \otimes \check\sigma_2) &= \alpha^2\eta(\rho_1 \otimes \check\rho_2) + \alpha\beta\big(\eta(\rho_1 \otimes \check\tau_2) + \eta(\tau_1 \otimes \check\rho_2)\big) + \beta^2\eta(\tau_1 \otimes \check\tau_2),\\
\eta(\sigma_1 \otimes \check\sigma_1) &= \alpha^2\eta(\rho_1 \otimes \check\rho_1) + 2\alpha\beta\eta(\rho_1 \otimes \check\tau_1) + \beta^2\eta(\tau_1 \otimes \check\tau_1),\\
\eta(\sigma_2 \otimes \check\sigma_2) &= \alpha^2\eta(\rho_2 \otimes \check\rho_2) + 2\alpha\beta\eta(\rho_2 \otimes \check\tau_2) + \beta^2\eta(\tau_2 \otimes \check\tau_2).
\end{aligned}$$

Proposition 5.4 implies that

$$2\,\eta(\tau_1 \otimes \check\tau_2) \geqslant \eta(\tau_1 \otimes \check\tau_1) + \eta(\tau_2 \otimes \check\tau_2).$$

The theorem will therefore follow from:

Lemma 5.8. *With the preceding notation,*

$$\eta(\rho_1 \otimes \check\tau_2) + \eta(\tau_1 \otimes \check\rho_2) \geqslant \eta(\rho_1 \otimes \check\tau_1) + \eta(\tau_2 \otimes \check\rho_2).$$

Proof. Write $\rho_1 = \bigoplus_{i\in I}\xi_i$ and $\rho_2 = \bigoplus_{j\in J}\theta_j$, where ξ_i and θ_j are indecomposable. Thus

$$\eta(\rho_1\otimes\check{\tau}_1) = \sum_{i\in I}\alpha_i\eta(\xi_i\otimes\check{\tau}_1), \quad \alpha_i = \dim\xi_i/\dim\rho_1,$$

and $\sum_{i\in I}\alpha_i = 1$. We have a similar formula for each of the three other terms in the inequality to be proved. Combining these, and writing $\beta_j = \dim\theta_j/\dim\rho_2$, the desired relation reduces to

$$\sum_{i\in I}\alpha_i\eta(\xi_i\otimes\check{\tau}_2) + \sum_{j\in J}\beta_j\eta(\tau_1\otimes\check{\theta}_j) \geqslant \sum_{i\in I}\alpha_i\eta(\xi_i\otimes\check{\tau}_1) + \sum_{j\in J}\beta_j\eta(\tau_2\otimes\check{\theta}_j).$$

We multiply each sum over i by $1 = \sum_j\beta_j$ and each in j by $1 = \sum_i\alpha_i$. Comparing the $\alpha_i\beta_j$-term on either side, we see it is enough to prove that

$$\eta(\xi_i\otimes\check{\tau}_2) + \eta(\tau_1\otimes\check{\theta}_j) \geqslant \eta(\xi_i\otimes\check{\tau}_1) + \eta(\tau_2\otimes\check{\theta}_j), \quad i\in I,\ j\in J.$$

The choice of (τ_1,τ_2) gives

$$\begin{aligned}\eta(\xi_i\otimes\check{\tau}_2) &\geqslant \eta(\tau_1\otimes\check{\tau}_2), \qquad i\in I,\\ \eta(\tau_1\otimes\check{\theta}_j) &\geqslant \eta(\tau_1\otimes\check{\tau}_2), \qquad j\in J.\end{aligned}$$

We now apply Proposition 5.5 to get

$$\begin{aligned}\eta(\xi_i\otimes\check{\tau}_1) &\leqslant \max\big\{\eta(\xi_i\otimes\check{\tau}_2),\eta(\tau_2\otimes\check{\tau}_1)\big\} &&= \eta(\xi_i\otimes\check{\tau}_2),\\ \eta(\tau_2\otimes\check{\theta}_j) &\leqslant \max\big\{\eta(\tau_2\otimes\check{\tau}_1),\eta(\tau_1\otimes\check{\theta}_j)\big\} &&= \eta(\tau_1\otimes\check{\theta}_j),\end{aligned}$$

whence the lemma follows. □

This completes the proof of Theorem 5.6 and the main assertion of Theorem B. The second assertion of Theorem B is Proposition 5.4. □

5.5. To prove Theorem BS, we can pass directly from the end of Section 5.1 to the start of Section 5.4. From there on, the argument is identical: one simply replaces η by ς and "indecomposable" by "irreducible" throughout.

6. **Upper bounds**

We prove Theorems C and CS.

6.1. We use a combinatorial device. Let $A = \mathbb{Z}\big[\mathbb{R}\big]$ be the integral group ring of the additive group of real numbers. We write the elements of A as finite formal sums of symbols $[\alpha]$, $\alpha\in\mathbb{R}$. The ring A comes equipped with two canonical homomorphisms

$$\begin{aligned} d: A &\longrightarrow \mathbb{Z}, \\ [\alpha] &\longmapsto 1, \end{aligned} \qquad \text{and} \qquad \begin{aligned} v: A &\longrightarrow \mathbb{R}, \\ [\alpha] &\longmapsto \alpha. \end{aligned}$$

There is a unique bi-additive map $A\times A\to A$, denoted $(x,y)\mapsto x\vee y$, so that

$$[\alpha]\vee[\beta] = \big[\max\{\alpha,\beta\}\big], \quad \alpha,\beta\in\mathbb{R}.$$

Let A^+ be the set of elements $\sum_\alpha c_\alpha[\alpha]$ such that $c_\alpha = 0$ if $\alpha < 0$ and $c_\alpha \geqslant 0$ otherwise.

Proposition 6.1. *If $\sigma, \tau \in A^+$, then*

$$v(\sigma \vee \tau) \leqslant d(\tau)v(\sigma) + d(\sigma)v(\tau) - \min\{v(\sigma), v(\tau)\}.$$

Proof. If either σ or τ is the zero element of A, the assertion is trivial. We therefore assume that both $\sigma, \tau \in A^+$ are non-zero and proceed by induction on the integer $d = d(\sigma+\tau) \geqslant 2$. In the first case $d = 2$, we have $\sigma = [\alpha]$, $\tau = [\beta]$, for some positive real numbers α, β. The assertion is

$$\max\{\alpha, \beta\} \leqslant \alpha + \beta - \min\{\alpha, \beta\}.$$

This holds with equality. For the general inductive step, we may assume by symmetry that $\sigma = \sigma_1+\sigma_2$, for non-zero elements σ_i of A^+. By inductive hypothesis,

$$v(\sigma_i \vee \tau) \leqslant d(\tau)v(\sigma_i) + d(\sigma_i)v(\tau) - \min\{v(\sigma_i), v(\tau)\}, \quad i = 1, 2.$$

Adding and using the inductive hypothesis, we get

$$\begin{aligned} v(\sigma \vee \tau) &\leqslant v(\sigma_1 \vee \tau) + v(\sigma_2 \vee \tau) \\ &\leqslant d(\tau)v(\sigma) + d(\sigma)v(\tau) - \min\{v(\sigma_1), v(\tau)\} - \min\{v(\sigma_2), v(\tau)\} \\ &\leqslant d(\tau)v(\sigma) + d(\sigma)v(\tau) - \min\{v(\sigma), v(\tau)\}, \end{aligned}$$

as required. □

Remark 6.2. If we fix positive integers d_1, d_2, and real numbers v_1, v_2, there exist $\sigma_1, \sigma_2 \in A^+$ such that $d_i = d(\sigma_i)$, $v_i = v(\sigma_i)$, and

$$v(\sigma_1 \vee \sigma_2) = d_2 v_1 + d_1 v_2 - \min\{v_1, v_2\}.$$

In other words, the inequality of the proposition is optimal.

6.2. We prove Theorem CS. Recall that the assertion of the theorem concerning irreducible representations has been proved in Lemma 2.4.

A representation $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}$ gives an element $\mathbf{S}(\sigma) = \dim(\sigma)[\varsigma(\sigma)]$ of the ring A of 5.1. For $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$, we define $\mathbf{S}(\sigma) \in A^+$ by

$$\mathbf{S}(\sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_r) = \sum_{i=1}^{r} \mathbf{S}(\sigma_i), \quad \sigma_i \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}.$$

This definition gives

$$\left.\begin{aligned} v\big(\mathbf{S}(\sigma)\big) &= \mathrm{Sw}(\sigma), \\ d\big(\mathbf{S}(\sigma)\big) &= \dim\sigma, \end{aligned}\right. \qquad \sigma \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}.$$

We know that $\varsigma(\sigma \otimes \tau) \leqslant \max\{\varsigma(\sigma), \varsigma(\tau)\}$ when both representations σ, τ are irreducible. In our present notation, this says

$$\mathrm{Sw}(\sigma \otimes \tau) \leqslant v\big(\mathbf{S}(\sigma) \vee \mathbf{S}(\tau)\big), \quad \sigma, \tau \in \widehat{\mathcal{W}}_F^{\mathrm{irr}}.$$

Consequently, if $\rho, \theta \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$, then

$$\begin{aligned}\mathrm{Sw}(\rho\otimes\theta) &\leqslant v\big(\mathbf{S}(\rho)\vee\mathbf{S}(\theta)\big)\\ &\leqslant d\big(\mathbf{S}(\theta)\big)v\big(\mathbf{S}(\rho)\big)+d\big(\mathbf{S}(\rho)\big)v\big(\mathbf{S}(\theta)\big)-\min\big\{v\big(\mathbf{S}(\rho)\big),v\big(\mathbf{S}(\theta)\big)\big\}\\ &=\dim(\theta)\mathrm{Sw}(\rho)+\dim(\rho)\mathrm{Sw}(\theta)-\min\{\mathrm{Sw}(\rho),\mathrm{Sw}(\theta)\},\end{aligned}$$

as required to prove Theorem CS. □

6.3. We can use exactly the same argument to prove Theorem C once we establish:

Proposition 6.3. *If* $\mathsf{R}, \mathsf{T} \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ *are indecomposable, then*

$$\eta(\mathsf{R}\otimes\mathsf{T}) \leqslant \max\{\eta(\mathsf{R}),\eta(\mathsf{T})\}.$$

Proof. Let $\Sigma \in \widehat{\mathcal{W}}_F^{\mathcal{D}}$ be indecomposable. Thus

$$\eta(\mathsf{R}\otimes\mathsf{T}) \leqslant \max\{\eta(\mathsf{R}\otimes\Sigma),\eta(\check{\Sigma}\otimes\mathsf{T})\},$$

by Proposition 5.5. Taking for Σ the trivial character of $\mathcal{W}_F$, we get the proposition. □

This completes the proof of Theorem C. □

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