EXISTENCE OF AN ALGEBRA NORM ON CERTAIN SUBALGEBRAS OF C(X)

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Abstract. Let $X$ be a topological space and $A$ be a subalgebra of $C(X)$. We provide conditions on $A$ so that it admits an algebra norm (not necessarily complete). When $X$ is hemicompact and $(A, (p_n))$ is a functionally continuous regular Fréchet function algebra on $X$, we show that $A$ is a Q-algebra and does not contain any unbounded function if it admits an algebra norm. Moreover, any uniform norm on $A$ is continuous. Finally we prove that in some particular cases, the existence of an algebra norm on $A$ leads to the compactness of $X$.

1. Introduction and Preliminaries

If $X$ is a compact Hausdorff space then $C(X)$, the algebra of all continuous complex valued functions on $X$, is obviously a Banach algebra with the supremum norm and any Banach algebra norm on $C(X)$ is equivalent to the supremum norm. When $X$ is an arbitrary topological space, it has been shown in [7] and [8] that $C(X)$ admits an algebra norm (not necessarily complete) if and only if it does not contain any unbounded function (also see [5]). For a completely regular space $X$, this is indeed equivalent to the compactness of the
real compactification of $X$. The proof of this result in [8] is based on the following theorem of Kaplansky.

**Theorem 1.1.** [6] Let $(A, \|\cdot\|)$ be a commutative Banach algebra which is isometrically isomorphic to $C_0(X)$, for some locally compact space $X$. Then any algebra norm $\|\cdot\|$ on $A$ satisfies $\|f\| \geq \|f\|$, for every $f \in A$.

In this paper we consider the same problem for certain subalgebras of $C(X)$, where $X$ is mainly a hemicompact space. In general, the existence of an algebra norm on a subalgebra $A$ of $C(X)$ does not imply that all elements in $A$ are bounded functions. For example, $\|f\| = \sup \{|f(z)| : |z| < 1\}$ defines an algebra norm on the algebra Hol($\mathbb{C}$) of all holomorphic functions on $\mathbb{C}$.

Before proving our results, we state some preliminaries. For more details one can refer to [1], [3] and [4].

By a hemicompact space we mean a Hausdorff topological space $X$ with a sequence $(K_n)$ of increasing compact subsets of $X$ such that every compact subset of $X$ is contained in some $K_n$. The sequence $(K_n)$ is called an admissible exhaustion of $X$. A $k$-space is a Hausdorff space in which every subset intersecting each compact subset in a closed set is itself closed.

In this paper we assume that all algebras are unital.

By a Fréchet algebra we mean a locally multiplicatively convex topological algebra which is also metrizable and complete. Thus the topology of a Fréchet algebra can be defined by an increasing sequence of submultiplicative seminorms. Unlike the theory of Banach algebras, in the theory of Fréchet algebras we do not know whether each complex homomorphism on a Fréchet algebra is continuous or not (Michael’s problem). A commutative Fréchet algebra $(A, (p_n))$ is called functionally continuous if all complex homomorphisms on $A$ are automatically continuous. The set of all nonzero complex homomorphisms on $A$ is denoted by $S_A$ and the set of all nonzero continuous complex homomorphisms on $A$, which is called the spectrum of $A$, is denoted by $M_A$ or $M(A, (p_n))$. As usual, we always endow $M_A$ with the Gelfand topology and for every $x \in A$, the Gelfand transform of $x$. A commutative
Fréchet algebra $A$ is regular on its spectrum if for any closed subset $F$ of $M_A$ and $\varphi \in M_A \setminus F$ there is an element $x \in A$ with $\hat{x}(\varphi) = 1$ and $\hat{x}(F) = \{0\}$. It is well known that each commutative regular Fréchet algebra is (Gelfand) normal, that is, for every pair of disjoint closed subsets $K$ and $F$ of $M_A$ there exists $x \in A$ such that $\hat{x}(F) = \{0\}$ while $\hat{x}(K) = \{1\}$.

A topological algebra $A$ is called a $Q$-algebra if the set $A^{-1}$ of all invertible elements of $A$ is an open set. A uniform topological (Fréchet) algebra is a topological (Fréchet) algebra whose topology is determined by a separating family $\rho$ of uniform seminorms, that is, $p(x^2) = p(x)^2$ for all $p$ in $\rho$ and $x \in A$. For example, if $X$ is a hemicompact $k$-space with an admissible exhaustion $(K_n)$, then $(C(X), (\|\cdot\|_{K_n}))$ is a uniform Fréchet algebra, where $\|\cdot\|_{K_n}$ is the supremum norm on $K_n$. Indeed, a commutative algebra $A$ is a uniform Fréchet algebra if and only if there is a hemicompact space $X$ such that $A$ is topologically and algebraically isomorphic to a point separating and complete subalgebra of $C(X)$ which also contains the constants ([4, Theorem 4.1.3]).

Similarly by a Fréchet function algebra on a hemicompact space $X$ we mean a point separating subalgebra $A$ of $C(X)$ which contains the constants and is a Fréchet algebra under some topology such that all evaluation homomorphisms $\varphi_x$, for $x \in A$, are continuous. That is, $\varphi_x \in M_A$ for all $x \in A$. In fact, the class of all commutative semisimple unital Fréchet algebras coincides with the class of Fréchet function algebras, since the spectrum of each commutative Fréchet algebra is a hemicompact space.

If $(A, (p_n))$ is a Fréchet function algebra on a hemicompact space $X$ and $(K_n)$ is an admissible exhaustion of $X$ then it is easy to see that for each $n$ there is an integer $m \geq n$ such that

$$\|f\|_{K_n} \leq p_m(f)$$

for all $f \in A$ [9]. This shows that the identity map from $(A, (p_n))$ into $(C(X), \|\cdot\|_{K_n})$ is continuous. For each $n$, let $i(n) \geq n$ be the smallest integer for which the above inequality holds. Since $(p_n)$ is an increasing sequence and $(p_{i(n)})$ generates the same topology, we may assume that $i(n) = n$. 

Existence of an algebra norm
2. The Results

When $X$ is a hemicompact $k$-space with an admissible exhaustion $(K_n)$, the existence of an algebra norm on the uniform Fréchet algebra $(C(X), (\| \cdot \|_{K_n}))$ implies the compactness of $X$. Because, as it was mentioned before, in this case all functions in $C(X)$ are bounded by [7] or [8]. That is, $C_b(X) = C(X)$. Since $(C_b(X), (\| \cdot \|_{X}))$ is a semisimple Banach algebra, by Carpenter’s Theorem, it has a unique topology as a Fréchet algebra. Hence the topologies induced by the sequence $(\| \cdot \|_{K_n})$ of seminorms and the supremum norm $\| \cdot \|_X$ are equivalent on $C(X)$. That is, $(C(X), (\| \cdot \|_{K_n}))$ is a Banach algebra and thus $X$ is compact (see [4]).

Now let $A$ be a subalgebra of $C(X)$ for some topological space $X$ and set $A_b = A \cap C_b(X)$. As the following proposition shows, if there exists an algebra norm $\| \cdot \|$ on $A$ such that either all evaluation homomorphisms are $\| \cdot \|$ continuous or the set $A^{-1}$ is open with respect to this norm, then $A_b = A$.

**Proposition 2.1.** Let $S$ be a non-empty set and let $A$ be an algebra of functions on $S$. If there exists an algebra norm on $A$ such that each evaluation functional is continuous then all functions in $A$ are bounded. The same conclusion holds if there is a topology on $A$ making it a $Q$-algebra.

**Proof.** Let $\| \cdot \|$ be an algebra norm on $A$ such that $\varphi_x$ is continuous on $(A, \| \cdot \|)$ for all $x \in X$. The completion $A$ of $(A, \| \cdot \|)$ is a Banach algebra and since every $\varphi_x$ is $\| \cdot \|$ continuous on $A$ it can be extended to a $\| \cdot \|$ continuous homomorphism on $A$. Hence for $f \in A$, $f(X)$ is contained in $\sigma_A(f)$, the spectrum of $f$ in $A$. Since $\sigma_A(f)$ is compact, $f$ is a bounded function.

In a $Q$-algebra each element has a bounded spectrum, so that the result follows from the inclusion $f(X) \subseteq \sigma_A(f)$. $\square$

An immediate consequence of the above proposition is that when $(A, (\| \cdot \|_{K_n}))$ is a uniform Fréchet algebra on a hemicompact space $X$ (with $(K_n)$ as an admissible exhaustion) such that $(A, \tau)$ is a $Q$-algebra, for some topology $\tau$, then $(A, (\| \cdot \|_{K_n}))$ is a Banach algebra.
Existence of an algebra norm

Because \( A_b = A \) by Proposition 2.1, and since \( (A, \| \cdot \|_{K_n}) \) is a uniform Fréchet algebra, it is easy to verify that \( (A_b, \| \cdot \|_X) \) is a Banach algebra. Hence by the Carpenter’s theorem the topologies induced by \( \| \cdot \|_X \) and \( \| \cdot \|_{K_n} \) are equivalent on \( A \). That is, \( A \) is a Banach algebra.

It is well known that for any algebra norm \( \| \cdot \| \) on a regular uniform Banach algebra \( (A, \| \cdot \|) \) we have \( \| \cdot \| \leq \| \cdot \| \) (see [3, Corollary 4.1.28] or [2]). In the sequel we confine ourselves to hemi-compact spaces. In the first part of the following theorem we use a similar idea to [2] to get a result concerning the existence of an algebra norm on a functionally continuous regular Fréchet function algebra.

**Theorem 2.2.** Let \( X \) be a topological space and let \( (A, (p_n)) \) be a commutative semisimple unital Fréchet algebra whose spectrum is equal to \( X \). Suppose that \( A \) is regular and functionally continuous.

(i) If \( A \) admits an algebra norm then \( (A, (p_n)) \) is a \( Q \)-algebra.

(ii) Any uniform norm on \( A \) is continuous.

**Proof.** (i) Let \( \| \cdot \| \) be an algebra norm on \( A \) and let \( A \) be the completion of \( (A, \| \cdot \|) \). Take \( Y \) as the set of all points in \( X \) such that \( \varphi_x \) is \( \| \cdot \| \) continuous on \( A \). Note that since \( A \) is functionally continuous, \( Y \) can be identified with the spectrum of \( A \). We first show that \( Y \) is dense in \( X \). Assume that \( Y \neq X \) and choose, by the regularity of \( A \), an open subset \( G \) of \( X \) with \( G \subseteq X \setminus Y \). Since \( A \) is regular and hence normal, there exists an element \( f \in A \) with \( f|_Y = 1 \) and \( f|_G = 0 \). We claim that \( f \) is invertible in \( A \). If \( f \) is not so, there exists an element \( y \) in the spectrum of the Banach algebra \( A \) with \( y(f) = 0 \). That is, \( y \in Y \) and \( f(y) = 0 \) which is impossible. Now let \( z \in G \) and choose an element \( g \in A \) with \( g|_{X \setminus G} = 0 \) and \( g(z) = 1 \) so that \( \text{supp } g \subseteq G \) and \( g \neq 0 \). Then \( f\hat{g} = 0 \) on \( X \) and hence \( fg = 0 \) in \( A \) and consequently in \( A \). But this is impossible because \( f \) is invertible in \( A \) and \( g \neq 0 \). Thus \( Y = X \).

As we noted before \( Y \) can be identified with the spectrum of \( A \) and so \( Y \) is a compact space with respect to the Gelfand topology from \( A \) which is the same as the relative topology from \( X \). Hence \( Y \) is a compact subset of \( X \), so that \( Y = X \). It now follows that \( A \)
is a Q-algebra (see [10] or [1, 4.12-3]). In particular, each element of \( A \) has a finite spectral radius and \( \| f \|_X = r_A(f) = r_{\hat{A}}(f) \leq |f| \),
for all \( f \in A \).

(ii) Let \(|\cdot|\) be a uniform norm on \( A \). By Theorem 1 in [2], \(|\cdot|\) is an
algebra norm and so by (i), \((A, (p_n))\) is a Q-algebra and \( \| f \|_X \leq |f| \),
for all \( f \in A \). Let \( N \in \mathbb{N} \) be such that \( V = \{ f \in A : p_N(1 - f) < 1/N \} \subseteq A^{-1}. \) If \( f \in A \) and \( p_N(f) = 0 \) then for each non zero scalar \( \lambda, 1 - \lambda^{-1}f \in V \) and so \( \lambda \notin \sigma_A(f) \). Therefore, \( r_A(f) = r_{\hat{A}}(f) = |f| = 0 \), where \( A \) is the completion of \((A, |\cdot|)\). Hence \( f = 0 \). This
shows that \( p_N \) is indeed a norm.

Now let \( f \) be a non zero element of \( A \). Then for each scalar \( \lambda \) with \(|\lambda| > 1\) we have \( p_N(\frac{f}{Np_N(f)}) < 1/N \) and so \( \lambda 1 - \frac{f}{Np_N(f)} \in A^{-1}. \)
This shows that \( r_A(\frac{f}{Np_N(f)}) = r_{\hat{A}}(\frac{f}{Np_N(f)}) = |\frac{f}{Np_N(f)}| \leq 1 \). Note that
since \(|\cdot|\) is a uniform norm it follows easily that \( |f| \leq p_N(f) \), for all
\( f \in A \). \( \square \)

In the following theorem we obtain a similar result for regular
Fréchet function algebras with locally compact spectrum without
the functional continuity assumption.

**Theorem 2.3.** Let \( X \) be a hemi-compact space and let \((A, (p_n))\) be a
regular Fréchet function algebra whose spectrum is locally compact.
If there is a continuous algebra norm on \( A \) then all functions in \( A \)
are bounded.

**Proof.** Let \( \| \cdot \| \) be a continuous algebra norm on \( A \) and let \((A, \| \cdot \|)\)
be the completion of \((A, |\cdot|)\). Then the inclusion map \( \text{id} : (A, (p_n)) \rightarrow
(A, \| \cdot \|) \) is continuous and has a dense range so that its transpose
map \( \text{id}^* : M(\hat{A}, \| \cdot \|) \rightarrow M(A, (p_n)), \) defined by \( \varphi \mapsto \varphi |A, \) is a con-
tinuous injective map. Since \( M(A, (p_n)) \) is locally compact, it follows
from Theorem 2.6 in [9] that \( \text{id}^*(M(\hat{A}, \| \cdot \|)) \) is dense in \( M(A, (p_n)) \) and
hence \( M(A, (p_n)) \) is compact, since \( M(\hat{A}, \| \cdot \|) \) is compact. Therefore,
\((A, (p_n))\) is a Q-algebra by [10] and all elements of \( A \) are bounded
by Proposition 2.1. Note that in this case, \((A, \| \cdot \|_X)\) may not be a
Banach algebra. \( \square \)
**Theorem 2.4.** Let $X$ be a hemicompact space, $(K_n)$ an admissible exhaustion of $X$ and $(A, (p_n))$ a functionally continuous regular Fréchet function algebra on $X$. If the sequence $(p_n)$ is such that for $f \in A$ and $n \in \mathbb{N}$, $f|_{K_n} = 0$ implies $p_n(f) = 0$, then $A$ admits an algebra norm if and only if $X$ is compact.

**Proof.** Assume that there is an algebra norm on $A$. Then by Theorem 2.2, $(A, (p_n))$ is a Q-algebra and hence the set of all invertible elements of $A$ is open. So that there is an integer $N$ such that $V = \{ f \in A : p_N(1 - f) < 1/N \} \subseteq A^{-1}$. Now if $X$ is not compact, we can choose $x \in X \setminus K_N$. The regularity of $A$ shows that there is an element $f \in A$ with $f(\varphi_y) = 1, y \in K_N$ and $f(\varphi_x) = 0$. Hence $(1 - f)|_{K_N} = 0$ and so by the hypothesis $p_N(1 - f) = 0$, which shows that $f$ is invertible in $A$. But this is impossible because $f(x) = 0$. \[ \square \]

Let $X$ be a hemicompact $k$-space, $(K_n)$ an admissible exhaustion of $X$ and $(A_n)$ a sequence of Banach function algebras such that for each $n$, $A_n$ is a Banach function algebra on $K_n$ with respect to $\| \cdot \|_n$, $A_{n+1}|_{K_n} \subseteq A_n$ and $\| f|_{K_n} \|_n \leq \| f \|_{n+1}$, for all $f \in A_{n+1}$. Then the subalgebra

$$A = \{ f \in C(X) : f|_{K_n} \in A_n, \ n \in \mathbb{N} \}$$

of $C(X)$ is a Fréchet algebra with respect to the topology defined by the sequence $(p_n)$ of seminorms, where $p_n(f) = \| f|_{K_n} \|_n$, $f \in A$. Moreover, if $A$ separates the points of $X$ then it is a Fréchet function algebra on $X$ (see [9]). Clearly, in this case for each $n$ and $f \in A$, $f|_{K_n} = 0$ implies $p_n(f) = 0$. Thus if $A$ is regular and functionally continuous then the existence of an algebra norm on $A$ is equivalent to the compactness of $X$ by Theorem 2.4. An example of this situation is described as follows:

**Example 2.5.** Let $(X, d)$ be a hemicompact metric space with the admissible exhaustion $(K_n)$ and let $0 < \alpha \leq 1$. For each $n$, let $A_n = \text{Lip}(K_n, \alpha)$ be the Banach algebra of all Lipschitz functions of order $\alpha$ on $K_n$ with pointwise addition and multiplication, which is
endowed with the following norm:

$$\|f\|_n = \|f\|_{K_n} + \sup_{x,y\in K_n} \frac{|f(x) - f(y)|}{d^n(x,y)}$$

By the above argument the subalgebra $A = \text{FLip}(X, \alpha) = \{ f \in C(X) : f|_{K_n} \in A_n, n \in \mathbb{N}\}$ of $C(X)$ is a Fréchet function algebra on $X$. Since $A$ is a symmetric *-algebra with $f \mapsto f$ as an involution, it is functionally continuous. Moreover, $M_A$ is homeomorphic to $X$ [9] and it is easy to see that $A$ is regular on its spectrum. Since $M_A \cong X$, the algebra $A$ does not admit a Banach algebra norm if $X$ is not compact. Now Theorem 2.4 shows that in this case $A$ does not admit an algebra norm (not necessarily complete).

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