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Author(s):

H.H. Kim

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THE RESIDUAL SPECTRUM OF $U(n,n)$; CONTRIBUTION FROM BOREL SUBGROUPS

H.H. KIM

To Freydoon Shahidi on his 70th birthday

Abstract. In this paper we study the residual spectrum of the quasi-split unitary group $G = U(n,n)$ defined over a number field F, coming from the Borel subgroups, $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_T$. Due to lack of information on the local results, that is, the image of the local intertwining operators of the principal series, our results are incomplete. However, we describe a conjecture on the residual spectrum and prove a certain special case by using the Knapp-Stein R-group of the unitary group.
Keywords: Automorphic representation, spectral decomposition, residual spectrum.
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1. Introduction

In this paper we study the residual spectrum of the quasi-split unitary group $G = U(n,n)$ defined over a number field F, coming from the Borel subgroups, $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_T$. (See the introduction of [18] for the definition.) Its root system over F is C_n and we can apply the technique in [22]. Due to lack of information on the local results, that is, the image of the local intertwining operators of the principal series, our results are incomplete. However, we describe a conjecture on the residual spectrum and prove a certain special case as in [22] by using the Knapp-Stein R-group of the unitary group studied by Keys [15].

More precisely, let $G = U(n,n)$ with respect to E/F, be a quadratic extension of number fields, and let $\mathbb{A}_F$ be the ring of adeles. Let ${}^LG = GL(2n,\mathbb{C}) \rtimes W_F$ denote its dual group. Let $\mu_1,\ldots,\mu_k$ be distinct grössencharacters of E such that $\mu_i|_{N(\mathbb{A}_E^\times)} = 1$ but $\mu_i|_{\mathbb{A}_F^\times} \neq 1$. We note that this means that $\mu_i|_{\mathbb{A}_F^\times} = \eta_{E/F}$, the quadratic character corresponding to E/F by the class field theory. Let $\nu_1,\ldots,\nu_l$ be distinct grössencharacters of E such that $\nu_j|_{\mathbb{A}_F^\times} = 1$.

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Then the character $\chi = \chi(\underbrace{\mu_1,\ldots,\mu_1}_{r_1},\ldots,\underbrace{\mu_k,\ldots,\mu_k}_{r_k},\underbrace{\nu_1,\ldots,\nu_1}_{s_1},\ldots,\underbrace{\nu_l,\ldots,\nu_l}_{s_l})$, where $r_1+\cdots+r_k+s_1+\cdots+s_l=n$, defines a character of $T(F)\backslash T(\mathbb{A}_F)$, where T is the maximal split torus in G. An Eisenstein series attached to a character of $T(F)\backslash T(\mathbb{A}_F)$ will contribute to the residual spectrum only if the character is of the above type (Proposition 4.8).

The Arthur parameter of interest to us is a homomorphism

$$\psi : W_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

which factors through $GL(2r_1,\mathbb{C})\times\cdots\times GL(2r_k,\mathbb{C})\times GL(2s_1,\mathbb{C})\times GL(2s_l,\mathbb{C})\rtimes W_F \hookrightarrow GL(2n,\mathbb{C})\rtimes W_F$. By Jacobson-Morozov theorem, $\psi|_{SL(2,\mathbb{C})}$ is determined by distinguished unipotent orbit $\prod_{i=1}^k O_i \times \prod_{j=1}^l O'_j$, where O_i's are distinguished unipotent orbits in $O(2r_i,\mathbb{C})$, $i=1,\ldots,k$ and O_j's in $Sp(2s_j,\mathbb{C})$; $\psi|_{W_F}$ is defined using μ_i's and ν_j's (see Section 3.1). To ψ, we can define the set $L^2(G(F)\backslash G(\mathbb{A}))_\psi$ (see (3.4)).

In Section 4, by assuming a local conjecture on the image of the local intertwining operators of the principal series, we describe a conjecture on the intersection

$$L^2(G(F)\backslash G(\mathbb{A}))_\psi \cap L^2_{dis}(G(F)\backslash G(\mathbb{A}))_T.$$

We hope that the technique of [13] in the case of symplectic and odd orthogonal groups may be applied in our case.

In Section 5, we consider a special case as in [22], that is, for each $i=1,\ldots,k$, let O_i be the unipotent orbit of $O(2r_i,\mathbb{C})$ attached to principal Jordan block $(2r_i-1,1)$. Let O'_j, $j=1,\ldots,l$, be the principal unipotent orbit of $Sp(2s_j,\mathbb{C})$, i.e., the one attached to the Jordan block $(2s_j)$. To ψ, Arthur associates a Langlands' parameter ϕ_ψ. We construct the representations in Π_{ϕ_ψ} as residues of Eisenstein series associated to the character χ. Using certain identities satisfied by local intertwining operators which was proved in [16], we then verify Arthur's multiplicity formula for these square integrable residues. (See Section 3.1 for Arthur's multiplicity formula.) It is remarkable that in this case also as in [22] the formula itself appears in the corresponding residue of the Eisenstein series. We note that the local R-group $C_{\phi_{\psi_v}}$ for the parameter ϕ_ψ is the Knapp-Stein R-group of the unitary principal series $I_v = Ind_{B_0(F_v)}^{M(F_v)}\chi_v$, where M is the Levi-subgroup whose L-group is ${}^LM^0 = Cent(im\,\phi_\psi^+, {}^LG)$. Here ϕ_ψ^+ is the non-tempered part of ϕ_ψ. M is isomorphic to $Res_{E/F}GL_{n_1}\times\cdots\times Res_{E/F}GL_{n_k}\times U(k,k)$ for some $n_1,\ldots,n_k$. The R-group has been calculated by Keys [15] and is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\cdots\times\mathbb{Z}/2\mathbb{Z}$ if v is inert in E, and is generated by the sign changes $c_{r_1+\cdots+r_i}$ such that μ_{iv}'s are distinct and $\mu_{iv}|_{F_v^\times}$ is non-trivial.

In Section 6 we determine the residual spectrum of $U(2,2)$ coming from the conjugacy class of Borel subgroups. Konno [24] obtained all residual spectrum of $U(2,2)$. However, his result on the residual spectrum, coming from Borel subgroup, is not a suitable form for Arthur parameters. So we redo the calculations imitating [17]. We should also note that Konno's results do not show the global obstruction condition, i.e., Arthur's condition.

At the suggestion of the referee, we would like to review some recent relevant (by no means exhaustive) works on the residual spectrum: In [20], we gave a conjecture on the residual spectrum from arbitrary parabolic subgroups, assuming some local and global conjectures. The local conjecture is about normalized local intertwining operators, similar to Conjecture 4.10 of this paper. It is conjectured that the residual spectrum is parametrized by unipotent orbits as in the Borel case. In [21], we showed the relationship among the residual spectrum coming from Borel subgroups in an isogeny class. In a monumental work in [5], J. Arthur proved the endoscopic classification of the discrete spectrum of symplectic and orthogonal groups over a number field. In [33], C.P. Mok followed Arthur closely and gave the endoscopic classification of quasi-split unitary groups. Based on Arthur's classification, Mœglin [29, 30] gave a conjectural description of the residual spectrum in terms of its transfer to the general linear group. She proves a special case of unitary groups and cohomological representations at the infinity. Most recently, De Martino, Heiermann and Opdam [8] computed the spherical residual spectrum of arbitrary split reductive group from the trivial character of the torus, using graded affine Hecke algebras. We hope that their technique may be helpful in proving Conjecture 4.10. There have been some results on residual spectrum of inner forms of classical groups such as inner forms of GL_n [6], and inner forms of symplectic and orthogonal groups [10, 11, 12].

2. Preliminaries

Let F be a number field and E/F be a quadratic extension of number fields. Let W_F denote the Weil group of F and $\mathbf{G} = \mathbf{U}(n,n)$ be the quasi-split unitary group in $2n$ variables defined with respect to E/F. Let $G = \mathbf{G}(F)$. It is given as follows: Let J_n be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

Let $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Then

$$G = \left\{ g \in GL(2n) \middle| \, {}^t\bar{g} J'_{2n} g = J'_{2n} \right\},$$

where $x \longmapsto \bar{x}$ is the Galois automorphism of E/F. We note that $G(E) = GL(2n,E)$. Let T_d be the maximal F-split torus consisting of diagonal elements in G. Then

$$T_d = \left\{ t(\lambda_1,\ldots,\lambda_n) = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} \middle| \lambda_i \in F^\times \right\}.$$

The centralizer of T_d in G is the maximal torus T of diagonal elements:

$$T(F) = \left\{ t(\lambda_1,\ldots,\lambda_n) = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \bar{\lambda}_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \bar{\lambda}_2^{-1} & \\ & & & & & & & \bar{\lambda}_1^{-1} \end{pmatrix} \middle| \lambda_i \in E^\times \right\}.$$

Then the root system $\Phi(G,T)$ is of type A_{2n-1}. But the restricted root system $\Phi(G,T_d)$ is of type C_n. We choose the ordering on the restricted roots so that the Borel subgroup B is the subgroup of upper triangular matrices in G. Let Δ be the set of simple roots in $\Phi(G,T_d)$ given by $\Delta = \{\alpha_1,\ldots,\alpha_n\}$, with $\alpha_j = e_j - e_{j+1}$ for $1 \le j \le n-1$ and $\alpha_n = 2e_n$. We let $\langle , \rangle$ be the standard inner product on $\Phi(G,T_d)$.

The Weyl group is given by $W(G/T) \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. S_n acts by permutations on the λ_i, $i = 1,\ldots,n$. We will use standard cycle notation for the elements of S_n. Thus (ij) interchanges λ_i and λ_j. If c_i is the nontrivial element in the i-th copy of $\mathbb{Z}/2\mathbb{Z}$ then c_i takes λ_i to $\bar{\lambda}_i^{-1}$. The element c_i is called a sign change because its action on $\Phi(G,T)$ takes e_i to $-e_i$.

For each place v, we define $G_v = G(F_v) = G \otimes_F F_v$, $T_v = T(F_v)$. We note that $T(F_v) = \{t(\lambda_1,\ldots,\lambda_n) | \lambda_i \in E_v^\times\}$, where $E_v = E \otimes_F F_v$. E_v/F_v is a quadratic extension, except the case when v splits in E. In that case, $E_v \simeq F_v \oplus F_v$. In this case, we define the 'Galois' automorphism $x = (x_1,x_2) \longmapsto \bar{x} = (x_2,x_1)$.

If a place v of F is inert in E, then the place of E lying over v will be denoted by w. If v splits in E, then the places of E lying over v will be denoted by w_1 and w_2. We note that $G_v \simeq U(n,n)_{E_w/F_v}$ if v is inert in E and

$G_v \simeq GL(2n, F_v)$ if v splits in E. According to the class field theory, roughly half places of F split in E and roughly half of them are inert in E.

3. **Arthur parameters for the residual spectrum**

3.1. **Arthur parameters for the unitary groups.** Let F be a number field and let W_F denote the Weil group of F. For $G = U(n,n)$, we can take the dual group ${}^LG^0 = GL(2n,\mathbb{C})$ and ${}^LG = GL(2n,\mathbb{C}) \rtimes W_F$. Note that $W_F/W_E \simeq Gal(E/F) = \mathbb{Z}/2\mathbb{Z}$. Let w_σ be the non-trivial element in $Gal(E/F)$. We let W_E acts on $GL(2n,\mathbb{C})$ trivially and w_σ acts on $GL(2n,\mathbb{C})$ by $g \longmapsto J'_{2n}{}^t g^{-1} J'_{2n}{}^{-1}$, i.e., the multiplication in ${}^LG$ is given by

$$(g' \times w_\sigma)(g \times w) = g' J'_{2n}{}^t g^{-1} J'_{2n}{}^{-1} \times w_\sigma w,$$

for $w \in W_F$ and $(g' \times w')(g \times w) = g'g \times w'w$ for $w' \in W_E$. We recall the definition of J'_{2n}: Let J_n be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

Then $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$.

In the following, let $\mu_1, \ldots, \mu_k$ be distinct grössencharacters of E such that $\mu_i|_{N(\mathbb{A}_E^\times)} = 1$ but $\mu_i|_{\mathbb{A}_F^\times} \neq 1$ and let $\nu_1, \ldots, \nu_l$ be distinct grössencharacters of E such that $\nu_j|_{\mathbb{A}_F^\times} = 1$. Then the character
$\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{r_k}, \underbrace{\nu_1, \ldots, \nu_1}_{s_1}, \ldots, \underbrace{\nu_l, \ldots, \nu_l}_{s_l})$, where $r_1 + \cdots + r_k + s_1 + \cdots + s_l = n$, defines a character of $T(F)\backslash T(\mathbb{A}_F)$, where T is the maximal torus in G.

The Arthur parameter of interest to us is a homomorphism

$$\psi : W_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

which factors through

$$GL(2r_1,\mathbb{C}) \times \cdots \times GL(2r_k,\mathbb{C}) \times GL(2s_1,\mathbb{C}) \times GL(2s_l,\mathbb{C}) \rtimes W_F \hookrightarrow GL(2n,\mathbb{C}) \rtimes W_F,$$

as follows: (The usual definition of Arthur parameters uses Langlands' hypothetical group L_F. But since we are only dealing with Langlands' quotients which come from principal series, W_F is enough.)

(1) If $w \in W_E$,

$$\psi|_{W_F} : w \longmapsto \begin{pmatrix} \mu_1(w)I_{2r_1} & & & & & \\ & \ddots & & & & \\ & & \mu_k(w)I_{2r_k} & & & \\ & & & \nu_1(w)I_{2s_1} & & \\ & & & & \ddots & \\ & & & & & \nu_l(w)I_{2s_l} \end{pmatrix} \times w$$

(2)

$$\psi|_{W_F} : w_\sigma \longmapsto J_{r_1,\dots,r_k,s_1,\dots,s_l} {J'_{2n}}^{-1} \times w_\sigma,$$

where

$$J_{r_1,\dots,r_k,s_1,\dots,s_l} = \begin{pmatrix} J_{2r_1} & & & & & \\ & \ddots & & & & \\ & & J_{2r_k} & & & \\ & & & J'_{2s_1} & & \\ & & & & \ddots & \\ & & & & & J'_{2s_l} \end{pmatrix}.$$

(3) By Jacobson-Morozov theorem, $\psi|_{SL(2,\mathbb{C})}$ is determined by a unipotent orbit of $GL(2r_1,\mathbb{C}) \times \cdots \times GL(2r_k,\mathbb{C}) \times GL(2s_1,\mathbb{C}) \times \cdots \times GL(2s_l,\mathbb{C})$ of the form

$$\prod_{i=1}^{k} O_i \times \prod_{j=1}^{l} O'_j,$$

where O_i, $i = 1,\dots,k$, is a distinguished unipotent orbit of $O(2r_i,\mathbb{C}) \subset GL(2r_i,\mathbb{C})$ and O'_j, $j = 1,\dots,l$, is a distinguished unipotent orbit of $Sp(2s_j,\mathbb{C}) \subset GL(2s_j,\mathbb{C})$. Inside O_i, O'_j, we fix elements u_i, v'_j such that

$$\psi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \prod_{i=1}^{k} u_i \times \prod_{j=1}^{l} v'_j.$$

Remark 3.1. The above definition is a refinement of the usual definition which is:

(1) $\psi(W_F)$ is bounded and included in the set of semi-simple elements of ${}^LG$.
(2) $\psi|_{SL(2,\mathbb{C})}$ is algebraic.

We note that

$$\begin{aligned} &Cent(J_{r_1,\dots,r_k,s_1,\dots,s_l} {J'_{2n}}^{-1} \times w_\sigma, {}^LG^0) \cap \\ &\quad GL(2r_1,\mathbb{C}) \times \cdots \times GL(2r_k,\mathbb{C}) \times GL(2s_1,\mathbb{C}) \times \cdots \times GL(2s_l,\mathbb{C}) \times 1 \\ &\qquad = O(2r_1,\mathbb{C}) \times \cdots \times O(2r_k,\mathbb{C}) \times Sp(2s_1,\mathbb{C}) \times \cdots \times Sp(2s_l,\mathbb{C}). \end{aligned}$$

This is the reason we chose distinguished unipotent orbits from $O(2r_i,\mathbb{C})$ and $Sp(2s_j,\mathbb{C})$. Recall from Carter [7] that O is a distinguished unipotent orbit in $O(2n,\mathbb{C})$ if and only if it has a Jordan block $(1^{t_1},3^{t_3},5^{t_5},\ldots)$, where $t_i=0$ or 1; O is a distinguished unipotent orbit in $Sp(2n,\mathbb{C})$ if and only if it has a Jordan block $(2^{t_2},4^{t_4},6^{t_6},\ldots)$, where $t_i=0$ or 1.

Let $S_\psi = Cent(im\,\psi, {}^LG^0)$, $Z_G = Cent({}^LG, {}^LG^0)=\{\pm 1\}$ and

$$C_\psi = S_\psi / S_\psi^\circ Z_G.$$

Then S_ψ is a maximal reductive subgroup of

$$\prod_{i=1}^k Cent(u_i, O(2r_i,\mathbb{C})) \times \prod_{j=1}^l Cent(v_j', Sp(2s_j,\mathbb{C})).$$

Therefore $S_\psi^\circ = 1$, i.e., S_ψ is finite if and only if each u_i is a distinguished unipotent element in $O(2r_i,\mathbb{C})$ and v_j' in $Sp(2s_j,\mathbb{C})$. So:

Lemma 3.2. *In order that S_ψ is finite, $r_i \geq 2$ for $i=1,\ldots,k$, and $O_i \subset O(2r_i,\mathbb{C})$, $O_j' \subset Sp(2s_j,\mathbb{C})$ are distinguished.*

Now it is clear that $S_\psi/S_\psi^\circ Z_G$ is equal to

$$\prod_{i=1}^k Cent(u_i, O(2r_i,\mathbb{C}))/Cent(u_i, O(2r_i,\mathbb{C}))^\circ \times \prod_{i=1}^l Cent(v_j', Sp(2s_j,\mathbb{C}))/Cent(v_j', Sp(2s_j,\mathbb{C}))^\circ Z_G. \tag{3.1}$$

Here $Cent(u_i, O(2r_i,\mathbb{C}))/Cent(u_i, O(2r_i,\mathbb{C}))^\circ$ is t product of $\mathbb{Z}/2\mathbb{Z}$, where t is the number of i odd with $t_i>0$ in Jordan blocks;
$Cent(v_j', Sp(2s_j,\mathbb{C}))/Cent(v_j', Sp(2s_j,\mathbb{C}))^\circ$ is t product of $\mathbb{Z}/2\mathbb{Z}$, where t is the number of i even with $t_i>0$ in Jordan blocks.

For each place v of F, we have a local Arthur parameter $\psi_v : W_{F_v}\times SL_2(\mathbb{C}) \longrightarrow GL(2n,\mathbb{C})\rtimes W_{F_v}$ such that it commutes with the map $W_{F_v}\to W_F$ [37, p. 8]. As in the global case, we can then define S_{ψ_v}.

There are two cases.

Case 1. v is inert in E. In this case, $Gal(E_w/F_v)\simeq Gal(E/F)$ and therefore, $GL(2n,\mathbb{C})\rtimes W_{F_v}$ is defined in the same way as in the global case.

But in the local case, μ_{iv} or ν_{jv} may not be distinct. Suppose $\mu_{1v}=\mu_{2v}$. Then in the above formula (3.1),

$$Cent(u_1, O(2r_1,\mathbb{C}))/Cent(u_1, O(2r_1,\mathbb{C}))^\circ \times Cent(u_2, O(2r_2,\mathbb{C}))/Cent(u_2, O(2r_2,\mathbb{C}))^\circ$$

must be replaced by

$$Cent(u_1\times u_2, O(2r_1+2r_2,\mathbb{C}))/Cent(u_1\times u_2, O(2r_1+2r_2,\mathbb{C}))^\circ.$$

However, if $\mu_{1v} = \nu_{jv}$, then the above formula (3.1) still holds.

Case 2. v splits in E. In this case, $E_{w_i} = F_v$, where w_i, $i = 1, 2$, are the places of E over v. Therefore, the action of W_{F_v} in $GL(2n, \mathbb{C}) \rtimes W_{F_v}$ is trivial. Therefore, ψ_v is given by $\psi_v : W_{F_v} \times SL(2, \mathbb{C}) \longrightarrow GL(2n, \mathbb{C}) \times W_{F_v}$ such that

(1)

$$\psi_v|_{W_{F_v}} : w \longmapsto \begin{pmatrix} \mu_{1w_i}(w) I_{2r_1} & & & & & \\ & \ddots & & & & \\ & & \mu_{kw_i}(w) I_{2r_k} & & & \\ & & & \nu_{1w_i}(w) I_{2s_1} & & \\ & & & & \ddots & \\ & & & & & \nu_{lw_i}(w) I_{2s_l} \end{pmatrix} \times w,$$

where w_i is a place of E over v. We note that the map is well-defined, independent of the choice of w_i (see [37, p. 9]).

(2) $\psi_v|_{SL(2,\mathbb{C})}$ is defined by the unipotent orbit $\prod_{i=1}^{k} O_i \times \prod_{j=1}^{l} O'_j$, where O_i, $i = 1, \ldots, k$, be a distinguished unipotent orbit of $O(2r_i, \mathbb{C}) \subset GL(2r_i, \mathbb{C})$ and O'_j, $j = 1, \ldots, l$, is a distinguished unipotent orbit of $Sp(2s_j, \mathbb{C}) \subset GL(2s_j, \mathbb{C})$.

Then C_{ψ_v} is trivial in this case.

Now we recall Mœglin's reformulation of Arthur's conjecture [29]: It is a part of local Arthur's conjecture that for each irreducible character η_v of C_{ψ_v}, there exists an irreducible representation $\pi(\psi_v, \eta_v)$. For each v, let Π_{ψ_v} be the set of $\pi(\psi_v, \eta_v)$.

We define the global Arthur packet Π_ψ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each v, π_v belongs to Π_{ψ_v}.

Conjecture 3.3 (Arthur's conjecture (Global))**.** $\Pi_\psi \cap L^2_{dis}(G(F)\backslash G(\mathbb{A}))$ is nonempty if and only if S_ψ is finite, i.e., $S^\circ_\psi = 1$. We call such an Arthur parameter elliptic. For an elliptic Arthur parameter ψ, any $\pi \in \Pi_\psi$ occurs in $L^2_{dis}(G(F)\backslash G(\mathbb{A}))$ if and only if

$$\sum_{x \in C_\psi} \prod_v \eta_v(x_v) \neq 0, \tag{3.2}$$

where $\pi = \otimes_v \pi(\psi_v, \eta_v)$, $x = (x_v)$.

Note that, if C_ψ is abelian, (3.2) is equivalent to

$$\left(\prod_v \eta_v \right) |_{C_\psi} = 1. \tag{3.3}$$

Define

$$L^2(G(F)\backslash G(\mathbb{A}))_\psi = \Pi_\psi \cap L^2_{dis}(G(F)\backslash G(\mathbb{A})). \tag{3.4}$$

Let Π_{res_v} be the subset of Π_{ψ_v} which consists of the local components of the residual spectrum. We will parametrize the elements in Π_{res_v} and prove the multiplicity formula (3.3) under certain conjecture on local problems, namely, construct a set of characters $C_{res_v} \subset \widehat{C_{\psi_v}}$ and each character of C_{res_v} gives rise to an element in Π_{res_v}.

Remark 3.4. To any Arthur parameter ψ, Arthur associates a Langlands' parameter $\phi_\psi : W_F \longrightarrow {}^LG$ as follows:

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right).$$

Let $S_{\phi_\psi} = Cent(im\,\phi_\psi, {}^LG^0)$ and $C_{\phi_\psi} = S_{\phi_\psi}/S^\circ_{\phi_\psi} Z_G$. For each place v, we have $S_{\phi_{\psi_v}}$, $C_{\phi_{\psi_v}}$. For each v, there is a natural surjection $C_{\psi_v} \to C_{\phi_{\psi_v}}$. The parameter ϕ_{ψ_v} gives a L-packet $\Pi_{\phi_{\psi_v}}$ which consists of Langlands' quotients. It is a part of Arthur's original local conjecture that for each place v, there is a pairing $\langle\ ,\ \rangle$ on $C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}}$ and an enlargement Π_{ψ_v} of $\Pi_{\phi_{\psi_v}}$ which allows an extension of $\langle\ ,\ \rangle$ to $C_{\psi_v} \times \Pi_{\psi_v}$ such that $\pi \in \Pi_{\phi_{\psi_v}} \subset \Pi_{\psi_v}$ if and only if the function $\langle\ , \pi\rangle$ lies in the image of $\widehat{C_{\phi_{\psi_v}}}$ in $\widehat{C_{\psi_v}}$. Since C_{ψ_v} is abelian in our case, giving a pairing between C_{ψ_v} and Π_{ψ_v} is the same as giving a character of C_{ψ_v}.

In Section 5, we construct the representations in $\Pi_{\phi_{\psi_v}}$ as local component of the residual spectrum attached to special unipotent orbits. In this case, the pairing $\langle\ ,\ \rangle$ on $C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}}$ is given by the Knapp-Stein R-groups [16].

4. Residual spectrum of $U(n,n)$

We fix a non-trivial additive character $\psi_F = \otimes_v \psi_{F_v}$ of $\mathbb{A}_F/F$ and let $L_F(z,\mu)$ be the Hecke L-function over F with the ordinary Γ-factor so that it satisfies the functional equation $L_F(z,\mu) = \epsilon_F(z,\mu)L_F(1-z,\mu^{-1})$, where $\epsilon_F(z,\mu) = \prod_v \epsilon_F(z,\mu_v,\psi_{F_v})$ is the usual ϵ-factor, see [9, p. 159]. If μ is the trivial character μ_0, then we write simply $L_F(z)$ for $L_F(z,\mu_0)$.

We extend the coroots $\alpha^\vee : F^\times \to T_d$ to $\alpha^\vee : F^\times \to T$ as follows. For $\alpha = e_i - e_j$, $\alpha^\vee(\lambda) = t(1,\dots,\underset{i}{\lambda},\dots,\underset{j}{\lambda^{-1}},\dots,1) \in T(F)$ for $1 \le i < j \le n$. For $\alpha = e_i + e_j$, $\alpha^\vee(\lambda) = t(1,\dots,\underset{i}{\lambda},\dots,\underset{j}{\bar\lambda},\dots,1)$, for $1 \le i < j \le n$. For $\alpha = 2e_i$, $\alpha^\vee(\lambda) = t(1,\dots,\underset{i}{\lambda},\dots,1)$ for $1 \le i \le n$. Here dots represent 1.

Let $X(T)_F$ (resp. $X^*(T)_F$) be the group of F-characters (resp. cocharacters) of T. There is a natural pairing $\langle,\rangle : X(T)_F \times X^*(T)_F \to \mathbb{Z}$. For $\alpha,\beta \in \Phi(G,T_d)$, $\langle \beta, \alpha^\vee\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$, where $(\ ,\ )$ is the standard inner product in $\Phi(G,T)$. Let $\omega_i = e_1 + \cdots + e_i$. Then $\omega_1,\dots,\omega_n$ are the fundamental weights of G with respect to (G,T_d). Set $\mathfrak{a}^* = X(T)_F \otimes \mathbb{R}$, $\mathfrak{a}^*_{\mathbb{C}} = X(T)_F \otimes \mathbb{C}$, and $\mathfrak{a} = X^*(T)_F \otimes \mathbb{R} = Hom(X(T)_F,\mathbb{R})$, $\mathfrak{a}_{\mathbb{C}} = X^*(T)_F \otimes \mathbb{C}$. The positive Weyl chamber in $\mathfrak{a}^*$ is

$$\begin{aligned} C^+ &= \{\Lambda \in \mathfrak{a}^* \,|\, \langle \Lambda, \alpha^\vee \rangle > 0, \text{ for all } \alpha \text{ positive roots}\} \\ &= \{\sum_{i=1}^n a_i \omega_i \,|\, a_i > 0\}. \end{aligned}$$

Let $B = TU$ be the Borel subgroup, where U is the unipotent radical. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{A}_\infty)$. If $v < \infty$, let K_v be an open maximal compact subgroup of G_v such that for almost all v, $K_v = G(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers of F_v. Then $K = K_\infty \times \prod_{v<\infty} K_v$ is a maximal compact subgroup of $G(\mathbb{A}_F)$ and $G(\mathbb{A}_F) = B(\mathbb{A}_F)K$. The embedding $X(T)_F \hookrightarrow X(T)_{F_v}$ induces an embedding $\mathfrak{a}_v \hookrightarrow \mathfrak{a}$. There exists a homomorphism $H_B : T(\mathbb{A}_F) \to \mathfrak{a}$, defined by

$$exp(\langle \chi, H_B(t) \rangle) = \prod_v |\chi(t_v)|_v,$$

where $\chi \in X(T)_F$ and $t = (t_v)$. We will extend H_B to G by making it trivial on U and K, see [22].

4.1. **Definition of Eisenstein series.** For $\mu_1, \dots, \mu_n$ grössencharacters of E, we define a character $\chi = \chi(\mu_1, \dots, \mu_n)$ of $T(F)\backslash T(\mathbb{A}_F)$ by

$$\chi(\mu_1, \dots, \mu_n)(t(\lambda_1, \dots, \lambda_n)) = \mu_1(\lambda_1) \dots \mu_n(\lambda_n).$$

As in [22], we form the Eisenstein series:

$$E(g, f, \Lambda) = \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g),$$

where $f \in I(\Lambda, \chi) = Ind_B^G \chi \otimes exp(\Lambda, H_B(\ ))$. It converges absolutely for $Re\,\Lambda \in C^+ + \rho_B$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g, f, \Lambda)$ along B is given by

$$E_0(g, f, \Lambda) = \int_{U(F)\backslash U(\mathbb{A})} E(ug, f, \Lambda)\, du = \sum_{w \in W} M(w, \Lambda, \chi) f(g),$$

where

$$M(w, \Lambda, \chi) f(g) = \int_{wU(\mathbb{A})w^{-1} \cap U(\mathbb{A}) \backslash U(\mathbb{A})} f(w^{-1} ug)\, du.$$

Then $M(w, \Lambda, \chi)$ defines an intertwining map from $I(\Lambda, \chi)$ to $I(w\Lambda, w\chi)$ and satisfies a functional equation of the form

$$M(w_1 w_2, \Lambda, \chi) = M(w_1, w_2\Lambda, w_2\chi) M(w_2, \Lambda, \chi).$$

For $\chi = \chi(\mu_1, \dots, \mu_n)$,

$$\chi \circ \alpha^\vee = \begin{cases} \mu_i \mu_j^{-1}, & \text{for } \alpha = e_i - e_j \\ \mu_i \bar{\mu}_j, & \text{for } \alpha = e_i + e_j \text{ and } i < j \\ \mu_i|_{\mathbb{A}_F^\times}, & \text{for } \alpha = 2e_i, \end{cases}$$

where $\bar{\mu}$ denotes the character $\bar{\mu}(a) = \mu(\bar{a})$ for $a \in \mathbb{A}_E^\times$. We note that if μ is a grössencharacter of E, then $\tilde{\mu} = \mu|_{\mathbb{A}_F^\times}$ is a grössencharacter of F and its local component is $\tilde{\mu}_v = \mu_w|_{F_v^\times}$ if v is inert in E and $\tilde{\mu}_v = \mu_{w_1}\mu_{w_2}$ if v splits in E.

We have

$$M(w, \Lambda, \chi) = \otimes_v A(w, \Lambda, \chi_v).$$

We normalize the local intertwining operators $A(w, \Lambda, \chi_v)$ as in [16] for p-adic places and [4] for archimedean places: For any v, let

$$r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)}{L(\langle \Lambda, \alpha^\vee \rangle + 1, \chi_v \circ \alpha^\vee)\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee, \psi_{F_v})}.$$

Here the L-functions are defined as follows:

(1) The case $\alpha = e_i - e_j$, $i < j$. In this case,

$$L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee) = \begin{cases} L_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw}\mu_{jw}^{-1}), & \text{if } v \text{ is inert in } E \\ L_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_1}\mu_{jw_1}^{-1}) L_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_2}\mu_{jw_2}^{-1}), & \text{if } v \text{ splits in } E, \end{cases}$$

$$\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee, \psi_{F_v}) = \begin{cases} \lambda(E_w/F_v, \psi_{F_v})\epsilon_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw}\mu_{jw}^{-1}, \psi_{F_v} \circ Tr_{E_w/F_v}), & \text{if } v \text{ is inert in } E \\ \epsilon_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_1}\mu_{jw_1}^{-1}, \psi_{F_{w_1}})\epsilon_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_2}\mu_{jw_2}^{-1}, \psi_{F_{w_2}}), & \text{if } v \text{ splits in } E. \end{cases}$$

(2) The case $\alpha = e_i + e_j$ is similar.

(3) The case $\alpha = 2e_i$. In this case,

$$L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee) = \begin{cases} L_F(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw}|_{F_v^\times}), & \text{if } v \text{ is inert in } E \\ L_F(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_1}\mu_{iw_2}), & \text{if } v \text{ splits in } E, \end{cases}$$

$$\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee, \psi_{F_v}) = \begin{cases} \epsilon_F(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw}|_{F_v^\times}, \psi_{F_v}), & \text{if } v \text{ is inert in } E \\ \epsilon_E(\langle \Lambda, \alpha^\vee \rangle, \mu_{iw_1}\mu_{iw_2}, \psi_{F_v}), & \text{if } v \text{ splits in } E. \end{cases}$$

We normalize the intertwining operators $A(w, \Lambda, \chi_v)$ for all v by

$$A(w, \Lambda, \chi_v) = r_v(w) R(w, \Lambda, \chi_v).$$

Let $R(w, \Lambda, \chi) = \otimes_v R(w, \Lambda, \chi_v)$ and

$$r(w) = \Pi_v r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{L(\langle \Lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee)\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.$$

$R(w, \Lambda, \chi)$ satisfies the functional equation

$$R(w_1 w_2, \Lambda, \chi) = R(w_1, w_2\Lambda, w_2\chi) R(w_2, \Lambda, \chi)$$

for any w_1, w_2. Because of the normalization of the intertwining operators,

$$A(w, \Lambda, \chi_v) \prod_{\alpha>0, w\alpha<0} L_v(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)^{-1}$$

is holomorphic for any v. So for any v, $R(w,\Lambda,\chi_v)$ is holomorphic for each Λ with $Re(\langle\Lambda,\alpha^\vee\rangle) > -1$ and for all positive α with $w\alpha < 0$.

For $\alpha \in \Phi^+$, let $S_\alpha = \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} | \langle\Lambda,\alpha^\vee\rangle = 1\}$. We call S_α a singular hyperplane. We say that $E(g,f,\Lambda)$ has a pole of order l at Λ_0 if Λ_0 is the intersection of l singular hyperplanes in general position on which the Eisenstein series has a simple pole.

For $\Psi \subset \Phi^+$, we define $r(w,\Lambda,\Psi)$ by

$$r(w,\Lambda,\Psi) = \prod_{\alpha\in\Psi, w\alpha<0} \frac{L(\langle\Lambda,\alpha^\vee\rangle, \chi\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1,\chi\circ\alpha^\vee)\epsilon(\langle\Lambda,\alpha^\vee\rangle,\chi\circ\alpha^\vee)}.$$

Observe that we have suppressed the dependence of $r(w,\Lambda,\Psi)$ on χ.

4.2. **Definition of pseudo-Eisenstein series.** In order to apply the results in [28], we introduce pseudo-Eisenstein series, following Mœglin. For T a maximal split torus, a character χ of $T(F)\backslash T(\mathbb{A}_F)$ defines a cuspidal representation of T. For any $w \in W$, $wTw^{-1} = T$ and so $(T, w\chi)$ is conjugate to (T,χ). Let $I(\chi)$ be the set of entire functions ϕ of Paley-Wiener type such that $\phi(\Lambda) \in I(\Lambda,\chi)$ for each Λ. Let

$$\theta_\phi(g) = \left(\frac{1}{2\pi i}\right)^n \int_{Re\Lambda=\Lambda_0} E(g,\phi(\Lambda),\Lambda)\, d\Lambda,$$

where $\Lambda_0 \in \rho_B + C^+$. Let

$$L^2(G(F)\backslash G(\mathbb{A}))_{(T,\chi)},$$

be the space spanned by θ_ϕ for all $\phi \in I(w\chi)$ as $w\chi$ runs through all distinct conjugates of χ. Let $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(T,\chi)}$ be the discrete part of $L^2(G(F)\backslash G(\mathbb{A}))_{(T,\chi)}$. It is the set of iterated residues of $E(g,\phi(\lambda),\lambda)$ of order n and the residual spectrum attached to (T,χ). In order to decompose $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(T,\chi)}$, we use the inner product formula of two pseudo-Eisenstein series: Let χ and χ' be conjugate characters and $\phi \in I(\chi)$, $\phi' \in I(\chi')$. Then

$$\begin{aligned}\langle\theta_\phi,\theta_{\phi'}\rangle &= \tfrac{1}{(2\pi i)^n}\textstyle\int_{Re\Lambda=\Lambda_0}\sum_{w\in W(\chi,\chi')}(M(w^{-1},-w\bar\Lambda,w\chi)\phi'(-w\bar\Lambda),\phi(\Lambda))\,d\Lambda\\ &= \tfrac{1}{(2\pi i)^n}\textstyle\int_{Re\Lambda=\Lambda_0}\sum_{w\in W(\chi,\chi')}(M(w,\Lambda,\chi)\phi(\Lambda),\phi'(-w\bar\Lambda))\,d\Lambda,\end{aligned}$$

where $W(\chi,\chi') = \{w \in W | \, w\chi = \chi'\}$. Let D be the set of distinguished coset representatives in Proposition 4.9. Then $\{d\chi | \, d \in D\}$ is the set of distinct conjugates of χ.

In order to deal with the distinct conjugates of χ simultaneously, we consider

$$\sum_{d\in D}\left(\sum_{w\in W(\chi,d\chi)} M(w^{-1},w\Lambda,w\chi)\phi_d(w\Lambda)\right),$$

where $\phi_d \in I(d\chi)$. Since $W = \cup_{d\in D} W(\chi, d\chi)$, for simplicity, we write it as

$$E_0^{PS}(\phi,\Lambda,\chi) = \sum_{w\in W} M(w^{-1}, w\Lambda, w\chi)\phi(-w\Lambda).$$

Here $M(w^{-1}, w\Lambda, w\chi) = r(w, -\Lambda, \chi)R(w^{-1}, w\Lambda, w\chi)$, since $r(w, -\Lambda, \chi) = r(w^{-1}, w\Lambda, w\chi)$. We will take the iterated residues of $E_0^{PS}(\phi,\Lambda,\chi)$ in Section 4.4. We first review Mœglin's results on the residual spectrum of split classical groups attached to the trivial character of the maximal torus.

4.3. **Review of Mœglin's results.** In order to apply Mœglin's results, we recall some definitions. Let ${}^LG^0 = O(2n, \mathbb{C}), Sp(2n, \mathbb{C})$. Recall that unipotent orbits in ${}^LG^0$ are in 1 to 1 correspondence with partitions $(1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots)$ of $2n$ such that r_i is even for even i in the orthogonal case and for odd i in the symplectic case. For a unipotent orbit O in ${}^LG^0$, Mœglin formed a set $P(O)$ of ordered partitions as follows:

$\mathfrak{p} = (p_1, \ldots, p_r; q_1, \ldots, q_s) \in P(O)$ if and only if

(1) $(p_1, p_1, \ldots, p_r, p_r, q_1, \ldots, q_s)$ is O if we ignore the order.
(2) q_i are distinct and odd in the orthogonal case and even in the symplectic case.
(3) For all $\leq j \leq [\frac{s+1}{2}]$, $q_{2j-1} > q_{2j}$ and there does not exist $1 \leq k \leq [\frac{s+1}{2}]$ such that $q_{2j-1} > q_{2k-1} > q_{2j} > q_{2k}$.
(4) If there exists a $1 \leq k \leq s$ such that $q_{2j-1} > q_k > q_{2j}$, then $k < 2j - 1$.

We set $q_{s+1} = 0$ if s is odd. We can put an equivalence relation on $P(O)$ as follows: For $\mathfrak{p} = (p_1, \ldots, p_r; q_1, \ldots, q_s), \mathfrak{p}' = (p'_1, \ldots, p'_r; q'_1, \ldots, q'_s) \in P(O)$, $\mathfrak{p} \simeq \mathfrak{p}'$ if and only if for all $1 \leq i \leq [\frac{s+1}{2}]$, there exists $1 \leq j \leq [\frac{s+1}{2}]$ such that $q_{2i-1} = q'_{2j-1}, q_{2i} = q'_{2j}$. We note that $\{p_1, \ldots, p_r\} = \{p'_1, \ldots, p'_r\}$ as sets. We note that

Remark 4.1. For a distinguished unipotent orbit, we have $r = 0$. In that case, we write $\mathfrak{p} = (; q_1, \ldots, q_s)$.

Example 4.2. For a unipotent orbit of the form $(7, 5, 3, 1)$ in $O(16, \mathbb{C})$, there are two nonequivalent elements in $P(O)$, namely, (;7,5,3,1) and (;5,3,7,1).

For $\mathfrak{p} = (p_1, \ldots, p_r; q_1, \ldots, q_s)$, we set, for $2 \leq i \leq r$, $p'_i = p_1 + \cdots + p_{i-1}$ and $p'_1 = 0$, and for $1 \leq i \leq [\frac{s+1}{2}]$

$$T_i^d = \sum_{j=1}^r p_j + \sum_{1\leq l<i} \frac{q_{2l-1} + q_{2l}}{2},$$

$$T_i^f = \sum_{j=1}^r p_j + \sum_{1\leq l\leq i} \frac{q_{2l-1} + q_{2l}}{2}.$$

We recall the definition of $\Lambda_{\mathfrak{p}}$ and $w_{\mathfrak{p}}$: $\Lambda_{\mathfrak{p}} = (\Lambda_{\mathfrak{p},1}, \dots, \Lambda_{\mathfrak{p},n})$, where

$$\Lambda_{\mathfrak{p},p'_i+t} = \frac{p_i+1}{2} - t, \text{for } 1 \le i \le r \text{ and } 1 \le t \le p_i,$$
$$\Lambda_{\mathfrak{p},T^d_k+t} = \frac{q_{2k-1}+1}{2} - t, \text{for } 1 \le k \le [\frac{s+1}{2}] \text{ and } 1 \le t \le \frac{q_{2k_1}+q_{2k}}{2},$$

and $w_{\mathfrak{p}}$ is an element of the Weyl group given by:

$$w_{\mathfrak{p}}(p'_i+t) = p'_{i+1} - t + 1, \text{for } 1 \le i \le r \text{ and } 1 \le t \le p_i,$$
$$w_{\mathfrak{p}}(t) = -t, \text{for } 1 \le k \le [\frac{s+1}{2}] \text{ and } T^d_k < t \le T^d_k + \frac{q_{2k_1}-q_{2k}}{2},$$
$$w_{\mathfrak{p}}(T^d_k + \frac{q_{2k-1}-q_{2k}}{2} + t) = T^f_k - t + 1, \text{for } 1 \le k \le [\frac{s+1}{2}] \text{ and } 1 \le t \le q_{2k}.$$

Remark 4.3. All $\Lambda_{\mathfrak{p}}$ are conjugates and $w_{\mathfrak{p}}^2 = 1$. Let Λ_O be the conjugate of $\Lambda_{\mathfrak{p}}$ which is in the closure of the positive Weyl chamber.

We also define $\sigma(p_i)$ for $1 \le i \le r$ and σ_k for $1 \le k \le [\frac{s}{2}]$ and denote $Stab(\Lambda_{\mathfrak{p}}, \uparrow \mathfrak{p})$ be the subgroup of $Stab(\Lambda_{\mathfrak{p}})$ generated by these elements:

$$\sigma(p_i)(j) = j, \text{if } j \notin [p'_i+1, p'_{i+1}],$$
$$\sigma(p_i)(p'_i+t) = -(p'_{i+1} - t + 1), \text{if } t \in [1, p_i],$$
$$\sigma_k(j) = j, \text{if } j \notin [T^d_k + \frac{q_{2k-1}-q_{2k}}{2} + 1, T^f_k],$$
$$\sigma_k(T^d_k + \frac{q_{2k-1}-q_{2k}}{2} + t) = -(T^f_k - t + 1), \text{if } t \in [1, q_{2k}].$$

Let $A(O)$ be a finite abelian group generated by the order two elements $\sigma(p_1), \dots, \sigma(p_r), \sigma(q_1), \dots, \sigma(q_s)$. Let $\bar{A}(\mathfrak{p}) = A(O)/K_{\mathfrak{p}}$, where $K_{\mathfrak{p}}$ is generated by
$\sigma(q_{2i-1})\sigma(q_{2i})^{-1}$ for all $1 \le i \le [\frac{s+1}{2}]$. We set $\sigma(q_{s+1}) = 1$ if s is odd. We note that $|\bar{A}(\mathfrak{p})| = 2^{[\frac{s}{2}]}$.

4.3.1. *Local Theory in the split group case.* Let G be a split classical group $O(2n), O(2n+1)$ and let $I_v(\Lambda_{\mathfrak{p}}) = Ind_B^G\, exp(\langle \Lambda_{\mathfrak{p}}, H_B() \rangle)$ (normalized induction). The normalized intertwining operator $R_v(w_{\mathfrak{p}}, \Lambda)$ is not holomorphic at $\Lambda_{\mathfrak{p}}$ in general. Mœglin defined $R_v(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})$ as composition of several operators. Then we have:

(1) $R_v(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) I_v(\Lambda_{\mathfrak{p}})$ is a direct sum of $|\bar{A}(\mathfrak{p})|$ irreducible representations with multiplicity 1. Let $Unip(\mathfrak{p})$ be the set of the irreducible direct summands and $Unip(O) = \cup_{\mathfrak{p} \in P(O)} Unip(\mathfrak{p})$. Then the Iwahori-Matsumoto involution of elements in $Unip(O)$ is tempered.
(2) If $r = 0$, i.e., O is a distinguished unipotent orbit, then the Iwahori-Matsumoto involution of elements in $Unip(\mathfrak{p})$ is square integrable.

(3) $Unip(O)$ is exactly the set of irreducible representations of $G(n)$ whose infinitesimal character is Λ_O and whose Iwahori-Matsumoto involution is tempered.

(4) $Springer(O) \simeq \cup_{\mathfrak{p}\in P(O)}\widehat{\bar{A}(\mathfrak{p})}$, where $\widehat{\bar{A}(\mathfrak{p})}$ is the character group of $\bar{A}(\mathfrak{p})$. We recall that the Springer correspondence is an injective map from the characters of W, the Weyl group of ${}^LG^0$ into the set of pairs (O,η), where O is a unipotent orbit in ${}^LG^0$ and η is a character of $A(O)$. Given a unipotent orbit O in ${}^LG^0$, $Springer(O)$ is the set of characters of $A(O)$ which are in the image of the Springer correspondence. We also note that if $O = (q_1,\dots,q_s)$ is a distinguished unipotent orbit, $|Springer(O)| = {}_sC_{[\frac{s}{2}]}$ (see [19]).

(5) For each $\sigma \in Stab(\Lambda_\mathfrak{p},\uparrow \mathfrak{p})$, there is an intertwining operator $R_v(\sigma)$ of $R_v(w_\mathfrak{p},\Lambda_\mathfrak{p})I_v(\Lambda_\mathfrak{p})$ and $\sigma \longmapsto R_v(\sigma)$ is a homomorphism of groups.

(6) $\bar{A}(\mathfrak{p})$ is isomorphic to the quotient of $Stab(\Lambda_\mathfrak{p},\uparrow \mathfrak{p})$ by the subgroup generated by $\sigma(p_i)\sigma(p_j)^{-1}$ for $p_i = p_j$ and $\sigma(p_i)\sigma_k^{-1}$ for $p_i = q_{2k-1}$ or $p_i = q_{2k}$.

(7) For $X \in Unip(\mathfrak{p})$, let $R_v(\sigma)X = \eta^\mathfrak{p}_X(\sigma)X$. Then $\eta^\mathfrak{p}_X$ defines a character of $Stab(\Lambda_\mathfrak{p},\uparrow \mathfrak{p})$.

(8) By passing to quotient, $X \to \eta^\mathfrak{p}_X$ gives an isomorphism $Unip(\mathfrak{p}) \simeq \widehat{\bar{A}(\mathfrak{p})}$ which is extended canonically to

$$Unip(O) \simeq Springer(O),$$

by $X \to \eta_X$ in the sense that $|Unip(\mathfrak{p})\cap Unip(\mathfrak{p}')| = |\widehat{\bar{A}(\mathfrak{p})}\cap\widehat{\bar{A}(\mathfrak{p}')}|$ and for $X \in Unip(\mathfrak{p}) \cap Unip(\mathfrak{p}')$, $\eta^\mathfrak{p}_X = \eta^{\mathfrak{p}'}_X$.

(9) If $\mathfrak{p} \simeq \mathfrak{p}'$, then $Unip(\mathfrak{p}) \simeq \widehat{\bar{A}(\mathfrak{p})} = \widehat{\bar{A}(\mathfrak{p}')} \simeq Unip(\mathfrak{p}')$. In other words, up to isomorphism, $Unip(\mathfrak{p})$ depends only on the equivalence class of $\mathfrak{p}$.

4.3.2. *Global Theory for the split group case.* We only look at the residual spectrum. So in this section O will be a distinguished unipotent orbit, i.e., $r = 0$. Let $\mathfrak{p} = (;q_1,q_2,\dots,q_s) \in P(O)$. Let $S_\mathfrak{p}$ be the set of positive roots defined as follows:

$$\begin{cases} e_j - e_{j+1} & \text{for } T_i^d < j \le T_i^f - 1 \text{ and } e_{T_i^d+\frac{q_{2i}-q_{2i-1}}{2}} + e_{T_i^f}, \text{ where } 1 \le i \le [\dfrac{s}{2}] \\ e_j - e_{j+1} & \text{for } T^d_{\frac{s+1}{2}} < j < n \text{ and } 2e_n \text{ if } s \text{ is odd and } q_s > 1. \end{cases}$$

We note that $S_\mathfrak{p} \subset \{\alpha > 0 \mid w_\mathfrak{p}\alpha < 0,\ \langle\Lambda_\mathfrak{p},\alpha^\vee\rangle = 1\}$ and $S_\mathfrak{p}$ has exactly n elements. We will take the iterated residue of the Eisenstein series along the n singular hyperplanes $\langle\Lambda_\mathfrak{p},\alpha^\vee\rangle = 1$ for $\alpha \in S_\mathfrak{p}$.

Definition 4.4. For $\mathfrak{p} = (;q_1,q_2,\dots,q_s) \in P(O)$, we define

$$M'_\mathfrak{p} = GL(\tfrac{q_1+q_2}{2}) \times \cdots \times GL(\tfrac{q_{2[\frac{s+1}{2}]-1}+q_{2[\frac{s+1}{2}]}}{2}).$$

If s is odd (symplectic group case), we put the convention that $\frac{q_{2[\frac{s+1}{2}]-1}+q_{2[\frac{s+1}{2}]}}{2}$ is $\frac{q_s-1}{2}$.

Definition 4.5. Let $V(\mathfrak{p})=\{\Lambda_{\mathfrak{p}}\}$ and let $V'(\mathfrak{p})$ be the set of elements of the form $\Lambda_{\mathfrak{p}}+\eta$, where η is a character of $M'_{\mathfrak{p}}(\mathbb{A})$. We note that $V'(\mathfrak{p})$ is the intersection of the singular hyperplanes $\langle\Lambda,\alpha^\vee\rangle=1$, where
$\alpha\in\{e_j-e_{j+1}$ for $T_i^d<j\le T_i^f-1$, $i=1,..,[\frac{s}{2}]$ and $T_s^f<j<n\}$.

We denote the element in $V'(\mathfrak{p})$ as

$$\Lambda_{\mathfrak{p}}(z_1,\ldots,z_{[\frac{s+1}{2}]})=\Lambda_{\mathfrak{p}}+(\underbrace{z_1,\ldots,z_1}_{\frac{q_1+q_2}{2}},\ldots,\underbrace{z_{[\frac{s+1}{2}]},\ldots,z_{[\frac{s+1}{2}]}}_{\frac{q_{2[\frac{s+1}{2}]-1}+q_{2[\frac{s+1}{2}]}}{2}}).$$

Definition 4.6. For $1\le k\le[\frac{s+1}{2}]$, we define

$$V'_k(\mathfrak{p})=\{\Lambda_{\mathfrak{p}}(z_1,\ldots,z_{[\frac{s+1}{2}]})\in V'(\mathfrak{p}),\quad \text{such that } z_i=0 \text{ for all } i>k\}.$$

In particular, $V'_0(\mathfrak{p})=V(\mathfrak{p})$ and $V'_{[\frac{s+1}{2}]}=V'(\mathfrak{p})$.

Definition 4.7. We define $W(\uparrow,\mathfrak{p})$ to be the set of the Weyl group elements which send the positive roots of $M'_{\mathfrak{p}}$ to the positive roots of $M'_{\mathfrak{p}}$.

Let

$$d(\mathfrak{p},\Lambda)=\prod_{\alpha\in S_{\mathfrak{p}}}(\langle\Lambda_{\mathfrak{p}},\alpha^\vee\rangle-1).$$

Let $Unip$ be the submodule of $\otimes'_v R_v(w_{\mathfrak{p}},\Lambda_{\mathfrak{p}})I_v(\Lambda_{\mathfrak{p}})$ which is the sum of irreducible subrepresentations of type $\otimes'_v X_v$, where $X_v\in Unip(\mathfrak{p})$ for all v and there does not exist $\mathfrak{p}'>\mathfrak{p}$ and $X_v\in Unip(\mathfrak{p}')$ for all v.

Let $proj_{[\mathfrak{p}]}$ be the $G(\mathbb{A})$-projection $\otimes'_v R_v(w_{\mathfrak{p}},\Lambda_{\mathfrak{p}})I_v(\Lambda_{\mathfrak{p}})\longrightarrow Unip$. For $\phi\in PW$, the set of Paley-Wiener type functions, let

$$l_{\mathfrak{p}}(\phi,\Lambda)=\sum_{w\in W}r(w,-\Lambda)R(w_{\mathfrak{p}}w^{-1},w\Lambda)\phi(w\Lambda).$$

Then we have

(1) $r(w_{\mathfrak{p}},\Lambda)d(\mathfrak{p},\Lambda)$ is holomorphic at $\Lambda=\Lambda_{\mathfrak{p}}$ and its value is non-zero.
(2) The poles of $l_{\mathfrak{p}}(\phi,\Lambda)$ in a neighborhood of $\Lambda_{\mathfrak{p}}$ are contained in the local intertwining operators.
(3) $r(w,-\Lambda)$ is identically zero on $V'(\mathfrak{p})$ if $w\notin W(\uparrow,\mathfrak{p})$. So the restriction of $l_{\mathfrak{p}}(\phi,\Lambda)$ to $V'(\mathfrak{p})$ is given by

$$l_{\mathfrak{p}}(\phi,\Lambda)=\sum_{w\in W(\uparrow,\mathfrak{p})}r(w,-\Lambda)R(w_{\mathfrak{p}}w^{-1},w\Lambda)\phi(w\Lambda).$$

(4) $l_{\mathfrak{p}}(\phi,\Lambda_{\mathfrak{p}})$ can be defined inductively by restricting it to $V'_k(\mathfrak{p})$ from $k=[\frac{s+1}{2}]-1$ to $k=0$ and it is the iterated residue of $E_0^{PS}(\phi,\Lambda)$.

(5) $l_{\mathfrak{p}}(\phi, \Lambda_{\mathfrak{p}}) \in \otimes'_v R_v(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) I_v(\Lambda_{\mathfrak{p}})$. This depends only on ϕ and the equivalence class of $\mathfrak{p}$. Let $l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}}) = proj_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}})$.

(6) Let $\langle , \rangle_{dis}$ be the inner product in the discrete spectrum $L^2_{dis}(G(F)\backslash G(\mathbb{A}))$. Then

$$\langle \theta_{\phi'}, \theta_{\phi} \rangle_{dis} = \sum_{O \subset G^*(n)} \sum_{\mathfrak{p} \in P(O)} c_{\mathfrak{p}} \langle \langle l'_{[\mathfrak{p}]}(\phi', \bar{\Lambda}_{\mathfrak{p}}), l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}}) \rangle \rangle,$$

where O runs through the distinguished unipotent orbits in $G^*(n)$ and $\mathfrak{p}$ runs through the set of representatives in each equivalence classes in $P(O)$.

(7) For $\phi \in PW$, suppose $l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}})$ generates an irreducible representation. Then for all v finite places, let X_v be the local representation of G_v generated by $l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}})$. Then $X_v \in Unip(\mathfrak{p})$ and $\prod_v \eta_{X_v} = 1$.

(8) Conversely, suppose $\mathfrak{p} = (; q_1, \ldots, q_s) \in P(O)$ and $\pi = \otimes'_v X_v$ be an irreducible automorphic representation which satisfies; (a) $X_v \in Unip(\mathfrak{p})$ for all v; (b) X_v is spherical almost everywhere and at archimedean places; (c) $\prod_v \eta_{X_v} = 1$. Then there exists $\phi \in PW$ such that the representation generated by $l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}})$ is isomorphic to π.

(9) In fact, for an appropriate $\phi \in PW$,

$$l_{\mathfrak{p}}(\phi, \Lambda_{\mathfrak{p}}) = \sum_{\tau \in Stab(\Lambda_{\mathfrak{p}}, \uparrow \mathfrak{p})} R(\tau^{-1}) R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}) \phi(\Lambda_{\mathfrak{p}}).$$

4.4. **Residues of the Eisenstein series.** For χ a non-trivial character of $T(F)\backslash T(\mathbb{A}_F)$, we can assume, after conjugation, that $\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{r_k}, \underbrace{\nu_1, \ldots, \nu_1}_{s_1}, \ldots, \underbrace{\nu_l, \ldots, \nu_l}_{s_l})$, where μ_i's and ν_j's are distinct grössencharacters of E such that $\mu_i|_{\mathbb{A}_F^\times} \neq 1$ for $i = 1, \ldots, k$, $\nu_j|_{\mathbb{A}_F^\times} = 1$ for $j = 1, \ldots, l$, and $r_1 + \cdots + r_k + s_1 + \cdots + s_l = n$, $r_1 \geq \ldots \geq r_k$, $s_1 \geq \ldots \geq s_l$.

Let $E(g, f, \Lambda)$ be the Eisenstein series attached to the character χ.

Proposition 4.8. *The Eisenstein series has a pole of order n only if $r_k \geq 2$, and μ_i is a grössencharacter which satisfies $\mu_i|_{N(\mathbb{A}_E^\times)} = 1$ for $i = 1, \ldots, k$.*

Proof. Similar to the proof [22, Proposition 4.2]. ☐

We divide the set of positive roots Φ^+ as follows:

$$\Phi_1 = \{e_i \pm e_j,\ 1 \le i < j \le r_1\},$$
$$\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j},\ 1 \le i < j \le r_2\},$$
$$\vdots$$
$$\Phi_k = \{e_{r_1+\cdots+r_{k-1}+i} \pm e_{r_1+\cdots+r_{k-1}+j},\ 1 \le i < j \le r_k\},$$
$$\Phi_{k+1} = \{e_{r_1+\cdots+r_k+i} \pm e_{r_1+\cdots+r_k+j}, 1 \le i < j \le s_1, 2e_{r_1+\cdots+r_k+i}, i = 1,\dots,s_1\},$$
$$\vdots$$
$$\Phi_{k+l} = \{e_{r_1+\cdots+r_k+s_1+\cdots+s_{l-1}+i} \pm e_{r_1+\cdots+r_k+s_1+\cdots+s_{l-1}+j},\ 1 \le i < j \le s_1,$$
$$2e_{r_1+\cdots+r_k+s_1+\cdots+s_{l-1}+i},\ i = 1,\dots,s_1\},$$
$$\Phi_D = \Phi^+ - \bigcup_{i=0}^{k+l} \Phi_i.$$

Let $\widetilde{W_i}$ be the Weyl group corresponding to Φ_i for $i = 1,\dots,k$ and $W_i = \widetilde{W_i} c_{r_1+\cdots+r_i}$ for $i = 1,\dots,k$. Let W_i be the Weyl group corresponding to Φ_i for $i = k+1,\dots,k+l$. We note $W(\chi,\chi) = W_1 \times \cdots \times W_{k+l}$.

Let $\Lambda = \Lambda_1 + \cdots + \Lambda_{k+l}$, where $\Lambda_i = a_{r_1+\cdots+r_{i-1}+1} e_{r_1+\cdots+r_{i-1}+1} + \cdots + a_{r_1+\cdots+r_i} e_{r_1+\cdots+r_i}$, for $i = 1,\dots,k$, and for $j = 1,\dots,l$,

$$\Lambda_{k+j} = a_{r_1+\cdots+r_k+s_1+\cdots+s_{j-1}+1} e_{r_1+\cdots+r_k+s_1+\cdots+s_{j-1}+1} + \cdots$$
$$+ a_{r_1+\cdots+r_k+s_1+\cdots+s_j} e_{r_1+\cdots+r_k+s_1+\cdots+s_j}.$$

We recall the following well-known result, see [7, p. 47].

Proposition 4.9. *Let Δ be a set of simple roots and W be the associated Weyl group. Let w_α be the simple reflection with respect to $\alpha \in \Delta$. Then W is generated by the w_α, $\alpha \in \Delta$. Let θ be a subset of Δ and W_θ be the subgroup of W generated by the w_α, $\alpha \in \theta$. Then each coset wW_θ has a unique element d_θ characterized by any of the following equivalent properties:*

(1) $d_\theta \theta > 0$,
(2) d_θ *is of minimal length in* wW_θ,
(3) *For any* $x \in W_\theta$, $l(d_\theta x) = l(d_\theta) + l(x)$.

We apply Proposition 4.9 to $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$ and

$$\theta = \Delta - \{ e_{r_1} - e_{r_1+1}, e_{r_1+r_2} - e_{r_1+r_2+1}, \dots, e_{r_1+\cdots+r_k} - e_{r_1+\cdots+r_k+1}$$
$$+ e_{r_1+\cdots+r_k+s_1} - e_{r_1+\cdots+r_k+s_1+1}, \dots, e_{r_1+\cdots+r_k+s_1+\cdots+s_l}$$
$$- e_{r_1+\cdots+r_k+s_1+\cdots+s_l+1}\}. \tag{4.1}$$

Let D be the set of such distinguished coset representatives.

For $d \in D$, and $w_i \in W_i$, $i = 1,\dots,k+l$, we have

$$\{\alpha > 0 \,|\, dw_1 \dots w_{k+l}\alpha < 0\} = \cup_{i=1}^{k+l} \{\alpha \in \Phi_i \,|\, w_i\alpha < 0\} \cup \{\alpha \in \Phi_D \,|\, dw_1 \dots w_{k+l}\alpha < 0\}.$$

Then for a Paley-Wiener type function $\phi \in I(\chi)$, the constant term of pseudo-Eisenstein series is given by

$$E_0^{PS}(\phi,\Lambda,\chi) = \sum_{w\in W} r(w,-\Lambda,\chi)R(w^{-1},w\Lambda,w\chi)\phi(w\Lambda) = \prod_{i=1}^{k+l}\sum_{w_i\in W_i} r(w_i,-\Lambda_i,\Phi_i)$$

$$\cdot\left(\sum_{d\in D} r(dw_1\dots w_{k+l},-\Lambda,\Phi_D)R(w_{k+l}^{-1}\dots w_1^{-1}d^{-1},dw_1\dots w_{k+l}\Lambda,d\chi)\phi(dw_1\dots w_{k+l}\Lambda)\right).$$

We note that $w_1\dots w_k\chi=\chi$. By the cocycle relation, we have

$$R(w_{k+l}^{-1}\dots w_1^{-1}d^{-1},dw_1\dots w_{k+l}\Lambda,d\chi) = R(w_{k+l}^{-1}\dots w_1^{-1},w_1\dots w_{k+l}\Lambda,\chi)R(d^{-1},dw_1\dots w_{k+l}\Lambda,d\chi).$$

Let

$$f(w_1\dots w_{k+l}\Lambda) = \sum_{d\in D} r(dw_1\dots w_{k+l},-\Lambda,\Phi_D) R(d^{-1},dw_1\dots w_{k+l}\Lambda,d\chi)\phi(dw_1\dots w_{k+l}\Lambda).$$

Then we have

$$(4.2)\quad E_0^{PS}(\phi,\lambda,\chi) = \prod_{i=1}^{k+l}\sum_{w_i\in W_i} r(w_i,-\Lambda_i,\Phi_i)R(w_{k+l}^{-1}\dots w_1^{-1},w_1\dots w_{k+l}\Lambda,\chi)f(w_1\dots w_{k+l}\Lambda).$$

We note that it has the same normalizing factors as the Eisenstein series of $O(2r_1),\dots,O(2r_k)$ and $O(2s_1+1),\dots,O(2s_l+1)$ attached to the trivial character.

Let O_i's be distinguished unipotent orbits in $O(2r_i,\mathbb{C})$ for $i=1,\dots,k$ and O'_j in $Sp(2s_j,\mathbb{C})$ for $j=1,\dots,l$. Let $\mathfrak{p}_i\in P(O_i)$ for $i=1,\dots,k+l$, $\mathfrak{p}=\mathfrak{p}_1\times\cdots\times\mathfrak{p}_{k+l}$ and $w_{\mathfrak{p}}=w_{\mathfrak{p}_1}\times\cdots\times w_{\mathfrak{p}_{k+l}}$ and $\Lambda_{\mathfrak{p}}=\Lambda_{\mathfrak{p}_1}+\cdots+\Lambda_{\mathfrak{p}_{k+l}}$. Let

$$l_{\mathfrak{p}}(\phi,\Lambda,\chi) = \prod_{i=1}^{k+l}\sum_{w_i\in W_i} r(w_i,-\Lambda_i,\Phi_i)R(w_{\mathfrak{p}}w_{k+l}^{-1}\dots w_1^{-1},w_1\dots w_{k+l}\Lambda,\chi)f(w_1\dots w_{k+l}\Lambda).$$

Since $r(w_i,-\lambda_i,\Phi_i)$ is identically zero on $V'(\mathfrak{p}_i)$ if $w_i\notin W(\uparrow,\mathfrak{p}_i)$, the restriction of $l_{\mathfrak{p}}(\phi,\Lambda,\chi)$ to $V'(\mathfrak{p})=V'(\mathfrak{p}_0)\times\cdots\times V'(\mathfrak{p}_k)$ is given by

$$l_{\mathfrak{p}}(\phi,\Lambda,\chi)|_{V'(\mathfrak{p})} = \prod_{i=1}^{k+l}\sum_{w_i\in W(\uparrow,\mathfrak{p}_i)} r(w_i,-\Lambda_i,\Phi_i)R(w_{\mathfrak{p}}w_{k+l}^{-1}\dots w_1^{-1},w_1\dots w_{k+l}\Lambda,\chi)f(w_1\dots w_{k+l}\Lambda).$$

We note that $f(w_1 \dots w_{k+l}\Lambda)$ is holomorphic on $V'(\mathfrak{p})$. We also note that $R(w_{\mathfrak{p}}, \Lambda, \chi_v)$ is Not holomorphic at $\Lambda_{\mathfrak{p}}$. We now suppose the local problem is solved, i.e., we can define the local intertwining operator $R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi_v)$ as composition of several operators as in [13]. We apply Mœglin's results and define $l_{\mathfrak{p}}(\phi, \Lambda_{\mathfrak{p}}, \chi)$ inductively. But the order of induction will matter. Among $V'(\mathfrak{p}_i)$'s, we can shuffle the segments. By shuffling, we mean the following:

Let $\mathfrak{p}_1 = (; q_1, \dots, q_s)$ and $\mathfrak{p}_2 = (; q_1', \dots, q_t')$ be two chains. By shuffling of $\mathfrak{p}_1 \times \mathfrak{p}_2$, we mean any permutation on segments so that

(1) $(q_1, q_2), \dots, (q_{2[\frac{s-1}{2}]-1}, q_{2[\frac{s-1}{2}]})$ appear in that order, and
(2) $(q_1', q_2'), \dots, (q'_{2[\frac{t-1}{2}]-1}, q'_{2[\frac{t-1}{2}]})$ appear in that order.

Take a shuffling of segments in such a way that it satisfies a certain condition, that is, the non-vanishing of the normalized intertwining operators, see [13]. It will correspond to starting with a conjugate of χ. If there is no confusion, we will still write it as χ. Then

$$\langle \theta_{\phi'}, \theta_{\phi} \rangle_{dis} = \sum_{i=1}^{k+l} \sum_{O_i} \sum_{\mathfrak{p}} c_{\mathfrak{p}} \langle \langle l'_{[\mathfrak{p}]}(\phi', \bar{\Lambda}_{\mathfrak{p}}, \chi), l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}}, \chi) \rangle \rangle,$$

where O_i runs through distinguished unipotent orbits in $O(2r_i, \mathbb{C})$ for $i = 1, .., k$ and $Sp(2s_i, \mathbb{C})$ for $i = k+1, \dots, k+l$. $\mathfrak{p} = \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_{k+l} \in P(O_1) \times \cdots \times P(O_{k+l})$. We have

$$l_{[\mathfrak{p}]}(\phi, \Lambda_{\mathfrak{p}}, \chi) \in \otimes_v' R_v(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi) I_v(\Lambda_{\mathfrak{p}}, \chi_v).$$

We need to analyze the image of the local intertwining operator $R_v(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi)$ $I_v(\Lambda_{\mathfrak{p}}, \chi_v)$. We give a conjecture on this in the next section.

4.5. **Conjecture on the local problem.** We can define $R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi_v)$ as a composition of several operators ([28, 13]). Then we need to study its image $R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi_v) I_v(\Lambda_{\mathfrak{p}}, \chi_v)$. We hope that the technique of [13] in the case of symplectic and odd orthogonal groups may be applied in our case.

Case 1. v splits in E. In this case $G_v = GL(2n, F_v)$. We expect that $R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi_v) I_v(\Lambda_{\mathfrak{p}}, \chi_v)$ is irreducible.

Case 2. v is inert in E. Then in light of [13], we expect the following:

Conjecture 4.10. (1) $R(w_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \chi_v) I_v(\Lambda_{\mathfrak{p}}, \chi_v)$ is semi-simple;
(2) Let $Unip(\mathfrak{p}, \chi_v)$ be the set of direct summands of $R_v(w_{\mathfrak{p}}, \lambda_{\mathfrak{p}}, \chi) I_v(\lambda_{\mathfrak{p}}, \chi_v)$ and $Unip(O_1, \dots, O_{k+l}, \chi_v)$ be the set of union of $Unip(\mathfrak{p}, \chi_v)$ as $\mathfrak{p}_i$ runs through $P(O_i)$ for $i = 1, \dots, k+l$. Then $Unip(O_1, \dots, O_{k+l}, \chi_v)$ is parametrized by

$$C(O_1, \dots, O_{k+l}, \chi_v) =$$
$$[Springer(O_1) \times \cdots \times Springer(O_k)] \times [Springer(O_{k+1} \times \cdots \times Springer(O_{k+l})],$$

where in [ ]: If $\mu_{1v} = \mu_{2v} \neq \mu_{iv}$ for $i = 0, 3, \ldots, k$, then we replace $Springer(O_1) \times Springer(O_2)$ by

$$C(O_1, O_2, \mu_{1v}) \\ =\{\eta \in Springer(O) : \ \eta|_{A(O_i)} \in Springer(O_i), \text{ for } i = 1, 2\},$$

where O is a unipotent orbit of $O(2(r_1 + r_2), \mathbb{C})$ by combining O_1, O_2.

In other words, $C(O_1, \ldots, O_{k+l}, \chi_v) \subset C_{\psi_v}$, and

$\Pi_{res_v} = Unip(O_1, \ldots, O_{k+l}, \chi_v) = \{\pi(\psi_v, \eta_v)|\ \eta_v \in C(O_1, \ldots, O_{k+l}, \chi_v)\}$.

Example 4.11. Let $\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{10}, \underbrace{\mu_2, \ldots, \mu_2}_{4})$, μ_1 and μ_2 are distinct quadratic grössencharacter of E such that $\mu_i|_{N(\mathbb{A}_E^\times)} = 1$ but $\mu_i|_{\mathbb{A}_F^\times} \neq 1$. Let $O_1 = (7, 5, 3, 1)$ is a unipotent orbit in $O(16, \mathbb{C})$ and $O_2 = (5, 3)$ is a unipotent orbit in $O(8, \mathbb{C})$. Then for a non-archimedean place v, if $\mu_{1v} \neq \mu_{2v}$, then Π_{res_v} is parametrized by $Springer(O_1) \times Springer(O_2)$. It has 12 elements. Let $\mu_{1v} = \mu_{2v}$. Let $O = (7, 5, 5, 3, 3, 1)$. Then $A(O)$ is an abelian group generated by order 2 elements $\sigma(1), \sigma(3), \sigma(5), \sigma(7)$. Thus $Springer(O) = \{\eta \in \widehat{A(O)} : \eta(\sigma(7)) = \eta(\sigma(1))\}$. Therefore, $C(O_1, O_2, \chi_v) = \{\eta \in Springer(O) : \ \eta|_{A(O_i)} \in Springer(O_i), i = 1, 2\}$ and $C(O_1, O_2, \chi_v)$ has 4 elements.

$\sigma(7)$	1	-1	1	-1
$\sigma(5)$	1	1	-1	-1
$\sigma(3)$	1	1	-1	-1
$\sigma(1)$	1	-1	1	-1

Remark 4.12. Special case; the result of Kudla-Sweet [25].

Let $\chi = \chi(\mu, \ldots, \mu)$, where μ is a grössencharacter of E such that (1) $\mu|_{N(\mathbb{A}_E^\times)} = 1$, $\mu|_{\mathbb{A}_F^\times} \neq 1$ or (2) $\mu|_{\mathbb{A}_F^\times} = 1$. Let O be a distinguished unipotent orbit of the form (p, q) in $O(2n, \mathbb{C})$ in the first case and (p, q) or (p) in $Sp(2n, \mathbb{C})$ in the second case. Then the above local problem is solved in [25].

4.6. **Conjecture on the residual spectrum.** In light of the local conjecture, we have the following description of the residual spectrum coming from Borel subgroups.

Conjecture 4.13. Let $\mu_1, \ldots, \mu_k$ be distinct grössencharacters of E such that $\mu_i|_{N(\mathbb{A}_E^\times)} = 1$ but $\mu_i|_{\mathbb{A}_F^\times} \neq 1$. Let $\nu_1, \ldots, \nu_l$ be distinct grössencharacters of E such that $\nu_j|_{\mathbb{A}_F^\times} = 1$. Then the character χ is given as follows:

$$\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{r_k}, \underbrace{\nu_1, \ldots, \nu_1}_{s_1}, \ldots, \underbrace{\nu_l, \ldots, \nu_l}_{s_l}),$$

where $r_1 + \cdots + r_k + s_1 + \cdots + s_l = n$, and the residual spectrum attached to the conjugacy class of (T, χ) is parametrized by the distinguished unipotent orbits in $O(2r_i,\mathbb{C})$, $i=1,\ldots,k$, and $Sp(2s_j,\mathbb{C})$, $j=1,\ldots,l$. More specifically, distinguished unipotent orbits $O_i\in O_{2r_i}(\mathbb{C})$, $i=1,\ldots,k$, $O_0\in O_{2r_0+1}(\mathbb{C})$, and χ give a quadratic unipotent Arthur parameter ψ. Let $C_{res_v}=[Springer(O_1)\times\cdots\times Springer(O_{k+l})]\subset C_{\psi_v}$ and $\Pi_{res_v}=Unip(O_1,\ldots,O_{k+l},\chi_v)\subset\Pi_{\psi_v}$ for all non-archimedean places. For each $X\in\Pi_{res_v}$, there is a character $\eta_X\in C_{res_v}$ which satisfies Arthur's conjecture, i.e.,

$$L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(T,\chi)}\cap L^2(G(F)\backslash G(\mathbb{A}))_\psi,$$

is the set of $\pi=\otimes'_v X_v$, where X_v satisfies the following conditions:

(1) there exists $\mathfrak{p}_i\in P(O_i)$, $i=1,\ldots,k+l$, such that $X_v\in Unip_v(\mathfrak{p}_1,\ldots,\mathfrak{p}_{k+l},\chi_v)$ for all v.
(2) X_v is spherical for almost all v and archimedean places.
(3) $\prod_v\eta_{X_v}$ is trivial on C_ψ.

5. **Special case as in** [22]

We restrict ourselves to the case where the unipotent orbits $O_i\subset O(2r_i,\mathbb{C})$ have Jordan blocks $(2r_i-1,1)$ for $i=1,\ldots,k$, and $O'_j\subset Sp(2s_j,\mathbb{C})$ have Jordan blocks $(2s_j)$ for $j=1,\ldots,l$, i.e., the ones with the most weighted Dynkin diagrams (cf. [7]). We construct the representations in $\Pi_{\phi_{\psi_v}}$ as local components of the residue of the Eisenstein series and show that they satisfy Arthur's conjecture.

In this case, each $P(O_i)$, $i=1,\ldots,k+l$, has only one element $\mathfrak{p}_i$ and $\Lambda_{\mathfrak{p}_i}=\Lambda_{i,0}$ is the half–sum of (positive) roots in $\Phi_i, i=1,\ldots,k+l$. Let $\Lambda_0=\Lambda_{1,0}+\ldots+\Lambda_{k+l,0}$. Recall the following two lemmas from [22]. Let C be the set spanned by $c_{r_1},c_{r_1+r_2},\ldots,c_{r_1+\cdots+r_k}$, where c_i's are sign changes in the Weyl group: its action on $\Phi(G,T_d)$ takes e_i to $-e_i$.

Lemma 5.1 ([22, Lemma 4.7]). *$\tilde{D}=DC$ is the set of distinguished coset representatives for $W/W_1\ldots W_{k+l}$, i.e., $d\in\tilde{D}$ if and only if $d(\Phi_1\cup\ldots\cup\Phi_{k+l})>0$.*

Lemma 5.2 ([22, Lemma 4.8]). *For each $d\in D$,*

$$r(dcw_0,\Lambda_0,\Phi_D)=r(dc'w_0,\Lambda_0,\Phi_D),$$

for any $c,c'\in C$.

By Lemma 5.1, for $d\in D$, $c\in C$ and $w_i\in W_i$ for $i=1,\ldots,k+l$, we have

$$\{\alpha>0|dcw_1\ldots w_{k+l}\alpha<0\}=\cup_{i=1}^{k+l}\{\alpha\in\Phi_i|\,w_i\alpha<0\}\cup\{\alpha\in\Phi_D|\,dcw_1\ldots w_{k+l}\alpha<0\}.$$

Then the constant term of pseudo-Eisenstein series is given by

$$E_0^{PS}(\phi,\Lambda,\chi)=\prod_{i=1}^{k}\sum_{w_i\in\widetilde{W_i}}\prod_{i=k+1}^{k+l}\sum_{w_i\in W_i}r(w_i,-\Lambda_i,\Phi_i)$$
$$\cdot\left(\sum_{d\in D}\sum_{c\in C}r(dw_1\dots w_{k+l},-\Lambda,\Phi_D)R(w_{k+l}^{-1}\dots w_1^{-1}c^{-1}d^{-1},dcw_1\dots w_{k+l}\Lambda,d\chi)\right.$$
$$\left.\phi(dcw_1\dots w_{k+l}\Lambda)\right).$$

We note that $cw_1\dots w_k\chi=\chi$. Let

$$f(cw_1\dots w_{k+l}\Lambda)=\sum_{d\in D}r(dcw_1\dots w_{k+l},-\Lambda,\Phi_D)$$
$$R(d^{-1},dcw_1\dots w_{k+l}\Lambda,d\chi)\phi(dcw_1\dots w_{k+l}\Lambda).$$

Let $V'_{i0}=\{\Lambda_i|\,\langle\Lambda_i,\alpha^\vee\rangle=1$ for $\alpha\in\theta\cap\Phi_i\}$ and $V'_0=\{\Lambda\in\mathfrak{a}^*_{\mathbb{C}}|\,\Lambda=\Lambda_1+\cdots+\Lambda_{k+l},\Lambda_i\in V'_{i0}\}$. We note that $V'_0=\{\Lambda\in\mathfrak{a}^*_{\mathbb{C}}|\,\langle\Lambda,\alpha^\vee\rangle=1$ for $\alpha\in\theta\}$.

Let $V_{i0}=\{\Lambda_i|\,\langle\Lambda_i,\alpha^\vee\rangle=1$ for α simple roots in $\Phi_i\}$ and $V_0=\{\Lambda\in\mathfrak{a}^*_{\mathbb{C}}|\,\Lambda=\Lambda_1+\cdots+\Lambda_{k+l},\Lambda_i\in V_{i0}\}$. We note that $V_0=\{\Lambda_0\}$. Since for each $d\in D$, $d\alpha>0$ for all $\alpha\in\theta$, $f(cw_1\dots w_{k+l}\Lambda)$ is holomorphic on V'_0. Therefore we can take iterated residue of $E_0^{PS}(\phi,\Lambda,\chi)$ at Λ_0. Let $l_0(\phi,\Lambda_0,\chi)$ be the iterated residue. Since $c\Lambda_0=\Lambda_0$, by Lemma 5.2, it is given by

(5.1)
$$l_0(\phi,\Lambda_0,\chi)=R(w_0,\Lambda_0,\chi)\prod_{i=1}^{k}(1+R(c_{r_1+\cdots+r_i},\Lambda_0,\chi))f(\Lambda_0)\in\otimes'_vR(w_0,\Lambda_0,\chi_v),$$

where $w_0=w_{1,0}\dots w_{k+l,0}$ and $w_{i,0}$ is the longest element in $\widetilde{W_i}$ for $i=1,\dots,k$ and W_i for $i=k+1,\dots,k+l$. $l_0(\phi,\Lambda_0,\chi)$ spans the part of the residual spectrum attached to Λ_0. Here we note that we defined the local intertwining operator $R(w_0,\Lambda_0,\chi_v)$ on $Ind_M^G\Lambda_0\otimes Ind_B^M\chi_v$, where M is defined below and $R(c_{r_1+\cdots+r_i})=R(c_{r_1+\cdots+r_i},\Lambda_0,\chi_v)$ defines a self intertwining operator for $Ind_B^M\chi_v$. Recall from Remark 3.4 the associated Langlands' parameter ϕ_ψ, i.e. the homomorphism $\phi_\psi:W_F\longrightarrow{}^LG$ defined by

$$\phi_\psi(w)=\psi\left(w,\begin{pmatrix}|w|^{\frac12}&0\\0&|w|^{-\frac12}\end{pmatrix}\right).$$

Its non–tempered part is $\phi_\psi^+=\exp\langle\Lambda_0,H_B(\quad)\rangle$.

Let ${}^LM^0=Cent(im\phi_\psi^+,{}^LG^0)$. Since $(\Lambda_0,e_i)=0$ for $i=r_1,r_1+r_2,\dots,r_1+\cdots+r_k$, the Levi subgroup M which has ${}^LM^0$ as its L–group, will be, up to isomorphism, of the form $Res_{E/F}GL_{n_1}\times\cdots\times Res_{E/F}GL_{n_r}\times U(k,k)$, where

$n_1, \ldots, n_r$ are determined by Λ_0. The parameter Λ_0 is in the positive Weyl chamber of the split component of M.

We need to analyze the image of the intertwining operator

$$\otimes'_v R(w_0, \Lambda_0, \chi_v) I(\Lambda_0, \chi_v).$$

For each place v, decompose ϕ_{ψ_v} as $\phi_{\psi_v} = \phi^\circ_{\psi_v} \cdot \phi^+_{\psi_v}$ as in [2]. The parameter $\phi^\circ_{\psi_v}$ factors through ${}^L M^0$ and is the Langlands parameter for the (tempered) constituents of the unitary principal series $I_v = \mathrm{Ind}_{B_0(F_v)}^{M(F_v)} \chi_v = \oplus_i \pi_{v,i}$, of $M(F_v)$, where $B_0 = B \cap M$. For each $\pi_{v,i}$, let $\Pi_{v,i} = J(\pi_{v,i} \otimes \exp\langle \Lambda_0, H_P(\ )\rangle)$ be the corresponding Langlands quotient, where $P = MN$. Then for each v the L–packet parameterized by ϕ_{ψ_v} is $\Pi_{\phi_{\psi_v}} = \{\Pi_{v,i}\}$. The R–group for the parameter ϕ_{ψ_v}, i.e. $C_{\phi_{\psi_v}}$ is the same as the R–group of I_v for each v in the sense of Knapp–Stein. We divide into two cases:

Case 1. v is inert in E. By [15, Theorem 3.7], the R–group $C_{\phi_{\psi_v}}$ of I_v is a subgroup of the group generated by the sign changes $c_i, i = r_1, r_1 + r_2, r_1 + \cdots + r_k$, a product of 2–groups. In fact, it is generated by the sign changes $c_{r_1+\cdots+r_i}$ for which μ_{iv}'s are distinct and $\mu_{iv}|_{F_v^\times}$ is non-trivial. Moreover, if the sign change $c_{r_1+\ldots+r_i}$ in (5.1) does not belong to $C_{\phi_{\psi_v}}$ for some i, then the normalized operator $R(c_{r_1+\ldots+r_i})$ acts like identity.

Case 2. v splits in E. In this case, $C_{\phi_{\psi_v}} = 1$.

Let $\pi(\chi_v) = \{\pi_{v,i}\}$. Then, given a place v, Keys and Shahidi [16] defined a pairing $\langle\ ,\ \rangle$ on $C_{\phi_{\psi_v}} \times \pi(\chi_v)$. We extend the pairing $\langle\ ,\ \rangle$ to $C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}}$ as in Arthur [2, p. 9] by setting $\langle \tau_v, \Pi_{v,i}\rangle = \langle \tau_v, \pi_{v,i}\rangle$. This can further be extended to $C_{\psi_v} \times \Pi_{\phi_{\psi_v}}$, using the surjection $C_{\psi_v} \to C_{\phi_{\psi_v}}$ for each v [2, p. 11]. Let $\Pi = \otimes_v \Pi_{v,i}$ where almost all $\Pi_{v,i}$ are spherical. Then $\Pi \in \Pi_{\phi_\psi}$, the (global) L–packet of ϕ_ψ. Finally set $\langle \tau, \Pi\rangle = \prod_v \langle \tau_v, \Pi_{v,i}\rangle$, where τ_v is the image of τ under the map $C_\psi \to C_{\psi_v}$. As in [22, p. 421], $\langle \cdot, \Pi\rangle$ is well-defined.

Applying (5.1) to $\Pi = \otimes_v \Pi_{v,i} \in \Pi_{\phi_\psi}$ now implies that the residue is equal to

$$\sum_{d \in D} (*) R(dw_0, \Lambda_0, \chi) \sum_{x \in C_{\phi_\psi}} \langle x, \Pi\rangle \Pi.$$

It is now clear that since C_{ϕ_ψ} is abelian, the residue is non–zero if and only if $\langle \cdot, \Pi\rangle$ is the trivial character. We state this as

Theorem 5.3. Π *appears in* $L^2_{dis}(G(F)\backslash G(\mathbb{A}))$ *if and only if* $\langle \cdot, \Pi\rangle$ *is the trivial character.*

This proves the global Arthur conjecture on the multiplicity formula (3.3) for the residual spectrum. According to Conjecture 4.13, the full residual spectrum attached to the Arthur parameter ψ can be described as follows:

Let $\mu_{i_1,v}, \ldots, \mu_{i_s,v}$ be the set of distinct characters such that $\mu_{i_j,v}|_{F_v^\times}$ is non-trivial. Then if v is inert in E, $C_{\phi_{\psi_v}}$ is spanned by the order two elements

$c_{r_1+\cdots+r_{i_1}}, \ldots, c_{r_1+\cdots+r_{i_s}}$. On the other hand, by Conjecture 4.13, $C_{res_v} = C(\mu_{i_1,v})\times\ldots\times C(\mu_{i_s,v})\times C_0$, where $C(\mu_{i_j,v}) \simeq \mathbb{Z}/2\mathbb{Z}$ and C_0 is a set determined by the remaining μ_{iv}'s and O'_j, $j=1,\ldots,l$. We note that $C_{\phi_{\psi_v}} \simeq C(\mu_{i_1,v}) \times \ldots \times C(\mu_{i_s,v})$.

Example 5.4. Let $\chi = \chi(\mu_1,\mu_1,\mu_2,\mu_2)$, where μ_1,μ_2 are distinct grössencharacters of E such that $\mu_i|_{N(\mathbb{A}_F^\times)} = 1$ but $\mu_i|_{\mathbb{A}_F^\times} \neq 1$ for each $i=1,2$. Let O_1, O_2 be the distinguished unipotent (3,1) in $O(4,\mathbb{C})$. Then if v is inert in E, $\mu_{1v} \neq \mu_{2v}$ and $\mu_{iv}|_{F_v^\times}$ is non-trivial for $i=1,2$, $C_{res_v} = C_{\phi_{\psi_v}} \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. On the other hand, if v is inert in E, $\mu_{1v}|_{F_v^\times} = 1$ and $\mu_{2v}|_{F_v^\times} \neq 1$, then $C_{\phi_{\psi_v}} \simeq \mathbb{Z}/2\mathbb{Z}$ but $C_{res_v} \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. If v splits in E, $C_{\phi_{\psi_v}} = C_{res_v} = 1$.

6. Special case of $G = U(2,2)$

As we remarked in the introduction, the results in [24] are stated in such a way that it is hard to see Arthur's multiplicity formula. So we redo the calculations, using the same method as in [17]. We will get all residual spectrum, coming from the Borel subgroup.

Let $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$, $\alpha_3 = e_1 + e_2$ and $\alpha_4 = 2e_1$ be the restricted positive roots. Let $\beta_1 = e_1$. Let σ, τ be the simple reflections with respect to α_1, α_2, respectively. Let P_1 be the Siegel parabolic subgroup, i.e., $P_1 = M_1N_1$, $M_1 = Res_{E/F}GL_2$. Let P_2 be the non-Siegel parabolic subgroup, i.e., $P_2 = M_2N_2$, $M_2 = Res_{E/F}GL_1 \times U(1,1)$.

Let μ,ν be grössencharacters of E. Then $\chi = \chi(\mu,\nu)$ defines a unitary character of $T(F)\backslash T(\mathbb{A}_F)$. Let $E(g,f,\Lambda)$ be the Eisenstein series attached to χ and $f \in I(\Lambda,\chi) = Ind_B^G\, \chi\otimes exp(\langle\Lambda, H_B(\ )\rangle)$. There are 4 singular hyperplanes to consider: $S_i = \{\Lambda \in \mathfrak{a}_{\mathbb{C}}^*|\ \langle\Lambda,\alpha_i^\vee\rangle = 1\}$ for $i=1,2,3,4$.

As we see in [17], we have to consider the residue

$$Res_{\beta_1} Res_{S_1} E(g,f,\Lambda),$$
$$Res_{\rho_B} Res_{S_1} E(g,f,\Lambda),$$
$$Res_{\alpha_3} Res_{S_2} E(g,f,\Lambda).$$

Then the following characters of $T(F)\backslash T(\mathbb{A}_F)$ contribute the residual spectrum ($*$ is some constant.):

(1) $\chi = \chi(\mu,\mu)$, $\mu|_{\mathbb{A}_F^\times} = 1$. In this case, the residue at ρ_B is square integrable and it is given by

$$(*)\ R(\sigma\tau\sigma\tau, \rho_B, \chi)f.$$

(2) $\chi = \chi(\mu,\nu)$, $\mu|_{\mathbb{A}_F^\times} = 1$, $\nu|_{\mathbb{A}_F^\times} = 1$ and μ,ν are distinct. In this case, the residue at α_3 is square integrable and it is given by

$$(*)\ R(\tau\sigma\tau, \alpha_3, \chi)(1 + R(\sigma,\alpha_3,\chi))f.$$

(3) $\chi=\chi(\mu,\mu)$, $\mu|_{N(\mathbb{A}_E^\times)}=1$ but $\mu|_{\mathbb{A}_F^\times}\neq 1$. In this case, the residue at β_1 is square integrable and it is given by

$$(*)R(\sigma\tau\sigma,\beta_1,\chi)(1+R(\tau,\beta_1,\chi))f.$$

We analyze the image of the intertwining operators by inducing in stages:

Case 1. Since $\sigma\tau\sigma\tau$ is the longest element in the Weyl group, $R(\sigma\tau\sigma\tau,\rho_B,\chi_v)I(\rho_B,\chi_v)$ is the Langlands' quotient which is the one dimensional representation $\mu_v\circ det$.

Case 2. $I(\tau\sigma\tau,\alpha_3,\chi_v)=Ind_{P_1}^G\,\alpha_3\otimes Ind_B^{M_1}\,\chi_v$.

Here $\pi=Ind_B^{M_1}\,\chi_v$ is irreducible. Since $R(\sigma,\alpha_3,\chi_v)$ is a self-intertwining operator on $\pi=Ind_B^{M_1}\,\chi_v$, it acts like a scalar. But because of the normalization, it acts like 1. α_3 is in the positive Weyl chamber of the split component of M_1. Therefore the image of the intertwining operator is the Langlands' quotient $J(\pi\otimes exp\langle\alpha_3,H_{P_1}(\ )\rangle)$.

Case 3. $I(\sigma\tau\sigma,\beta_1,\chi_v)=Ind_{P_2}^G\,\beta_1\otimes Ind_B^{M_2}\,\chi_v$.

If v splits in E, then $Ind_B^{M_2}\,\chi_v$ is irreducible. If v is inert, by the result of Keys [15] on R-group, $Ind_B^{M_2}\,\chi_v$ is a sum of two irreducible representations if and only if $\mu_v|_{F_v^\times}\neq 1$. Let $Ind_{B_0}^{U(1,1)}\,\mu_v=\pi_+(\mu_v)\oplus\pi_-(\mu_v)$, as in [16], i.e., with $\pi_+(\mu_v)$ generic with respect to η_v. Let $\epsilon(\pi_+(\mu_v))=1$ and $\epsilon(\pi_-(\mu_v))=-1$. Observe that for almost all v, $\pi_+(\mu_v)$ is spherical. If $\mu_v|_{F_v^\times}=1$, then $Ind_{B_0}^{U(1,1)}\,\mu_v$ is irreducible. In this case, we take $\pi_-(\mu_v)=0$. Let $\pi(\mu_v)=\{\pi_+(\mu_v),\pi_-(\mu_v)\}$ and if $\pi_v\in\pi(\mu_v)$, let $\epsilon(\pi_v)$ be the corresponding sign. Let $J_\pm(\mu_v)$ be the Langlands' quotients of $Ind_{P_i}^G\,|\ |_v\mu_v\times\pi_\pm(\mu_v)$, respectively. Let $J(\mu_v)=\{J_+(\mu_v),J_-(\mu_v)\}$.

Observe that $R(\tau,\beta_1,\chi_v)$ is the normalized intertwining operator for $Ind_{B_0}^{U(1,1)}\mu_v$. By [16, Theorem 5.1],

$$R(\tau,\beta_1,\chi_v)f_v=\begin{cases} f_v, & \text{for } f_v\in\pi_+(\mu_v)\\ -f_v, & \text{for } f_v\in\pi_-(\mu_v).\end{cases}$$

Then we define $J(\mu)$ to be the collection

$$J(\mu)=\{\Pi=\otimes\Pi_v\,|\,\Pi_v\in J(\mu_v)\text{ for all } v,\ \Pi_v=J_+(\mu_v)\text{ for almost all } v,\ \prod_v\epsilon(\pi_v)=1\}.$$

We note that $\prod_v\epsilon(\pi_v)$ is well-defined and independent of the choice of η.

Therefore we have the following theorem.

Theorem 6.1. *The residual spectrum $\oplus_{(T,\chi)}V_{(T,\chi)}$ of $U(2,2)$ coming from the Borel subgroup, where*

$$V_{(T,\chi)}=L^2_{dis}(G(F)\backslash G(\mathbb{A}_F))_{(T,\chi)},$$

is given as follows:

(1) $\chi = \chi(\mu, \mu)$, $\mu|_{\mathbb{A}_F^\times} = 1$. $V_{(T,\chi)}$ *consists of the one-dimensional representation* $\mu \circ det$.
(2) $\chi = \chi(\mu, \nu)$, $\mu|_{\mathbb{A}_F^\times} = 1$, $\nu|_{\mathbb{A}_F^\times} = 1$ *and* μ, ν *are distinct.* $V_{(T,\chi)}$ *is irreducible and it consists of the Langlands' quotient* $\otimes_v' J(\pi_v \otimes exp\langle \alpha_3, H_{P_1}(\ )\rangle)$, *where* $\pi_v = Ind_B^{P_1} \chi_v$.
(3) $\chi = \chi(\mu, \mu)$, $\mu|_{N(\mathbb{A}_E^\times)} = 1$ *but* $\mu|_{\mathbb{A}_F^\times} \neq 1$. $V_{(T,\chi)} = J(\mu)$.

Remark 6.2. Notice the difference between $U(2,2)$ and Sp_4. In Sp_4 case, the contribution from α_3 is zero.

Remark 6.3. In $U(2,2)$ case, Conjecture 4.13 on the residual spectrum is proved since all three cases belong to the special case in Section 5.

In this special case, the Arthur parameters are given as follows:

Theorem 6.4. (1) $\chi = \chi(\mu, \mu)$, $\mu|_{\mathbb{A}_F^\times} = 1$. *The Arthur parameter is given by the unipotent orbit* $O = (4) \subset Sp(4, \mathbb{C})$.
(2) $\chi = \chi(\mu, \nu)$, $\mu|_{\mathbb{A}_F^\times} = 1$, $\nu|_{\mathbb{A}_F^\times} = 1$, *and* μ, ν *are distinct. The Arthur parameter is given by the unipotent orbit* $O = (2) \times (2) \subset Sp(2, \mathbb{C}) \times Sp(2, \mathbb{C})$.
(3) $\chi = \chi(\mu, \mu)$, $\mu|_{N(\mathbb{A}_E^\times)} = 1$ *but* $\mu|_{\mathbb{A}_F^\times} \neq 1$. *The Arthur parameter is given by the unipotent orbit* $O = (3, 1) \subset O(4, \mathbb{C})$.

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(Henry H. Kim)
E-mail address: henrykim@math.toronto.edu