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COMPUTING LOCAL COEFFICIENTS VIA TYPES AND COVERS: THE EXAMPLE OF $SL(2)$

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Abstract. We illustrate a method of computing Langlands-Shahidi local coefficients via the theory of types and covers.
Keywords: Local factors, supercuspidal representations, types and covers.
MSC(2010): Primary: 11F70; Secondary: 22E50.

1. **Introduction**

The purpose of this paper is to illustrate a method of computing the Langlands-Shahidi *local coefficients* [18] using the theory of *types* and *covers* [6]. Local coefficients are important in number theory in that they are related to the theory of *local factors*. The theory of types and covers on the other hand provides a systematic way of studying the smooth representation theory of reductive p-adic groups via the representation theory of compact open subgroups (cf. Section 3.2).

Shahidi defines a specific list of local factors (L- and ϵ-factors) (see [20]) using these local coefficients through the method developed in [21, 18]– what is known as the *Langlands-Shahidi method*. Another well known method for constructing local factors is the *Rankin-Selberg method* [12]. Although these two methods are completely different, they are expected to be consistent with the putative Local Langlands Correspondence. In particular, whenever both constructions are possible in a given situation, the resulting local factors ought to be the same, perhaps up to normalization of certain measures used in the Langlands-Shahidi method. This is non-trivial and is known only in a limited number of cases; for example, see [19] where equality is proved in the context of local factors attached to a pair (π_1, π_2) of representations of general linear groups. This latter result of Shahidi combined with the theory of types and

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covers led to an explicit formula for conductors of pairs [4]. We also refer the reader to [2] and [15] in which ϵ-factors of pairs are computed.

Our approach in this paper is to tackle the problem of computing local coefficients through the theory of types and covers. In fact such an approach goes back at least as far as the work of Casselman [8], where local coefficients in the context of unramified principal series representations are explicitly computed. The calculation there relies on finding the effect of the intertwining operator (cf. Section 3.1) on the subspace of vectors fixed by the Iwahori subgroup (cf. [8]). One now knows that the role played by (the trivial representation of) the Iwahori subgroup in Casselman's calculation is an extreme instance of the more general theory of types and covers. It is our belief that certain test functions (cf. Section 3.3) that arise naturally in the theory of types and covers can be used to compute local coefficients in more general situations. We illustrate this approach using the example of $SL(2)$, an example which is of course well-known through other methods.

In addition to his important contributions to the Langlands program, Freydoon Shahidi has played a critical role in the mathematical development of both authors. He was the thesis advisor of the first author, who has benefited from his mentoring and from the great breadth of his mathematical knowledge. The second author has learned much of what he knows about the local Langlands program from Shahidi. Beyond this, we are both grateful for our lifelong friendship with Freydoon, his wife Guity and his family. *Tavalodat Mobarak Freydoon!*

2. Background and preliminaries

Suppose F is a non-archimedean local field with ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ denote the unique maximal ideal in $\mathfrak{o}_F$ and let q denote the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$. We fix a generator ϖ of $\mathfrak{p}_F$ and normalize the absolute value $\|\cdot\|$ of F so that $\|\varpi\| = q^{-1}$. Let us now consider a F-quasisplit connected reductive group $\mathbf{G}$ and let $\mathbf{P}$ be an F-parabolic subgroup of $\mathbf{G}$. We fix a F-Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and assume $\mathbf{P} \supset \mathbf{B}$. We write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{T}$ is a maximal torus defined over F and $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let $\mathbf{P} = \mathbf{MN}$ be a Levi decomposition with $\mathbf{M} \supset \mathbf{T}$ and $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Then $\mathbf{U} \supset \mathbf{N}$. We assume that the parabolic subgroup $\mathbf{P}$ is maximal; we let $\overline{\mathbf{P}} = \mathbf{M}\overline{\mathbf{N}}$ denote the unique standard F-parabolic subgroup opposed to $\mathbf{P}$ (cf. [1, Section 20.5]). Let $G, B, T, U, P, M, N,$ and $\overline{N}$ denote the corresponding groups of F-rational points. Let $N_G(M)$ denote the normalizer of M in G. Our assumption that P is maximal implies that the quotient group $N_G(M)/M$ is of order two; we write w_0 for a representative of the non-trivial element of $N_G(M)/M$ in $N_G(M)$.

For any topological group H, we write $\widehat{H}$ to denote the group of continuous homomorphisms from H to $\mathbb{C}^\times$. In particular, one has the character groups $\widehat{F}$

and $\widehat{\mathfrak{o}_F^\times}$. Since F is a union of compact open subgroups and $\mathfrak{o}_F^\times$ is compact, all characters in $\widehat{F}$ and $\widehat{\mathfrak{o}_F^\times}$ respectively are unitary. Now fix a non-trivial character $\psi \in \widehat{F}$, then it defines a non-degenerate character of the maximal unipotent subgroup U of G as explained in [11, Section 2]. We denote this character by ψ^G. One can then arrange the choice of $w_0 \in N_G(M)/M$ so that it is *compatible* with ψ^G [10, Section 1.2], in the sense that its restriction to the maximal unipotent radical $U \cap M$ has the following property:

$$\psi^G(u) = \psi^G(w_0^{-1}uw_0), u \in U \cap M.$$

Throughout the rest of the paper, we fix such a representative w_0. We write ψ^M to denote the restriction of ψ^G to $U \cap M$. Now, let (σ, V_σ) be a smooth irreducible representation of M which is *generic* (with respect to ψ^M); i.e., there is a nonzero linear functional on V_σ that transforms according to ψ^M on $U \cap M$. Such a functional is called a ψ^M*-Whittaker functional*; by local uniqueness [23, 16], the space of ψ_M-Whittaker functionals is of dimension one. Let us fix a basis vector Ω^M for this one-dimensional space.

If we write $\mathrm{Rat}(\mathbf{M})$ for the group of F-rational characters of $\mathbf{M}$, then each $\chi \in \mathrm{Rat}(\mathbf{M})$ defines a continuous homomorphism $\|\chi\| : M \longrightarrow \mathbb{C}^\times$ via $\|\chi\|(m) = \|\chi(m)\|, m \in M$. Let

$$^0M = \bigcap_{\chi \in \mathrm{Rat}(\mathbf{M})} \mathrm{Ker}\|\chi\|;$$

and let $X(M)$ denote the group of continuous homomorphisms of M into $\mathbb{C}^\times$ which are trivial on ^0M – called the group of unramified characters of M (cf. [26, p. 239]). Let A_G denote the F-points of the maximal split torus in the center of G. Then, since M is maximal, the quotient $M/A_G{}^0M$ is isomorphic to $\mathbb{Z}$. Therefore we may identify the subgroup $X^G(M) \subset X(M)$ consisting of unramified characters of M that are trivial on A_G with $\mathbb{C}^\times$ or equivalently, using an exponential map, with $\mathbb{C}$ modulo a lattice. In [20, Section 1] Shahidi makes this identification explicit through the choice of a certain fundamental weight $\tilde{\alpha}$; we denote this identification here by $s \mapsto \chi_s \in X^G(M)$, $s \in \mathbb{C}$. (In the notation of loc.cit., $\chi_s(m) = \|\tilde{\alpha}(m)\|^s$.)

Let ι_P^G denote the functor of normalized parabolic induction and for any smooth (not necessarily generic) representation τ of M, let $\mathcal{F}_P(\tau)$ denote the space of $\iota_P^G(\tau)$. If τ is irreducible, one has [18] the usual intertwining operator $A(s, \tau, w_0) : \iota_P^G(\tau \otimes \chi_s) \longrightarrow \iota_P^G(w_0(\tau \otimes \chi_s))$ given by

$$A(s, \tau, w_0)f(g) = \int_N f(w_0^{-1}ng)dn,\ f \in \mathcal{F}_P(\tau \otimes \chi_s),$$

for $Re(s) \gg 0$; the function $s \mapsto A(s, \tau, w_0)$ admits a meromorphic continuation [18, Section 2.2]. (In fact $s \mapsto A(s, \tau, w_0)$ is a rational function in a certain precise sense [26, Section IV].) We recall this theory in detail for $G = SL(2, F)$ in Section 3.1 below.

Now for σ generic as above, it is a theorem of Rodier (cf. [9]) that the dimension of the space of ψ^G-Whittaker functionals on $\mathcal{F}_P(s, \sigma \otimes \chi_s)$ is the same as the dimension of the space of ψ^M-Whittaker functionals on V_σ; consequently this dimension is one. One may define a basis vector [9], Ω_s, for the one-dimensional space of ψ^G-Whittaker functionals on $\mathcal{F}_P(s, \sigma \otimes \chi_s)$ by the formula

$$\Omega_s(f) = \int_N \Omega^M(f(w_0^{-1}u))\overline{\psi^G(u)}du, f \in \mathcal{F}_P(\sigma \otimes \chi_s).$$

This integral may not converge for all f but can be extended to the whole space as a principal value integral. Further $s \mapsto \Omega_s$ is a holomorphic function [9, Proposition 2.1]. One also has the following convenient formula for Ω_s as a principal value integral [9, Corollary 2.3]: Given a compact open subgroup K of G, there exists a suitably large compact open subgroup $N_* \subset N$ such that

$$\Omega_s(f) = \int_{N_*} \Omega^M(f(w_0^{-1}u))\overline{\psi^G(u)}du \tag{2.1}$$

for all s and for all $f \in \mathcal{F}_P(\sigma \otimes \chi_s)^K$.

One similarly defines Ω'_s on $\mathcal{F}_P(w_0(\sigma \otimes \chi_s))$ via

$$\Omega'_s(f) = \int_N \Omega^M(f(w_0 u))\overline{\psi^G(u)}du. \tag{2.2}$$

as a principal value integral in the above sense.

Then by the aforementioned result of Rodier there is a non-zero constant $C_\psi(s, \sigma)$ called the *local coefficient* [18] satisfying

$$C_\psi(s, \sigma)(\Omega'_s \circ A(s, \sigma, w_0)) = \Omega_s. \tag{2.3}$$

The local coefficient is of great importance in number theory and is closely related to the theory of local factors *à la* Langlands-Shahidi [21, 20].

3. **A calculation for** $SL(2)$

In this section, we illustrate the use of the theory of types and covers in the computation of local coefficients using the example of $SL(2)$. To be precise, with notation as in the previous section, take $G = SL(2, F)$ and $P = B$, where B is the subgroup of F-points of the Borel subgroup of upper triangular matrices. Then $M = T$ and $N = U$, and explicitly

$$\begin{aligned} T &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \big| a \in F^\times \right\}; \\ U &= \left\{ u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \big| x \in F \right\}; \\ \overline{U} &= \left\{ \overline{u}(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \big| x \in F \right\}. \end{aligned}$$

In this case the character χ_s referred to above is given by the formula

$$\chi_s \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = \|a\|^s;$$

by extending χ_s trivially to U, we may also view it as a character of B. Let $K = G(\mathfrak{o}_F)$ and let $W = N_G(T)/T = \{1, w_0\}$, where

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the representative as fixed in Section 2 in the present context. Now, any $\tilde{\eta} \in \widehat{F^\times}$ (not necessarily unitary) defines a representation σ of M which is trivially supercuspidal and generic. It is our intention to calculate $C_\psi(s, \sigma)$ under the assumption $\eta^2 \neq 1$, where $\eta = \tilde{\eta}|_{\mathfrak{o}_F^\times}$. To that end, we start by recalling the theory of intertwining operators in the situation at hand.

3.1. **The intertwining operator.** We keep the notation from the previous section; in particular, we fix a character η of $\mathfrak{o}_F^\times$. For any smooth representation (π, V) of G, let V_U be the Jacquet module and let π_U denote the natural action of T on V_U. We write ι to denote the induction functor ι_B^G and $\mathcal{F}(\cdot)$ (instead of $\mathcal{F}_B(\cdot)$) to denote the corresponding representation space.

We recall the structure of $\iota(\tilde{\eta})_U$ for any extension $\tilde{\eta}$ of η to $F^\times$. First, let δ denote the modulus character of B; explicitly $\delta(a) = \|a\|^2$. Let $\mathcal{F}(\tilde{\eta})_{w_0}$ denote the B-subspace of functions in $\mathcal{F}(\tilde{\eta})$ supported on the big cell Bw_0N. We have a filtration

$$\{0\} \subseteq \mathcal{F}(\tilde{\eta})_{w_0} \subseteq \mathcal{F}(\tilde{\eta})$$

by B-stable subspaces and the map $f \mapsto f(1)$ induces a T-isomorphism between

$$(\mathcal{F}(\tilde{\eta})/\mathcal{F}(\tilde{\eta})_{w_0})_U \simeq \mathbb{C}_{\tilde{\eta}\delta^{1/2}}.$$

In short, we have an exact sequence

$$0 \longrightarrow (\mathcal{F}(\tilde{\eta})_{w_0})_U \longrightarrow \mathcal{F}(\tilde{\eta})_U \longrightarrow \mathbb{C}_{\tilde{\eta}\delta^{1/2}} \longrightarrow 0. \tag{3.1}$$

The Jacquet module of $\mathcal{F}(\tilde{\eta})_{w_0}$ can be realized using the B-intertwining map

$$a_{w_0}(\tilde{\eta}) : \mathcal{F}(\tilde{\eta})_{w_0} \longrightarrow \mathbb{C}_{\tilde{\eta}^{-1}\delta^{1/2}} \text{ given by } f \mapsto \int_U f(w_0 u) du.$$

Note that this integral is well defined since the function $n \mapsto f(w_0 n)$ belongs to $C_c^\infty(U)$ for $f \in \mathcal{F}(\tilde{\eta})_{w_0}$. Moreover, one checks that the kernel of $a_{w_0}(\tilde{\eta})$ is precisely $\mathcal{F}(\tilde{\eta})_{w_0}(U)$. Thus $a_{w_0}(\tilde{\eta})$ induces the isomorphism

$$(\mathcal{F}(\tilde{\eta})_{w_0})_U \simeq \mathbb{C}_{\tilde{\eta}^{-1}\delta^{1/2}}$$

which we continue to denote as $a_{w_0}(\tilde{\eta})$.

Now, if $\eta^2 \neq 1$, then $\tilde{\eta} \neq \tilde{\eta}^{-1}$; such a character is called a *regular* character. Therefore the sequence (3.1) splits and $a_{w_0}(\tilde{\eta})$ extends to all of $\mathcal{F}(\tilde{\eta})_U$ to give a well defined element in $\mathrm{Hom}_T(\mathcal{F}(\tilde{\eta})_U, \mathbb{C}_{\tilde{\eta}^{-1}\delta^{1/2}})$. Then, by Frobenius reciprocity, this determines a unique non-zero intertwining operator

$$A(\tilde{\eta}, w_0) : \mathcal{F}(\tilde{\eta}) \longrightarrow \mathcal{F}(\tilde{\eta}^{-1})$$

such that for all $f \in \mathcal{F}(\tilde{\eta})_{w_0}$

$$A(\tilde{\eta}, w_0)(f)(1) = \int_U f(w_0 u) du.$$

If we fix an extension $\tilde{\eta}$, then any other extension of η is of the form $\tilde{\eta}\|\cdot\|^s$, and the operator denoted as $A(s, \tilde{\eta}, w_0)$ in Section 2 is nothing but $A(\tilde{\eta}\|\cdot\|^s, w_0)$. It is for example shown in [8] that $A(s, \tilde{\eta}, w_0)$ varies holomorphically with s, or in the language of algebraic geometry, that it is a regular function on the domain of regular characters.

3.2. **Types and covers.** Let for the moment G be the set of F-points of an arbitrary connected reductive group over F. By a *cuspidal pair* in G we mean a pair (L, τ) in G, where L is the F-points of a F-Levi subgroup of G and τ is a supercuspidal representation of L. Two such pairs $(L_i, \tau_i), i = 1, 2$, are said to be *inertially equivalent* if there exist $g \in G$ and $\chi \in X(L)$ such that $L_2 = L_1^g = g^{-1} L_1 g$ and σ_2 is equivalent to the representation $\sigma_1^g \otimes \chi$: $x \mapsto \sigma_1(gxg^{-1})\chi(x)$ of L_2. We denote by $[(L, \tau)]$ the G-inertial equivalence class of a cuspidal pair (L, τ) in G. Let $\mathfrak{B}(G)$ denote the set of inertial equivalence classes of cuspidal pairs in G. For each $\mathfrak{s}$ in $\mathfrak{B}(G)$, one defines a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ as follows: a smooth representation Π belongs to $\mathfrak{R}^{\mathfrak{s}}(G)$ if and only if each irreducible subquotient π of Π has *inertial support* $\mathfrak{s}$. (See [6, Definition 1.1].) It is a fundamental result of Bernstein that $\mathfrak{R}(G)$ decomposes into a product of subcategories

$$\mathfrak{R}(G) \cong \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G). \tag{3.2}$$

Let $\mathcal{H} = \mathcal{H}(G)$ be the space of smooth compactly supported complex valued functions on G. After fixing a Haar measure μ on G, this becomes a convolution algebra relative to μ via

$$f \star g(y) = \int_G f(x) g(x^{-1} y) d\mu(x), f, g \in \mathcal{H}(G), y \in G.$$

It is a well-known fact that $\mathfrak{R}(G)$ can be identified with the category of non-degenerate left $\mathcal{H}$-modules. (See for example [3, Chapter 1, Section 4.2, Proposition 1].)

If $e \in \mathcal{H}$ is a nonzero idempotent, then $e \star \mathcal{H} \star e$ is a subalgebra of $\mathcal{H}$ with unit element e. If $(\pi, V) \in \mathfrak{R}(G)$, then the subspace $\pi(e)V = e \star V$ carries a natural $e\mathcal{H}e$-module structure. Let $\mathfrak{R}_e(G)$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are those representations V generated over G by the subspace $e \star V$, i.e., $V = \mathcal{H} \star e \star V$. The idempotent e is said to be *special* if the functor $V \mapsto e \star V$ is an equivalence of categories $\mathfrak{R}_e(G) \cong e\mathcal{H}e$-Mod. It follows from [6, (3.12)] that an idempotent e is special if and only if there is a finite subset $\mathfrak{S}(e)$ of $\mathfrak{B}(G)$ such that

$$\mathfrak{R}_e(G) = \prod_{\mathfrak{s} \in \mathfrak{S}(e)} \mathfrak{R}^{\mathfrak{s}}(G).$$

Conversely, it is shown in [6, Proposition (3.13)] that, for $\mathfrak{s} \in \mathfrak{B}(G)$, there exists a special idempotent $e \in \mathcal{H}$ so that $\mathfrak{s} = \mathfrak{S}(e)$. In particular, the category $\mathfrak{R}^{\mathfrak{s}}(G)$ is equivalent to the category $e\mathcal{H}e$-Mod of left, unital $e\mathcal{H}e$-modules.

A useful way of producing special idempotents is through the representation theory of compact open subgroups of G. Let (λ, W) be a smooth irreducible representation of a compact open subgroup J of G. Then W is finite dimensional and we may define the function $e_\lambda : G \longrightarrow \mathbb{C}$ by

$$e_\lambda(x) = \begin{cases} \frac{\dim W}{\mu(K)} \operatorname{tr}(\lambda(x^{-1})), & \text{if } x \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $e_\lambda \in \mathcal{H}(G)$ and it is an idempotent. The pair (J, λ) is said to be a *type* for $\mathfrak{s} \in \mathfrak{B}(G)$, or simply a $\mathfrak{s}$-*type*, if the idempotent e_λ is special and $\mathfrak{S}(e_\lambda) = \mathfrak{s}$ (cf. [6, Definition (4.1)]). Equivalently, the pair (J, λ) is a $\mathfrak{s}$-type if and only if for every irreducible object (π, V) in $\mathfrak{R}(G)$, we have $(\pi, V) \in \mathfrak{R}^{\mathfrak{s}}(G)$ if and only if $V^\lambda = e_\lambda \star V \neq 0$ [6, (4.2)].

In practice, the algebra $e_\lambda \mathcal{H} e_\lambda$ is not easy to compute. Fortunately there is an associated algebra $\mathcal{H}(G, \lambda)$ about which we know much more. In fact, in all known cases, this algebra is known to be an affine Hecke algebra. We define $\mathcal{H}(G, \lambda)$ as follows: Let $(\check{\lambda}, \check{W})$ denote the contragredient of (λ, W), then $\mathcal{H}(G, \lambda)$ is the space of compactly supported functions $f : G \longrightarrow \operatorname{End}_{\mathbb{C}}(\check{W})$ that satisfy $f(hxk) = \check{\lambda}(h) f(x) \check{\lambda}(k)$, $x \in G, h, k \in J$. This becomes a unital (associative) algebra under the standard convolution operation (cf. [6, Section 2]). Further, for any smooth representation (π, V) of G, there is a natural left $\mathcal{H}(G, \lambda)$-module structure on the space of λ-coinvariants $V_\lambda = \operatorname{Hom}_J(W, V)$. One has that (see [6, (2.13)])

$$V^\lambda \simeq V_\lambda \otimes_{\mathbb{C}} W. \tag{3.3}$$

Since the map $V \mapsto e_\lambda \star V$ is an equivalence of categories $\mathfrak{R}_{e_\lambda}(G) \cong e_\lambda \mathcal{H} e_\lambda$-Mod for a $\mathfrak{s}$-type (J, λ), it now follows that the map $V \mapsto V_\lambda$ is an equivalence of categories $\mathfrak{R}^{\mathfrak{s}}(G) \cong \mathcal{H}(G,\lambda)$-Mod. This latter equivalence has proved to be useful in a variety of contexts [5, 4, 14, 27].

The notion of a *cover* [6, Section 8] is used for the construction of types and it captures parabolic induction in the framework of module theory. In what follows we recall the key features of the theory of types and covers in the context of this paper. Keeping this in mind, we revert to our initial set-up with $G = SL(2,F)$; in particular we identify T with $F^{\times}$ when no confusion can arise.

Let $\tilde{\eta}_i, i = 1,2$, be characters of T, then $\tilde{\eta}_1$ is T-inertially equivalent to $\tilde{\eta}_2$ if and only if there exist $s \in \mathbb{C}$ such that $\tilde{\eta}_2 = \tilde{\eta}_1\chi_s$. Thus the T-inertial equivalence class of a character $\tilde{\eta}$ of T is determined by the restriction, η, of $\tilde{\eta}$ to $\mathfrak{o}_F^{\times}$. We denote this class by $\mathfrak{t}_\eta$. On the other hand with $\tilde{\eta}_i, i = 1,2$, as above, $\tilde{\eta}_2$ is G-inertially equivalent to $\tilde{\eta}_1$ if and only if $\eta_2 = \eta_1^{\pm 1}$. With η and $\tilde{\eta}$ as above, let $\mathfrak{s}_\eta$ be the corresponding G-inertial equivalence class. Then the associated subcategories of $\mathfrak{R}(T)$ and $\mathfrak{R}(G)$, respectively, are given as follows:

(i) $\mathcal{R}^{\mathfrak{t}_\eta}(T)$ is the full subcategory of $\mathcal{R}(T)$ whose objects (π, V) have the property that $\pi(x)v = \eta(x)v, x \in {}^0T, v \in V$,
(ii) $\mathcal{R}^{\mathfrak{s}_\eta}(G)$ is the full subcategory of $\mathcal{R}(G)$ whose irreducible objects are precisely those that appear as a sub-quotient of some $\iota_Q^G(\tilde{\eta}\chi_s)$, where Q is either B or $\overline{B}$.

For η as above, set $n_\eta = 1$ if $1 + \mathfrak{p}_F \subset \ker\eta$; otherwise it is defined to be the smallest integer n so that $1 + \mathfrak{p}_F^n \subset \ker\eta$. Let (J_η, λ_η) be as in [13]. In particular, J_η is the compact open subgroup given by

$$J_\eta = \left\{[c_{ij}] \in G | c_{11}, c_{22} \in \mathfrak{o}_F^{\times}, c_{12} \in \mathfrak{o}_F, c_{21} \in \mathfrak{p}_F^{n_\eta}\right\},$$

and λ_η is a function on J_η given by

$$\lambda_\eta([c_{ij}]) = \eta(c_{11}).$$

It is proved in [13] that the pair $(J,\lambda) = (J_\eta, \lambda_\eta)$ is a G-cover for $({}^0T, \eta)$. In our situation it means that the pair (J,λ) has the following properties:

(1) $J = (J \cap \overline{U})^0T(J \cap U)$.
(2) $J \cap \overline{U}, J \cap U \subset \ker\lambda$; $\lambda|_{{}^0T} = \eta$.
(3) There are positive integers n_1, n_2 and invertible elements $f_1, f_2 \in \mathcal{H}(G,\lambda)$ such that f_1, f_2 are supported on the double cosets $J\Pi^{n_1}J, J\Pi^{-n_2}J$ where $\Pi = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$.

In particular, it follows from [6, Theorem (8.3)] that (J_η, λ_η) is a type for $\mathfrak{s}_\eta$.

An important consequence of (3) from above is that, for $Q = B, \overline{B}$, there is a support preserving injective algebra map (cf. [6, Corollary (7.12)])

$$t_Q : \mathcal{H}(T,\eta) \longrightarrow \mathcal{H}(G,\lambda_\eta)$$

that realizes the parabolic induction functor ι_Q^G at the level of Hecke algebras. We once again refer the reader to [6, Corollary (8.4)] for its precise meaning.

Further, when $\eta^2 \neq 1$, the maps t_Q, $Q = B, \overline{B}$, are known to be isomorphisms [13, Section 3.1, Corollary] in which case the cover $(J, \lambda) = (J_\eta, \lambda_\eta)$ is said to be a *split* cover.

3.3. **The main calculation.** We continue with the notation of Section 3.2 and assume $\eta^2 \neq 1$. Having fixed w_0 and B, in what follows, we suppress them from the various notations for the sake of brevity. For example, as before we write $C_\psi(s, \tilde{\eta})$ for the local coefficient and $\mathcal{F}(s, \tilde{\eta})$ to denote the induced space $\mathcal{F}_B(\tilde{\eta}|\cdot|^s)$. We also have the G-isomorphism

$$A(s, \tilde{\eta}) : \mathcal{F}(s, \tilde{\eta}) \longrightarrow \mathcal{F}(-s, \tilde{\eta}^{-1}).$$

Since the cover splits, it follows from [6, Corollary (12.4)] that the induction functor ι gives an equivalence of categories. Therefore $\mathcal{F}(s, \tilde{\eta})_\lambda$ and $\mathcal{F}(-s, \tilde{\eta}^{-1})_\lambda$ are both one dimensional. On the other hand the representation $\lambda|_{{}^0T} = \eta$ is one dimensional and consequently the corresponding λ-isotypic subspaces $\mathcal{F}(s, \tilde{\eta})^\lambda$ and $\mathcal{F}(-s, \tilde{\eta}^{-1})^\lambda$ are also both one dimensional. Let us describe these explicitly:

(1) For $\mathcal{V} = \mathcal{F}(s, \tilde{\eta})$, let $f \in \mathcal{V}$ be given by

$$f(g) = \begin{cases} \tilde{\eta}\chi_s \delta^{1/2}(b)\lambda_\eta(j) & \text{if } g = bj \in BJ \\ 0 & \text{otherwise.} \end{cases}$$

(2) For $\mathcal{V}' = \mathcal{F}(-s, \tilde{\eta}^{-1})$, let $f' \in \mathcal{V}'$ be given by

$$f'(g) = \begin{cases} \tilde{\eta}^{-1}\chi_{-s} \delta^{1/2}(b)\lambda_\eta(j) & \text{if } g = bw_0 j \in Bw_0 J \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{\eta}$ (resp. $\tilde{\eta}^{w_0} = \tilde{\eta}^{-1}$) agrees with λ_η on $B \cap J$ (resp. $w_0^{-1}Bw_0 \cap J$), it follows that both f and f' are well-defined and that they belong to the corresponding λ-isotypic subspaces. More precisely, in (1), $\mathcal{V}^\lambda = \mathbb{C}f$; and in (2), $(\mathcal{V}')^\lambda = \mathbb{C}f'$.

We are now ready for the calculation.

Fix a Haar measure dx on F, or equivalently, a measure du (resp. $d\bar{u}$) on U (resp. $\overline{U}$). Set $d^\times x = \frac{dx}{\|x\|}$; then $d^\times x$ is an invariant measure on $F^\times$. Since $A(s, \tilde{\eta})$ is a G-map, it follows from the preceding discussion that $A(s, \tilde{\eta})f = af'$ for some complex constant a. We may evaluate both sides of this equation at w_0 to obtain $a = (A(s, \tilde{\eta})f)(w_0)$;

$$\begin{aligned} a &= \int_U f(w_0 u w_0)du \\ &= \tilde{\eta}(-1)\int_{\overline{U}} f(\bar{u})d\bar{u} \\ &= \tilde{\eta}(-1)\mathrm{vol}(\overline{U} \cap J). \end{aligned}$$

Here, we note that we pick up a factor of $\tilde{\eta}(-1)$ in the second equality since $w_0^2 = -1$.

Let us take $\psi \in \widehat{F}$ to be trivial on $\mathfrak{o}_F$, but not on $\mathfrak{p}_F^{-1}$. Since f' is supported in Bw_0J, it follows that the function $u \mapsto f'(w_0u)$ is supported on $U \cap J$, and consequently (cf. (2.2), Section 2)

$$\begin{aligned} \Omega_s'(f') &= \int\limits_{U\cap J} f'(w_0u)\psi^{-1}(u)du \\ &= \int\limits_{U\cap J} du \\ &= \mathrm{vol}(U\cap J). \end{aligned}$$

It remains to compute $\Omega_s(f)$. By definition (cf. (2.1), Section 2), there exists a suitably large compact open subgroup $U_* \subset U$ such that

$$\Omega_s(f) = \int\limits_{U_*} f(w_0u)\psi^{-1}(u)du.$$

One observes that $w_0u \in BJ$ if and only if $u \in \overline{B}w_0(J\cap\overline{U})$. Take

$$u = u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

then for $x \neq 0$, we have

$$u(x) = \begin{pmatrix} x & 0 \\ 1 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

It follows that $w_0u(x) \in BJ$ if and only if $x^{-1} \in \mathfrak{p}_F^{n_\eta}$. In particular, for such x,

$$f(w_0u(x)) = \tilde{\eta}\chi_s\delta^{1/2}(x^{-1}).$$

Therefore

$$\begin{aligned} \Omega_s(f) &= \int\limits_{\{\mathfrak{p}_F^{-k}-\{0\}\}\cap\{x:x^{-1}\in\mathfrak{p}_F^{n_\eta}\}} \tilde{\eta}(x^{-1})\|x\|^{-s-1}\overline{\psi(x)}dx \\ &= \int\limits_{\{\mathfrak{p}_F^{-k}-\{0\}\}\cap\{x:x^{-1}\in\mathfrak{p}_F^{n_\eta}\}} \tilde{\eta}(x^{-1})\|x\|^{-s}\overline{\psi(x)}d^\times x \end{aligned}$$

for some large positive integer k. For convenience, let $\mathcal{J}$ denote the above domain of integration. (We will soon see that the choice of k here is irrelevant.) For $m \geq 1$, set $\mathfrak{u}_F^m = 1 + \mathfrak{p}_F^m$. Note that $\mathcal{J}$ is invariant under multiplication by elements in $\mathfrak{o}_F^\times$; therefore for $u \in \mathfrak{u}_F^m$, $m \geq 1$, we obtain

$$\Omega_s(f) = \int\limits_{\mathcal{J}} \tilde{\eta}(xu)^{-1}\overline{\psi(xu)}\|x\|^{-s}d^\times x.$$

Now, integrating both sides of the above equation over u and interchanging the order of integration, we obtain

$$\Omega_s(f) = \frac{1}{v_m} \int_{\mathcal{J}} \tilde{\eta}(x)^{-1} \|x\|^{-s} \left(\int_{\mathfrak{u}_F^m} \tilde{\eta}(u)^{-1} \overline{\psi(xu)} d^\times u \right) d^\times x,$$

where $v_m = \mathrm{vol}(\mathfrak{u}_F^m)$. In particular, choosing $m = n_\eta$, and writing $u = 1 + y$, we see that

$$\Omega_s(f) = \frac{1}{v_m} \int_{\mathcal{J}} \tilde{\eta}(x)^{-1} \|x\|^{-s} \overline{\psi(x)} \left(\int_{\mathfrak{p}_F^m} \overline{\psi(xy)} dy \right) d^\times x.$$

Since the character $y \mapsto \psi(xy)$ has level $-\mathrm{ord}_F(x)$, we see that the inside integral is zero unless $m = n_\eta \geq -\mathrm{ord}_F(x)$. This combined with the fact that x lies in $\mathcal{J}$ forces $\mathrm{ord}_F(x) = -n_\eta$. Let $c = \varpi^{-n_\eta}$. Thus

$$\begin{aligned} \Omega_s(f) &= \int_{\mathfrak{p}_F^{-n_\eta} - \mathfrak{p}_F^{-n_\eta+1}} \tilde{\eta}(x^{-1}) \|x\|^{-s} \overline{\psi(x)} d^\times x \\ &= \int_{c\mathfrak{o}_F^\times} \tilde{\eta}(x^{-1}) \|x\|^{-s} \overline{\psi(x)} d^\times x \\ &= \tilde{\eta}(c^{-1}) q^{-n_\eta s} \int_{\mathfrak{o}_F^\times} \eta(x^{-1}) \overline{\psi(cx)} d^\times x. \end{aligned} \tag{3.4}$$

Note that the above integral (which is effectively a sum) is the *Gauss sum* of η relative to ψ.

Putting everything together we obtain from the definition of the local coefficient (see (2.3)) that

$$\tilde{\eta}(-1) \mathrm{vol}(\overline{U} \cap J) \mathrm{vol}(U \cap J) C_\psi(s, \tilde{\eta}) = \Omega_s(f),$$

where $\Omega_s(f)$ is given by (3.4). Now, if we normalize dx by setting $\mathrm{vol}(\mathfrak{o}_F) = 1$, then

$$\mathrm{vol}(U \cap J) = 1; \mathrm{vol}(\overline{U} \cap J) = q^{-n_\eta}.$$

Thus we have proved the following.

Proposition 3.1. *Suppose $\psi \in \widehat{F}$ is of level 0 and η is a character of $\mathfrak{o}_F^\times$ satisfying $\eta^2 \neq 1$ with level n. Assume that the measure dx is normalized as above. Let $\tilde{\eta} \in \widehat{F^\times}$ be any extension of η and let $C_\psi(s, \tilde{\eta})$ denote the corresponding local coefficient as defined in Section 2. Then*

$$C_\psi(s, \tilde{\eta}) = \tilde{\eta}(-c^{-1}) \tau(\eta, \psi, c) q^{-n(s-1)},$$

where $c = \varpi^{-n}$ *and* $\tau(\eta, \psi, c) = \int_{\mathfrak{o}_F^\times} \eta(x^{-1})\overline{\psi(cx)}dx.$

3.4. **Concluding remarks.**

(1) The above result should be compared with [17, Lemma 4.4], where Shahidi shows the equality of $C_\psi(s, \tilde{\eta})$ with the corresponding Hecke-Tate γ-factor [25, 24] up to normalization of the measure dx. The central idea in his proof is to reinterpret $C_\psi(s, \tilde{\eta})$ via the theory of Fourier transforms. (For a more general result in this context, see [19].) On the other hand, for $\eta^2 \neq 1$, one can recover this equality by comparing our expression for $C_\psi(s, \tilde{\eta})$ with that of the Hecke-Tate local factor [3, Section 23.6]. In general, as discussed in [22, Section 9], local coefficients can be realized as Fourier transforms of the measure that defines the corresponding intertwining operator by convolution.

(2) One can make a similar calculation when $\eta^2 = 1$ using the cover $(J, \lambda) = (J_\eta, \lambda_\eta)$, but it is more complicated as the cover is not split in this case; that is, the map t_B is not an isomorphism (see [13, Section 4, Proposition]). In addition, the intertwining map $A(s, \tilde{\eta})$, which may now have poles, is not an isomorphism and computing its effect on the λ-isotypic subspace $\mathcal{F}(s, \tilde{\eta})^\lambda$ is subtle. We omit this calculation here to keep our presentation simple. In fact, our original motivation was to study the problem of stability of local coefficients [11] via the theory of types and covers. In this regard, according to [7, Section 1.5, Theorem], one may always arrange the cover (if it exists) to be split after twisting the inducing representation σ (see Section 2 for notation) by a suitably highly ramified character. For example, in the case of $SL(2, F)$, given any quadratic η; i.e., $\eta^2 = 1$, we can twist it by a suitably highly ramified character so that the resulting character is not quadratic.

(3) Although there are technical difficulties in extending the method described above to the general situation of a p-adic group G and a maximal self-opposed F-Levi subgroup M, these difficulties do not appear to be insurmountable.

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