ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the

Iranian Mathematical Society

in Honor of Professor Freydoon Shahidi's 70th birthday

Vol. 43 (2017), No. 4, pp. 291–311

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 4, pp. 291–311
Online ISSN: 1735-8515

DISTINGUISHED POSITIVE REGULAR REPRESENTATIONS

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This paper is dedicated to Freydoon Shahidi on the occasion of his 70th birthday.

Abstract. Let G be a tamely ramified reductive p-adic group. We study distinction of a class of irreducible admissible representations of G by the group of fixed points H of an involution of G. The representations correspond to G-conjugacy classes of pairs (T, ϕ), where T is a tamely ramified maximal torus of G and ϕ is a quasicharacter of T whose restriction to the maximal pro-p-subgroup satisfies a regularity condition.

Under mild restrictions on the residual characteristic of F, we derive necessary conditions for H-distinction of a representation corresponding to (T, ϕ), expressed in terms of properties of T and ϕ relative to the involution.

We prove that if an H-distinguished representation arises from a pair (T, ϕ) such that T is stable under the involution and compact modulo $(T \cap H)Z$ (here, Z is the centre of G), then the representation is H-relatively supercuspidal.
Keywords: Distinguished representation, relatively supercuspidal.
MSC(2010): Primary: 22E50; Secondary: 20G05, 20G25.

1. Introduction

Let $G = \boldsymbol{G}(F)$ be the F-rational points of a connected reductive F-group $\boldsymbol{G}$ that splits over a tamely ramified extension of F, where F is a nonarchimedean local field of characteristic zero and odd residual characteristic.

Suppose that θ is an involution of G (that is, θ is an F-automorphism of $\boldsymbol{G}$ of order two) and $H = G^\theta$ is the group of θ-fixed points in G. A smooth representation π of G is said to be H-distinguished if the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear functionals on the space of π is nonzero. In this paper, we study distinction of particular irreducible admissible (complex) representations of G, which we refer to as positive regular representations.

We say that an irreducible admissible representation of G is *tame* if its cuspidal support consists of supercuspidal representations arising via Yu's construction [18]. This paper is one of a series of papers devoted to the general

Article electronically published on August 30, 2017.
Received: 21 July 2016, Accepted: 7 August 2016.

study of distinction of tame representations, the first of which is joint work with Hakim [4] on distinction of tame supercuspidal representations. Methods used here to obtain information about symmetry of tori and quasicharacters from which H-distinguished positive regular representations arise are also applied (with minor adjustments to the more general context of tame representations) in the study of distinction and tame types carried out in [13]. However, other more technically elaborate arguments also play an essential role there.

Before proceeding with a description of the contents of the paper, we point out that we assume throughout that the residual characteristic p of F is not a bad prime for $\boldsymbol{G}$, and also that p does not divide the order of the fundamental group of the derived group of $\boldsymbol{G}$. Recently, Kaletha [6] has shown that these conditions on p guarantee that the hypotheses assumed in [4] hold (see Section 3).

We now discuss the positive regular representations. A torus of G is called *tame* if it splits over a tamely ramified extension of F. Suppose that ϕ is a quasicharacter of a tame maximal torus T of G and T_{0+} is the maximal pro-p-subgroup of T. We say that the quasicharacter ϕ is *G-regular on T_{0+}* if ϕ does not agree on T_{0+} with the restriction of any quasicharacter of a twisted Levi subgroup of G that strictly contains T. (Here, a twisted Levi subgroup of G is a subgroup of the form $\boldsymbol{G}'(F)$, where $\boldsymbol{G}'$ is a connected reductive F-subgroup of $\boldsymbol{G}$ that becomes a Levi subgroup over a Galois extension of F.) It is easily seen that G-regularity is preserved under G-conjugation of pairs of the form (T, ϕ). Positive regular supercuspidal reprsentations are defined in Section 5. It follows from Kaletha's extensions of the results of [4] concerning equivalence of tame supercuspidal representations that the equivalence classes of positive regular supercuspidal representations of G are parametrized by the G-conjugacy classes of pairs (T, ϕ) such that T is an elliptic tame maximal torus and ϕ is G-regular on T_{0+}. (In [4] and [11], the positive regular supercuspidal representations were referred to as toral supercuspidals.)

The positive regular supercuspidals lie inside a much larger set of tame supercuspidals, the *regular supercuspidal* representations of [6]. As shown in [6], the regular supercuspidal representations are parametrized by G-conjugacy classes of pairs (T, ϕ) where T is a tame elliptic maximal torus of G and ϕ is a quasicharacter of T satisfying a root-theoretic regularity condition involving values of ϕ on all of T. (Our regularity condition is stronger than Kaletha's, and involves only the restriction of ϕ to the set of positive-depth elements of T, hence the terminology.) The parametrization of the positive regular supercuspidals referred to above is simply a special (and elementary) case of Kaletha's parametrization of regular supercuspidals.

Let $\mathcal{T}(G)$ be the set of pairs (T, ϕ) such that T is a tame maximal torus of G and ϕ is a quasicharacter of T that is G-regular on T_{0+}. Given $(T, \phi) \in \mathcal{T}(G)$, the pair (T, ϕ) belongs to $\mathcal{T}(M)$, where M is the centralizer of the maximal

F-split subtorus of T. Since T is elliptic in M, there is a unique equivalence class of positive regular supercuspidal representations of M associated with the M-conjugacy class of (T, ϕ). Parabolic induction gives rise to an irreducible admissible representation of G. The positive regular representations are the representations obtained through this process. Their equivalence classes are parametrized by the G-conjugacy classes in $\mathcal{T}(G)$ (Proposition 6.3).

Now we summarize the contents of the rest of the paper.

General notation and selected definitions appear in Section 2.

Section 3 begins with brief comments about the conditions on p referred to above and the hypotheses of [4]. Next, for ϕ a quasicharacter of a tame twisted Levi subgroup G' of G, we recall (from [11]) the notions of G-factorization of ϕ and of G-regularity of ϕ on the topologically unipotent subset G'_{0+} of G'. When G' is a tame maximal torus of G, a G-factorization is essentially the restriction to the maximal pro-p-subgroup of a Howe factorization (as defined in [6, Section 3.5]). Finally, after defining the notion of θ-symmetric G-factorization, we show that if G' is θ-stable and ϕ is trivial on $G' \cap H$, then ϕ has a θ-symmetric factorization on G'_{0+}.

Fix $(T, \phi) \in \mathcal{T}(G)$ and a point y in the (extended) Bruhat-Tits building $\mathcal{B}(T)$ of T. In Section 4, y and ϕ are used to define a compact open subgroup K_+ of G and a linear character $\vartheta(\phi)$ of K_+. In Proposition 4.4, we prove that (for any choice of y) if $\vartheta(\phi)$ is trivial on $K_+ \cap H$, then, upon replacing (T, ϕ) by a suitable K_+-conjugate, T is θ-stable and ϕ has a θ-symmetric G-factorization on T_{0+}. In addition, T contains G-regular θ-split elements. (Here, an element g of G is *θ-split* if $\theta(g) = g^{-1}$.) This is the main result of the section and is an essential tool (utilized in Section 7) in establishing necessary conditions for distinction of a positive regular representation associated with (T, ϕ).

Given $(T, \phi) \in \mathcal{T}(G)$, we denote a positive regular representation of G associated with the G-conjugacy class of (T, ϕ) by $\pi^G_{(T,\phi)}$. In Proposition 7.1, we prove that an H-distinguished positive regular representation is of the form $\pi^G_{(T,\phi)}$, where T is θ-stable, ϕ has a θ-symmetric G-factorization on T_{0+} and T contains θ-split G-regular elements. However, if a pair (T, ϕ) has these properties, this is not sufficient for distinction of $\pi^G_{(T,\phi)}$. But, as shown in Proposition 7.4, it is sufficient to guarantee distinction of $\pi^G_{(T,\chi\phi)}$ for some depth-zero quasicharacter χ of T. It is important to note that there may not exist θ-split G-regular elements in G. In this case, positive regular representations of G are never H-distinguished. This was previously proved for positive regular supercuspidals in [12].

The next definition is from [7] (see also [9]). Let Z be the centre of G.

Definition 1.1. An H-distinguished admissible representation π of G is said to be *H-relatively supercuspidal* if for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$ and every vector v

in the space of π, the function $g \mapsto \lambda(\pi(g^{-1})v)$ is compactly suported modulo HZ.

Kato and Takano in [7, Theorem 7.1] proved a p-adic symmetric space analogue of Jacquet's subrepresentation theorem which elucidates the importance of the problem of classifying relatively supercuspidal representations. A parabolic subgroup P of G is said to be *θ-split* if $P \cap \theta(P)$ is a Levi factor of P. The symmetric space subrepresentation theorem says that if π is an irreducible admissible H-distinguished representation of G, then there exists a θ-split parabolic subgroup P of G and an irreducible admissible $(P \cap H)$-relatively supercuspidal representation ρ of $P \cap H$ such that π occurs as a subrepresentation of $\operatorname{Ind}_P^G \rho$.

As indicated above, positive regular representations arise via parabolic induction from positive regular supercuspidals. What's more, the filtrations of their Jacquet modules arising from the Geometric Lemma ([2, Theorem 5.2]) involve positive regular representations of Levi subgroups. Consequently, if a parabolic subgroup P of G has a θ-stable Levi factor M, then we may apply Proposition 7.1 to establish necessary conditions for distinction of the Jacquet module $r_P(\pi)$ of a positive regular representation π. Combining this with the characterization of H-relatively supercuspidal representations given in [7, Theorem 6.9], we deduce (Proposition 8.3) that if T is θ-stable and compact modulo $(T \cap H)Z$ (that is, T is *θ-elliptic*) and $\pi^G_{(T,\phi)}$ is H-distinguished, then $\pi^G_{(T,\phi)}$ is H-relatively supercuspidal. Here is the main result of the paper (Theorem 8.5):

Theorem 1.2. *Let $(T, \phi) \in \mathcal{T}(G)$ and let M be the centralizer of the maximal F-split subtorus of T. Assume that T is θ-elliptic and $\phi \,|\, T_{0+} \cap H = 1$. Then there exists a depth-zero quasicharacter χ of T such that $\pi^M_{(T,\chi\phi)}$ is M^θ-distinguished. Furthermore,*

(1) $\operatorname{Hom}_{M^\theta}(\pi^M_{(T,\chi\phi)}, 1)$ *is isomorphic to a subspace of* $\operatorname{Hom}_H(\pi^G_{(T,\chi\phi)}, 1)$,

(2) $\pi^G_{(T,\chi\phi)}$ *is H-relatively supercuspidal.*

Further constructions of H-relatively supercuspidal tame representations are carried out in [14]. The arguments there are more complicated than the elementary arguments involving one dimensional types and the Geometric Lemma that we have employed in this paper.

2. **Notation and definitions**

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. If $\boldsymbol{G}$ is an F-group, then G denotes the group of F-rational points $\boldsymbol{G}(F)$ of $\boldsymbol{G}$. A connected reductive F-group $\boldsymbol{G}$ (or its F-rational points G) is *tame* if $\boldsymbol{G}$ splits over a tamely ramified extension of F.

For the remainder of the paper, we assume that $\boldsymbol{G}$ is a tame connected reductive F-group. We denote the centre of G by Z and the maximal F-split subtorus of Z by A_G.

If G' is a subgroup of G and $g \in G$, then ${}^gG' = gG'g^{-1}$. Let $C_G(g)$ and $C_G(G')$ denote the centralizers in G of g and G', respectively, and $N_G(G')$ the normalizer of G' in G. If ρ is a representation of G, then the function $x \mapsto {}^g\rho(x) := \rho(g^{-1}xg)$ on ${}^gG'$ defines a representation of ${}^gG'$.

A subgroup T of G is called a torus of G if $T = \boldsymbol{T}(F)$ for some F-torus $\boldsymbol{T}$ of $\boldsymbol{G}$. The same notational convention is applied to parabolic subgroups, their Levi factors, their unipotent radicals, and to twisted Levi subgroups. Recall from the introduction that a *twisted Levi subgroup* of $\boldsymbol{G}$ is a connected reductive subgroup of $\boldsymbol{G}$ that becomes a Levi subgroup of $\boldsymbol{G}$ over a finite extension of F. A (*tame*)*twisted Levi sequence* of G is a sequence $\vec{G} = (G^0, \ldots, G^d)$ where $G^0 \subset \cdots \subset G^d = G$ and each G^i $(0 \le i \le d-1)$ is a (tame) twisted Levi subgroup of G.

Given a Levi subgroup M of G, the set of parabolic subgroups having M as a Levi factor is denoted by $\mathcal{P}(M)$. If π is a smooth representation of G and P is a parabolic subgroup of G, the Jacquet module of π along P will be denoted by $r_P(\pi)$.

Let $\mathcal{B}(G)$ be the extended Bruhat-Tits building of G (cf. [3, Definition 7.4.2]). As in [10], we can associate to any point x in $\mathcal{B}(G)$ a parahoric subgroup $G_{x,0}$ of G and a filtration $\{\, G_{x,r} \mid r \in \mathbb{R},\ r \ge 0 \,\}$ of the parahoric. (The indexing of the filtration depends on a choice of affine roots, hence on a normalization of valuation on F, which for our purposes may be ignored.) For $r \in \mathbb{R}$ such that $r \ge 0$, let $G_{x,r^+} = \cup_{s>r} G_{x,s}$ and $G_{r^+} = \cup_{x \in \mathcal{B}(G)} G_{x,r^+}$.

If G' is a tame twisted Levi subgroup of G, then there exists a collection of embeddings $\mathcal{B}(G') \hookrightarrow \mathcal{B}(G)$ having a canonical image, see [17, 2.6.1]. All embeddings of buildings mentioned in the paper are assumed to be of this form. Given such an embedding and a point x in $\mathcal{B}(G')$, we typically identify x with its image in $\mathcal{B}(G)$. In that case, $G'_{x,r} = G_{x,r} \cap G'$, for $r > 0$.

Throughout the paper, we fix an involution θ of G. That is, θ is an F-automorphism of $\boldsymbol{G}$ of order two. If G' is a (not necessarily θ-stable) subgroup of G, let G'^{θ} be the group of θ-fixed elements of G'. Set $H = G^{\theta}$ Given $g \in G$, we define an involution $g \cdot \theta$ of g by $(g \cdot \theta)(x) = g\theta(g^{-1}xg)g^{-1}$, $x \in G$. Note that ${}^gH = G^{g \cdot \theta}$. The space G/H is often referred to as a *p-adic symmetric space*.

An element g of G is said to be *θ-split* if $\theta(g) = g^{-1}$. A torus T of G is said to be *θ-split* if every element of T is θ-split. If T is a θ-stable torus in G, let T^- be the maximal θ-split subtorus of T. When T is a θ-stable maximal torus of G, then A_T is θ-stable and we say that T is *θ-elliptic* if $A_T^- = A_G^-$. This is equivalent to compactness of $T/T^{\theta}Z$ (and of T^-/A_G^-). Recall that a maximal torus T of G is *elliptic* if T/A_G is compact. The θ-elliptic maximal tori are the symmetric space analogues of the elliptic maximal tori. Note that a maximal torus of G may have several θ-stable conjugates in G, some of which are θ-elliptic and others are not.

Remark 2.1. Suppose that G' is a connnected reductive p-adic group, $G = G' \times G'$ and θ is the involution of G given by $\theta(g_1, g_2) = (g_2, g_1)$, for g_1, $g_2 \in G'$. The θ-elliptic maximal tori of G are the θ-stable elliptic maximal tori of G.

A parabolic subgroup P of G is said to be θ-*split* if $P \cap \theta(P)$ is a Levi factor of P. In this case, P and $\theta(P)$ are opposite parabolic subgroups relative to the Levi subgroup $P \cap \theta(P)$.

Remark 2.2. If P is a proper θ-split parabolic subgroup of G and $M = P \cap \theta(P)$, then $M = C_G(A_M^-)$. In particular, A_M^- strictly contains A_G^-. This means that M does not contain any θ-elliptic maximal torus of G. (This is analogous to the fact that a proper Levi subgroup of G does not contain any elliptic maximal torus of G.)

3. **Quasicharacters and factorizations**

As mentioned in the introduction, in addition to the assumption that the residual characteristic p of F is odd, throughout the paper the following restrictions are imposed on p:

- (C1) p is not a bad prime for $\boldsymbol{G}$.
- (C2) p does not divide the order of the fundamental group of the derived group of $\boldsymbol{G}$.

Remark 3.1. Depending on the types of the components of $\boldsymbol{G}$, condition (C1) may exclude $p = 3$ and $p = 5$, see [16, I,4.3].

We refer to a smooth one-dimensional representation of G as a quasicharacter of G. Many of the results of [4] (also [11] and [12]) are proved subject to hypotheses on quasicharacters of G and of the groups occurring in tame twisted Levi sequences $\vec{G}$ of G. These hypotheses (the statements of which can be found in [4, Section 2.6]) are labelled $\mathrm{C}(G)$ and $\mathrm{C}(\vec{G})$.

Lemma 3.2 ([6, Lemma 3.3.3])**.** *If p satisfies* (C2)*, then Hypotheses* $\mathrm{C}(G)$ *holds and Hypothesis* $\mathrm{C}(\vec{G})$ *holds for any tame twisted Levi sequence $\vec{G}$ of G.*

In [18, Section 9], Yu defined the notion of G-generic quasicharacter of a tame twisted Levi subgroup. The definition involves the choice of a point in the Bruhat-Tits building of the twisted Levi subgroup. One useful consequence of Lemma 3.2 is that the G-genericity property is independent of the point (under the above restrictions on p), see [11, Lemma 4.7(2)].

Remark 3.3. Taking into account of Lemma 3.2, and the fact that we assume p is odd (this is used in many arguments involving involutions), when we apply arguments and results of [4] and [11], we need not mention hypotheses. Moreover, we may freely apply those results and definitions from [6] that require p to satisfy (C1) and (C2).

The *depth* of a quasicharacter ϕ of G is the smallest nonnegative real number r such that $\phi \,|\, G_{r^+} = 1$. (According to [4, Lemma 2.45], this is equivalent to the definition of depth of ϕ due to Moy and Prasad [10].)

Definition 3.4. Let ϕ be a quasicharacter of a tame twisted Levi subgroup G' of G. A *G-factorization* of ϕ (on G'_{0+}) is of a pair $(\vec{G}, \vec{\phi})$ consisting of a tame twisted Levi sequence $\vec{G} = (G^0, \dots, G^d)$ of G and a sequence $\vec{\phi} = (\phi_0, \dots, \phi_d)$, where, for $0 \leq i \leq d$, ϕ_i is a quasicharacter of G^i, satisfying:

(1) $G' \subset G^0$.
(2) $\phi = \prod_{i=0}^{d} \phi_i \,|\, G'$ ($\phi \,|\, G'_{0+} = \prod_{i=0}^{d} \phi_i \,|\, G'_{0+}$).
(3) $0 < r_0 < \cdots < r_{d-1}$, where r_i is the depth of ϕ_i for $0 \leq i < d$.
(4) For $0 \leq i < d$, ϕ_i is G^{i+1}-generic.
(5) If $d \geq 1$ and ϕ_d is nontrivial, then the depth r_d of ϕ_d satisfies $r_d > r_{d-1}$.

In two G-factorizations of a quasicharacter of a tame twisted Levi subgroup, the twisted Levi sequences are the same. Moreover, if d is the number of groups in the twisted Levi sequence, then, for $0 \leq i \leq d$, the two quasicharacters of the ith group of the twisted Levi sequence have equal depth. Further relations between the quasicharacters occurring the two G-factorizations are described in [11, Proposition 5.4].

In the case where ϕ is a quasicharacter of a tame maximal torus T of G, Kaletha ([6, Definition 3.5.1]) defines the notion of *Howe factorization* of (T, ϕ). (If $G = \boldsymbol{GL}_n(F)$, this definition agrees with the usual definition of Howe factorization.) A Howe factorization of (T, ϕ) involves a twisted Levi sequence $\vec{G} = (G^0, \dots, G^d)$ and a sequence of quasicharacters $\phi_0, \dots, \phi_d$ of $G^0, \dots, G^d$, respectively. There is also a depth-zero quasicharacter ϕ_{-1} of T, such that ϕ_{-1} is trivial on T_{0+} and $\phi = \prod_{i=-1}^{d} \phi_i \,|\, T$. For $0 \leq i < d$, the quasicharacters ϕ_i satisfy the same genericity requirement as in Definition 3.4. Thus, dropping the depth-zero quasicharacter from the Howe factorization of (T, ϕ) produces a G-factorization of ϕ. (We remark that, each of the quasicharacters ϕ_i, $0 \leq i \leq d$, in a Howe factorization have the additional property of restricting trivially to the simply connected cover of the derived group of $\boldsymbol{G}^i$.)

Kaletha shows ([6, Proposition 3.5.4]) that Howe factorizations exist (even when p does not satisfy condition (C2)). Returning to the more general setting where G' is not necessarily a maximal torus of G, we produce a G-factorization of ϕ by choosing a tame maximal torus $T \subset G'$ and then dropping the depth-zero quasicharacter from a Howe factorization of (T, ϕ). As a consequence, wherever results of [11] which assume existence of factorizations of quasicharacters are applied, it is now sufficient to assume that p satisfies (C1) (though usually (C2) is also required to guarantee that other hypotheses hold).

Definition 3.5. A quasicharacter ϕ of a twisted Levi subgroup G' of G is *G-regular on* G'_{0+} if for any quasicharacter of a twisted Levi subgroup of G strictly containing G', $(\chi \,|\, G')\phi$ has positive depth.

This regularity condition may also be stated in root-theoretic terms: if R is a root system of $\boldsymbol{T}$ in $\boldsymbol{G}$, then the subsystem R_{0+} defined in [6, Section 3.5] must equal R.

The following lemma is immediate from Definition 3.5 and the fact that if G' is a twisted Levi subgroup of a twisted Levi subgroup G'' of G, then G' is a twisted Levi subgroup of G.

Lemma 3.6. *If G' and G'' are twisted Levi subgroups of G such that $G' \subset G''$, and ϕ is a quasicharacter of G' that is G-regular on G'_{0+}, then ϕ is G''-regular on G'_{0+}.*

Recall that, as remarked above, the twisted Levi sequences occurring in two different factorizations of a quasicharacter are the same.

Lemma 3.7 ([11, Corollary 6.5, Lemma 5.6])**.** *Let ϕ be a quasicharacter of a tame twisted Levi subgroup G' of G. Let G^0 be the smallest Levi subgroup in the Levi sequence occurring in a G-factorization of ϕ. Then*

(1) *ϕ is G-regular on G'_{0+} if and only $G' = G^0$.*
(2) *If $g \in N_G(G')$ is such that $\phi \,|\, G'_{0+} = {}^g\phi \,|\, G'_{0+}$, then $g \in G^0$.*

Recall that we have fixed an involution θ of G.

Definition 3.8. Let ϕ be a quasicharacter of a (tame) twisted Levi subgroup G' of G.

(1) If $\theta(G') = G'$, then we say that ϕ is *θ-symmetric* (on G'_{0+}) if $\phi\circ\theta = \phi^{-1}$ ($\phi \circ \theta \,|\, G'_{0+} = \phi^{-1} \,|\, G'_{0+}$).
(2) We say that a G-factorization $(\vec{G}, \vec{\phi}) = ((G^0, \dots, G^d), (\phi_0, \dots, \phi_d))$ of ϕ is *θ-symmetric* (on G'_{0+}) if $\theta(G^i) = G^i$ and ϕ_i is θ-symmetric (on G^i_{0+}) for $0 \le i \le d$.

Lemma 3.9. *Let G' be a θ-stable tame twisted Levi subgroup of G and ϕ a quasicharacter of G' that is G-regular on G'_{0+}. If ϕ is θ-symmetric (on G'_{0+}), then ϕ has a θ-symmetric factorization (on G'_{0+}). Moreover, the set of θ-split elements g in the centre of G' satisfying $C_G(g) = G'$ is nonempty.*

Proof. Let

$$(\vec{G} = (G^0, \dots, G^d), \vec{\phi} = (\phi_0, \dots, \phi_d))$$

be a G-factorization of ϕ. Then

$$(\theta(\vec{G}), \theta(\vec{\phi}^{-1})) := ((\theta(G^0), \dots, \theta(G^d)), (\phi_0^{-1} \circ \theta, \dots, \phi_d^{-1} \circ \theta))$$

is a G-factorization of $\phi^{-1}\circ\theta$. If ϕ is θ-symmetric, then these are G-factorizations of the same quasicharacter of G'. By [12, Proposition 5.4], for $1 \le i \le d-1$, $\theta(G^i) = G^i$ and the depth r_i of ϕ_i is equal to the depth of $\phi_i^{-1} \circ \theta$, for $1 \le i \le d-1$.

$$\prod_{j=i} \phi_j \,|\, G^i_{r^+_{i-1}} = \prod_{j=i} \phi_j^{-1} \circ \theta \,|\, G^i_{r^+_{i-1}}, \qquad 1 \le i \le d.$$

That is, the quasicharacter $\prod_{j=i} \phi_j \,|\, G^i$ is θ-symmetric on $G^i_{r^+_{i-1}}$.

We use the above information to produce a θ-symmetric factorization of ϕ, as follows. Because ϕ_d is θ-symmetric on $G_{r^+_{d-1}}$, there exists a θ-symmetric quasicharacter $\dot\phi_d$ of G such that $\dot\phi_d \,|\, G_{r^+_{d-1}} = \phi_d \,|\, G_{r^+_{d-1}}$. (Existence of θ-symmetric extensions is straightforward – see [4, Remark 5.16]). Set $\phi'_{d-1} = \phi_{d-1}\dot\phi_d^{-1}\phi_d$. Next, observe that ϕ'_{d-1} is θ-symmetric on $G^{d-1}_{r^+_{d-2}}$. Fix a θ-symmetric quasicharacter $\dot\phi_{d-1}$ of G^{d-1} such that $\dot\phi_{d-1}$ extends $\phi'_{d-1} \,|\, G^{d-1}_{r^+_{d-2}}$. Continuing in this manner, we obtain θ-symmetric quasicharacters $\dot\phi_i$ of G^i, $1 \le i \le d$, such that

$$\dot\phi_i \,|\, G^i_{r^+_{i-1}} = \phi_i \prod_{j=i+1}^{d} \phi_j \dot\phi_j^{-1}, \qquad 1 \le i \le d.$$

Finally, let

$$\dot\phi_0 = \phi_0 \prod_{j=1}^{d} \phi_j \dot\phi_j^{-1}.$$

Note that $G' = G^0$, by the G-regularity of ϕ on G'_{0+}. Thus $\dot\phi_0 = \phi \prod_{j=1}^{d} \dot\phi_d^{-1}$, so θ-symmetry of $\dot\phi_0$ follows from that of ϕ and $\dot\phi_i$, $1 \le i \le d$. Set $\vec{\dot\phi} = (\dot\phi_0, \ldots, \dot\phi_d)$. Then as shown in the discussion following [11, Definition 5.3], $(\vec G, \vec{\dot\phi})$ is a G-factorization of ϕ.

If ϕ is θ-symmetric on G'_{0+}, then there exists a depth-zero quasicharacter η of G' such that $\phi\eta$ is θ-symmetric, so $\phi\eta$ has a G-factorization that is θ-symmetric. Upon multiplying the first quasicharacter in the corresponding sequence by η^{-1}, we obtain a G-factorization of ϕ that is θ-symmetric on G'_{0+}.

To obtain the second conclusion of the lemma, we may argue as in [12, Lemmas 5.3, 6.5, 6.6, Proposition 6.7]. The results of [12] are proved in the supercuspidal setting and involve θ-symmetric cuspidal G-data. However, the proofs depend only on θ-symmetry of the quasicharacters in the datum on the topologically unipotent sets of the twisted Levi subgroups. Thus, in order to see that the arguments of [12] give the desired result, existence of a G-factorization of ϕ that is θ-symmetric on G'_{0+} is sufficient. □

4. **Preliminary results**

In our study of distinguished representations we will work with linear characters of compact open subgroups of G that are associated to factorizations of quasicharacters of tori. We summarize some of their properties here.

Throughout this section, we fix a tame maximal torus T of G and a quasicharacter ϕ of T that is G-regular on T_{0+}.

Choose a G-factorization $(\vec G = (T = G^0, \ldots, G^d), \vec\phi = (\phi_0, \ldots, \phi_d))$ of ϕ.

Fix a sequence of embeddings

$$\mathcal{B}(T) = \mathcal{B}(G^0) \hookrightarrow \mathcal{B}(G^1) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(G^d) = \mathcal{B}(G).$$

For $0 \le i \le d-1$, let $s_i = r_i/2$, where r_i is the depth of ϕ_i. Fix $y \in \mathcal{B}(T)$. For $1 \le i \le d$, we define compact open subgroups K_+^i of G^i as in [18, Section 3]:

$$K_+^i = T_{0+} G^1_{y,s_0^+} G^2_{y,s_1^+} \cdots G^i_{y,s_{i-1}^+}.$$

(Here we are abusing notation and identifing y with its image in $\mathcal{B}(G^i)$ for $1 \le i \le d$.) Because the twisted Levi sequence $\vec{G}$ and the depths r_i are independent of the choice of G-factorization, the group K_+^i depends on y, on $\phi \,|\, T_{0+}$ and on the chosen sequence of embeddings of buildings, but not on the choice of factorization.

For $0 \le i \le d$, let $\hat{\phi}_i$ be the character of K_+ defined as in [18, Section 4]. In particular, $\hat{\phi}_i \,|\, K_+^i = \phi_i \,|\, K_+^i$. Set $\vartheta(\vec{\phi}) = \prod_{i=0}^d \hat{\phi}_i$. Then $\vartheta(\vec{\phi}) \,|\, T_{0+} = \phi \,|\, T_{0+}$.

Lemma 4.1. *Let $(\vec{G}, \vec{\phi}')$ be another G-factorization of ϕ. Then $\vartheta(\vec{\phi}) = \vartheta(\vec{\phi}')$.*

Proof. The proof is inductive, using the relations between two different G-factorizations of ϕ, as described in [11, Proposition 5.4]. The details are omitted. □

From this point on, we use the notation $\vartheta(\phi)$ instead of $\vartheta(\vec{\phi})$.

If $P = MN \in \mathcal{P}(M)$, let $\overline{P} = M\overline{N} \in \mathcal{P}(M)$ be the opposite parabolic subgroup. We say that a compact open subgroup K of G *decomposes with respect to* $(N, M, \overline{N})$ if $K = (K \cap N)(K \cap M)(K \cap \overline{N})$.

The next lemma is proved in [8, Lemma 6.3 and proof].

Lemma 4.2. *Let the notation be as above. Fix $P \in \mathcal{P}(M)$. Then*

1. *The group K_+ decomposes with respect to $(N, M, \overline{N})$.*
2. *For $0 \le i \le d$, $\hat{\phi}_i$ is trivial on $K_+ \cap N$ and $K_+ \cap \overline{N}$.*

Lemma 4.3. *Let K_+ be as above. Suppose that $\theta(M) = M$ and χ is a character of K_+ such that for every $P = MN \in \mathcal{P}(M)$, χ is trivial on $K_+ \cap N$ and $K_+ \cap \overline{N}$. If $\chi \,|\, (K_+ \cap M)^\theta = 1$, then $\chi \,|\, K_+^\theta = 1$.*

Proof. Let $J = K_+ \cap \theta(K_+)$. Since $K_+^\theta = J^\theta$ and $(K_+ \cap M)^\theta = (J \cap M)^\theta$, it suffices to prove that

$$J^\theta = (J \cap M)^\theta (\mathrm{Ker}\,\chi)^\theta.$$

Note that $\theta(P)$ and $\theta(\overline{P})$ are opposite parabolic subgroups in $\mathcal{P}(\theta(M)) = \mathcal{P}(M)$.

By Lemma 4.2(1), K_+ decomposes with respect to $(N, M, \overline{N})$ and also with respect to $(\theta(N), M, \theta(\overline{N}))$. A straightforward argument which combines this with injectivity of the product map $N \times M \times \overline{N} \to G$ shows that J decomposes with respect to $(N, M, \overline{N})$.

Let $U = \operatorname{Ker}\chi \cap \theta(\operatorname{Ker}\chi)$. By assumption, the subgroups $J \cap N$, $J \cap \overline{N}$, $J \cap \theta(N)$ and $J \cap \theta(\overline{N})$ belong to $\operatorname{Ker}\chi$. This, together with the fact that J decomposes with respect to $(N, M, \overline{N})$, can be used to show that $J = (J \cap M)U$.

Finally, because $J \cap M \cap U$ is a θ-stable subgroup of $M_{y,0^+}$, the image of the function $u \mapsto u\theta(u)^{-1}$ on $J \cap M \cap U$ is the set of all θ-split elements in $J \cap M \cap U$ (see, for example, [4, Proposition 2.12]). As U is normalized by $J \cap M$, it follows that $J^\theta = (J \cap M)U^\theta$ ([4, Lemma 2.9]). □

Proposition 4.4. *Let* $K = G^1_{y,s_0} G^2_{y,s_1} \cdots G^d_{y,s_{d-1}}$. *Then* $\vartheta(\phi) \,|\, K_+^\theta = 1$ *if and only if there exists* $k \in K$ *such that* ${}^kT$ *is* θ*-stable and* ${}^k\phi \,|\, {}^kT^\theta_{0+} = 1$. *Furthermore, if* k *is as above, then*

(1) *The quasicharacter* ${}^k\phi$ *has a* θ*-symmetric* G*-factorization on* ${}^kT_{0+}$.
(2) *The torus* ${}^kT$ *contains* G*-regular* θ*-split elements.*

Proof. First we prove that $\vartheta(\phi) \,|\, K_+^\theta = 1$ implies existence of existence of $k \in K$ such that ${}^kT$ is θ-stable and ${}^k\phi$ has a θ-symmetric G-factorization on ${}^kT_{0+}$.

The property $\vartheta(\phi) \,|\, K_+^\theta = 1$ corresponds to what is called weak compatibility in [4]. If the torus T is elliptic in G, existence of an element of K as above is a special case of [4, Proposition 5.7(1)]. The proof of [4, Proposition 5.7(1)] depends on the choice of $y \in \mathcal{B}(T)$ and on the factorization of ϕ occurring in the G-datum, but other properties of the cuspidal datum are not needed. Consequently, the proof carries over to nonelliptic T.

Taking into account Lemma 3.9, one direction of the proof of the proposition is now complete. Moreover, for the other direction, it suffices to show that if $k \in K$ is such that ${}^kT$ is θ-stable and ${}^k\phi \,|\, ({}^kT_{0+})^\theta = 1$, then $\vartheta(\phi) \,|\, K_+^\theta = 1$. Since $\vartheta(\phi) \,|\, K_+^\theta = 1$ if and only if $\vartheta(\phi) \,|\, K_+^{k \cdot \theta} = 1$, we may adjust our notation so that $k = 1$. That is, we assume that $\theta(T) = T$ and $\phi \,|\, T^\theta_{0+} = 1$.

By Lemma 3.9, there exists a G-factorization $(\vec{G}, \vec{\phi})$ of ϕ that is θ-symmetric on T_{0+}. By Lemma 4.1,

$$\vartheta(\phi) = \vartheta(\vec{\phi}) = \prod_{i=1} \hat{\phi}_i.$$

Hence it suffices to show that $\hat{\phi}_i \,|\, K_+^\theta = 1$ for $1 \leq i \leq d$. Let $M = C_G(A_T)$. Then $\theta(T) = T$ implies $\theta(M) = M$. Taking into account, Lemma 4.2(2) and Lemma 4.3, if $\hat{\phi}_i \,|\, (K_+ \cap M)^\theta = 1$, then $\hat{\phi}_i \,|\, K_+^\theta = 1$.

To complete the proof, we show that $\hat{\phi}_i \,|\, (K_+ \cap M)^\theta = 1$ for $1 \leq i \leq d$. Let $M^i = M \cap G^i$. Then M^i is a tame twisted Levi subgroup of M ([8, proof of Lemma 2.4]).

By definition of M, T is elliptic in M, hence also in M^i. By compactness of T/A_{M^i}, $M^i_{x_1,r^+} = M_{x_2,r^+}$ for all points x_1 and $x_2 \in \mathcal{B}(T)$. Because T is θ-stable and $y \in \mathcal{B}(T)$, we have $\theta(y) \in \mathcal{B}(T)$.

Using this information and the property $\theta(M^i_{y,r^+}) = M^i_{\theta(y),r^+}$, we conclude that M^i_{y,r^+} is θ-stable. It follows that $K_+ \cap M = T_{0^+} M^0_{y,r_0^+} \cdots M^d_{y,r_{d-1}^+}$ is θ-stable. An elementary argument using $K_+ \cap M \subset M_{y,0^+}$ and [4, Lemma 2.13] shows that if χ is a character of $K_+ \cap M$, then $\chi \,|\, (K_+ \cap M)^\theta = 1$ if and only if $\chi \circ \theta = \chi^{-1}$.

Equality of $\hat{\phi}_i \circ \theta$ and $\hat{\phi}_i^{-1}$ follows from the definition of $\hat{\phi}_i$ and θ-symmetry of ϕ_i on $M^i_{0^+}$. In particular, there exists a θ-stable subgroup U^i of $K_+ \cap M \cap \mathrm{Ker}\, \hat{\phi}_i$ that is normalized by $K_+ \cap M^i$, and

$$K_+ \cap M = (K_+ \cap M^i) U^i, \quad \hat{\phi}_i \,|\, K_+ \cap M^i = \phi_i \,|\, K_+ \cap M^i.$$

□

Let $M = C_G(A_T)$. For $1 \le i \le d$, let $M^i = C_{G^i}(A_T)$. Then, as observed in [8, Lemma 2.4], if $1 \le i \le j \le d$, M^i is a twisted Levi subgroup of M^j. The twisted Levi subgroups M^i of M are not necessarily distinct and form a generalized twisted Levi sequence in $M = M^d$, as defined on [18, p. 616].

Fix a commutative diagram of embeddings of buildings

$$\begin{array}{ccccccc} \mathcal{B}(T) = \mathcal{B}(G^0) & \hookrightarrow & \mathcal{B}(G^1) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{B}(G^d) \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{B}(T) = \mathcal{B}(M^0) & \hookrightarrow & \mathcal{B}(M^1) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{B}(M^d). \end{array}$$

In what follows below, we use the embeddings in the top row of the diagram when defining K_+.

Remark 4.5. In [8], the results of Kim and Yu concerning types and G-covers are proved subject to a genericity assumption on the commutative diagram of embeddings. Our results are independent of the choice of commutative diagram and we don't require the results of [8] on types and covers, so we don't make any assumptions regarding genericity of the embeddings.

In general, the pair $((T = M^0, \ldots, M^d), (\phi_0 \,|\, M^0, \ldots, \phi_d \,|\, M^d))$ is not an M-factorization of ϕ. Starting with a G-factorization $(\vec{G}, \vec{\phi})$, we can produce an M-factorization $(\vec{M}, \vec{\phi}_M)$ of ϕ as follows. For $1 \le j \le d$, let S_j be the set of i such $G^i \cap M = M^j$. Set $\phi'_j = \prod_{i \in S_j} \phi_i \,|\, M^j$. The twisted Levi sequence $\vec{M}$ consists of the distinct groups M^i. If M^{i_j} is the jth group in the sequence $\vec{M}$, then the jth quasicharacter in the sequence $\vec{\phi}_M$ is ϕ'_{i_j}. Now, [8, Lemma 5.3] can be used to show the required genericity. The sequence of embeddings of buildings on the bottom row of the commutative diagram above restricts to a sequence of embeddings of the buildings of the twisted Levi subgroups occuring in $\vec{M}$. We use the M-factorization of ϕ, together with the point $y \in \mathcal{B}(T)$, to define a compact open subgroup $K_{M,+}$ and a character $\vartheta_M(\phi) = \vartheta_M(\vec{\phi}_M)$ of $K_{M,+}$.

Because we have chosen the M-factorization of ϕ in a way that is compatible with the original G-factorization, the following lemma follows easily from the definitions of $\vartheta(\phi)$ and $\vartheta_M(\phi)$ in terms of these factorizations.

Lemma 4.6. *$K_{M,+} = K_+ \cap M$ and $\vartheta(\phi) \,|\, K_{M,+} = \vartheta_M(\phi)$.*

Remark 4.7. The definition of $K_{M,+}$ involves a choice of $y \in \mathcal{B}(T)$, the depths of the quasicharacters occurring in an M-factorization of ϕ, and a sequence of embeddings of buildings of the twisted Levi subgroups appearing in the M-factorization. Because T is elliptic in M, if y is replaced by another point in $\mathcal{B}(T)$, or if the sequence of embeddings of buildings is changed, the associated subgroup $K_{M,+}$ does not change, see [18, Remark 3.4]. As observed in Lemma 4.1, $\vartheta_M(\phi)$ is independent of the choice of M-factorization of ϕ. Thus $K_{M,+}$ and $\vartheta_M(\phi)$ are dependent on T and on $\phi \,|\, T_{0+}$, but are independent of other choices.

When T is not elliptic in G, then both K_+ and $\vartheta(\phi)$ do depend on the choice of point y in $\mathcal{B}(T)$, and on the sequence of embeddings of buildings, although $K_+ \cap M = K_{M,+}$ and $\vartheta(\phi) \,|\, K_+ \cap M = \vartheta_M(\phi)$ do not.

5. **Positive regular supercuspidal representations**

Yu's construction [18] associates irreducible supercuspidal representations of G to cuspidal (generic) G-data. As stated in the introduction, we refer to such representations as tame supercuspidal representations.

In [4], an equivalence relation on the set of (reduced, generic) cuspidal G-data is defined. [4, Theorem 6.6], which was proved subject to hypotheses on quasicharacters of twisted Levi subgroups of G, says that two cuspidal G-data belong to the same equivalence class if and only if the associated tame supercuspidal representations of G are equivalent. Recent work of Kaletha [6, Corollary 3.3.5] shows that the theorem holds without the hypotheses.

Each cuspidal G-datum includes a twisted Levi sequence. For the next definition, it is useful to recall that the twisted Levi sequences appearing in equivalent G-data are G-conjugate.

Definition 5.1. We say that a supercuspidal representation of G is *positive regular* if it arises via Yu's construction from a cuspidal G-datum having the property that the first twisted Levi subgroup in the sequence of twisted Levi subgroups occurring in the G-datum is a (tame) elliptic maximal torus in G.

Remark 5.2. In [4] and [11], the positive regular supercuspidal representations were referred to as *toral* supercuspidal representations. Since the term toral supercuspidal representation has been used by other authors to denote a particular proper subset of the set of positive supercuspidal representations, in order to avoid confusion, we do not use the term toral here.

Definition 5.3. Let $\mathcal{T}(G)$ be the set of pairs (T, ϕ), where T is a tame maximal torus in G and ϕ is a quasicharacter of T that is G-regular on T_{0^+}. Let $\mathcal{T}_{ell}(G) = \{ (T, \phi) \in \mathcal{T}(G) \mid T \text{ is elliptic} \}$.

Note that it is immediate from the definitions that $\mathcal{T}(G)$ and $\mathcal{T}_{ell}(G)$ are invariant under conjugation by G.

Let $(T, \phi) \in \mathcal{T}_{ell}(G)$. If $(\vec{G}, \vec{\phi})$ is a G-factorization of ϕ and 1 is the trivial character of T, then, in the terminology of [4], $(\vec{G}, 1, \vec{\phi})$ is a reduced, generic cuspidal G-datum.

An equivalence class of reduced, generic cuspidal G-data corresponds to an equivalence class of positive regular supercuspidal representations if and only if it contains a G-datum of the form $(\vec{G}, 1, \vec{\phi})$, where $(\vec{G}, \vec{\phi})$ is a G-factorization of a quasicharacter ϕ occurring in a pair $(T, \phi) \in \mathcal{T}_{ell}(G)$. Moreover, two such G-data are equivalent if and only if the associated pairs in $\mathcal{T}_{ell}(G)$ are G-conjugate. Therefore we have the following parametrization of the positive regular supercuspidal representations, a weaker version of which was stated, in different language, subject to hypotheses on quasicharacters, as [11, Corollary 9.4], and is a consequence of [6, Corollary 3.3.5, Proposition 3.5.4] and [4, Theorem 6.6]. As discussed below, the proposition is also obtained as a special case of Kaletha's parametrization of regular supercuspidal representations (cf. [6, Corollary 3.6.1]).

Proposition 5.4. *The set of equivalence classes of positive regular supercuspidal representations of G corresponds bijectively to the set of G-conjugacy classes of pairs in $\mathcal{T}_{ell}(G)$.*

Definition 5.5. Let $(T, \phi) \in \mathcal{T}_{ell}(G)$. We denote a positive regular supercuspidal representation whose equivalence class corresponds to the G-conjugacy class of (T, ϕ) by $\pi^G_{(T,\phi)}$.

We now comment on the relation between the positive regular supercuspidal representations and the regular supercuspidal representations of [6]. In [6, Section 3], Kaletha defines notions of regular and extra regular reduced generic cuspidal G-data. He proves that regularity is preserved under equivalence of G-data. The associated tame supercuspidal representations are called (*extra*) *regular* (*tame*) *supercuspidal representations*.

If T is an elliptic tame maximal torus of G and ϕ is a quasicharacter of T, Kaletha calls the pair (T, ϕ) *tame elliptic (extra) regular* if ϕ satisfies a particular root-theoretic regularity condition (cf. [6, Definition 3.4.5]). There is a pair (T, ϕ) associated to each (extra) regular reduced generic cuspidal G-datum, and Kaletha proves that such a pair is tame elliptic (extra) regular. Equivalence of such cuspidal G-data corresponds to G-conjugacy of the associated tame elliptic (extra) regular pairs. Furthermore, via a factorization process, called *Howe factorization*, each tame elliptic (extra) regular pair is associated to some (extra) regular tame elliptic pair. Consequently, as stated in [6, Corollary 3.6.1], this results in a bijective correspondence between the set of G-conjugacy classes of tame (extra) regular elliptic pairs and the set of equivalence classes of (extra) regular supercuspidal representations of G.

If $(T,\phi) \in \mathcal{T}_{ell}(G)$, then the fact that ϕ is G-regular on T_{0+} implies that (T,ϕ) is a tame elliptic (extra) regular pair. Thus Proposition 5.4 is a special case of [6, Corollary 3.6.1]. We have chosen the terminology positive regular supercuspidal representations because these are the regular supercuspidal representations corresponding to pairs (T,ϕ) to which the regularity condition is imposed on $\phi \,|\, T_{0+}$, that is, on the restriction of ϕ to the positive-depth elements in T.

6. Positive regular representations

Lemma 6.1. *Let $(T,\phi) \in \mathcal{T}(G)$. Let $M = C_G(A_T)$. Then*

1. *The pair (T,ϕ) belongs to $\mathcal{T}_{ell}(M)$.*
2. *If $\pi^M_{(T,\phi)}$ is as in Definition 5.1 and $P \in \mathcal{P}(M)$, then $\mathrm{Ind}_P^G \pi^M_{(T,\phi)}$ is irreducible.*

Proof. For the first part, note that, by Lemma 3.6, $(T,\phi) \in \mathcal{T}(M)$, and, by definition of M, T is elliptic in M.

For the second part, suppose that $g \in N_G(M)$ and ${}^g\pi^M_{(T,\phi)}$ is equivalent to $\pi^M_{(T,\phi)}\nu$ for some unramified quasicharacter ν of M. By [15, Corollary 2], to see that $\mathrm{Ind}_P^G \pi^M_{(T,\phi)}$ is irreducible, it suffices to show that $g \in M$. It follows from the definitions and the assumptions on g and (T,ϕ), together with the first part, that $({}^gT, {}^g\phi) \in \mathcal{T}(G) \cap \mathcal{T}_{ell}(M)$. Moreover, ${}^g\pi^M_{(T,\phi)}$ is equivalent to $\pi^M_{({}^gT,{}^g\phi)}$. Thus, by Kaletha's results (see Section 5 above), there exists $m \in M$ such that $mg \in N_G(T)$ and ${}^{mg}\phi = \phi\nu$. Restricting to T_{0+}, we have ${}^{mg}\phi \,|\, T_{0+} = \phi \,|\, T_{0+}$. Since ϕ is G-regular on T_{0+}, we may apply Lemma 3.7 to conclude that $mg \in M$. Hence $g \in M$. □

As a consequence of [2, Theorem 2.9] and Lemma 6.1, given $(T,\phi) \in \mathcal{T}(G)$, if $M = C_G(A_T)$, the equivalence class of $\mathrm{Ind}_P^G \pi^M_{(T,\phi)}$ is independent of the choice of $P \in \mathcal{P}(M)$.

Definition 6.2. We say that a representation π of G is a *positive regular representation* if π is equivalent to a representation of the form $\mathrm{Ind}_P^G \pi^M_{(T,\phi)}$, for some $(T,\phi) \in \mathcal{T}(G)$, where $M = C_G(A_T)$ and $P \in \mathcal{P}(M)$. In this case we write $\pi = \pi^G_{(T,\phi)}$.

Proposition 6.3. *Fix (T,ϕ) and $(T',\phi') \in \mathcal{T}(G)$. The representations $\pi^G_{(T,\phi)}$ and $\pi^G_{(T',\phi')}$ are equivalent if and only if the pairs (T,ϕ) and (T',ϕ') are G-conjugate.*

Proof. Because (cf. Lemma 6.1) the representations $\pi^G_{(T,\phi)}$ and $\pi^G_{(T',\phi)}$ are irreducible, they are equivalent if and only if their cuspidal supports coincide, that is, if and only if there exists $g \in G$ such that $M' = {}^gM$ and $\pi^{M'}_{(T',\phi')} \simeq {}^g(\pi^M_{(T,\phi)})$. Since ${}^g(\pi^M_{(T,\phi)}) \simeq \pi^{{}^gM}_{({}^gT,{}^g\phi)}$, this amounts to existence of $g \in G$ such that $M' = {}^gM$ and $\pi^{M'}_{(T',\phi')} \simeq \pi^{M'}_{({}^gT,{}^g\phi)}$. By Proposition 5.4, the pairs (T',ϕ') and $({}^gT,{}^g\phi)$ are M-conjugate. □

Lemma 6.4. *Let $(T,\phi) \in \mathcal{T}(G)$. If K_+ and $\vartheta(\phi)$ are as defined in Section 4, then the space of $\pi^G_{(T,\phi)}$ is generated by its $(K_+,\vartheta(\phi))$-isotypic subspace.*

Proof. Fix $P = MN$ and $\overline{P} = M\overline{N} \in \mathcal{P}(M)$ such that $P \cap \overline{P} = M$. Since (Lemma 6.1) $\operatorname{Ind}_P^G \pi^M_{(T,\phi)}$ is irreducible, it suffices to show that the $(K_+,\vartheta(\phi))$-isotypic subspace is nonzero.

Let $M = C_G(A_T)$, and define $K_{M,+}$ and $\vartheta_M(\phi)$ as in Section 4. The supercuspidal representation $\pi^M_{(T,\phi)}$ is obtained via compact induction from a representation κ_M of an open compact-mod-centre subgroup K_M of M that contains $K_{M,+}$. Moreover, the space of κ_M coincides with its $(K_{M,+},\vartheta_M(\phi))$-isotypic subspace (cf. [18, Proposition 4.4]). Hence the $(K_{M,+},\vartheta_M(\phi))$-isotypic subspace of $\pi^M_{(T,\phi)}$ is nonzero. Fix a nonzero vector v in this isotypic subspace. Then it follows from Lemma 4.6 that there exists a unique function f in the space of $\operatorname{Ind}_P^G \pi^M_{(T,\phi)}$ such that $f(1) = v$ and the support of f is equal to PK_+. This function is in the $(K_+,\vartheta(\phi))$-isotypic subspace. □

7. Distinction of positive regular representations

Here we derive necessary conditions for distinction of positive regular representations. These conditions are used in the next section to show that certain representations are relatively supercuspidal.

Proposition 7.1, Corollary 7.2 and Proposition 7.4 were proved in the context of positive regular supercuspidal representations in [4, Proposition 5.2(2)], [12, Proposition 5.7(2)], [12, Theorem 6.9], and [12, Proposition 6.11], (based on [4, Proposition 5.31]), respectively. Those results were proved subject to hypotheses on quasicharacters. Recall (see Remark 3.3) that, due to recent results of Kaletha, under the conditions we have imposed on p (listed at the beginning of Section 3) the hypotheses hold.

Proposition 7.1. *Let $(T,\phi) \in \mathcal{T}(G)$. Suppose that $\pi^G_{(T,\phi)}$ is H-distinguished. Then there exists $g \in G$ such that*

1. *$\theta({}^gT) = {}^gT$ and ${}^g\phi \,|\, {}^gT^\theta_{0+} = 1$.*
2. *The quasicharacter ${}^g\phi$ of ${}^gT$ has a G-factorization that is θ-symmetric on ${}^gT_{0+}$.*
3. *There exist G-regular θ-split elements in ${}^gT$.*

(4) *The group ${}^gM$ is θ-stable and ${}^g\phi$ has a ${}^gM$-factorization that is θ-symmetric on ${}^gT_{0+}$.*

Proof. Fix $y \in \mathcal{B}(T)$ and define K_+ and $\vartheta := \vartheta(\phi)$ as in Section 4. Let $\pi = \pi^G_{(T,\phi)}$ and let $V^{(K_+,\vartheta}$ be the (K_+,ϑ)-isotypic subspace of the space V of π. Fix a nonzero $\lambda \in \mathrm{Hom}_H(\pi, 1)$. By Lemma 6.4, there exists $g \in G$ such that the restriction of λ to $\pi(g)V^{(K_+,\vartheta)}$ is nonzero. This implies that ${}^g\vartheta \,|\, {}^gK_+^\theta = 1$. This is equivalent to $\vartheta \,|\, K_+^{g^{-1}\cdot\theta} = 1$, where $(g^{-1}\cdot\theta)(x) = g^{-1}\theta(gxg^{-1})g$ for $x \in G$.

According to Lemma 4.4, there exists $k \in K_+$ such that ${}^kT$ is $g^{-1}\cdot\theta$-stable, ${}^k\phi$ has a $g^{-1}\cdot\theta$-symmetric factorization on $({}^kT_{0+})^{g^{-1}\dot\theta}$ and ${}^kT$ contains G-regular $g^{-1}\cdot\theta$-split elements. After replacing g by kg, and rearranging the notation slightly, we obtain parts (2) and (3).

Part (1) is implied by part (3).

Part (4) is implied by part (2) and Lemma 4.6(4). □

In general, G need not contain θ-split G-regular elements. There are many examples of this sort, such as $G = \boldsymbol{GL}_{2n}(F)$, $H = \boldsymbol{Sp}_{2n}(F)$,

Corollary 7.2. *If there exist H-distinguished positive regular representations of G, then there exist θ-split G-regular elements.*

Remark 7.3. Suppose that F' is a finite, tamely ramified extension of F, $\mathfrak{R}_{F'/F}$ is restriction of scalars, $\boldsymbol{G} = \mathfrak{R}_{F'/F}\boldsymbol{GL}_n$, and θ is any involution of G. The converse of the corollary was proved in [12] in the positive regular supercuspidal setting. Thus existence of H-distinguished positive supercuspidal representations of $G = \boldsymbol{GL}_n(F')$ is equivalent to existence of θ-split tame elliptic G-regular elements in G. Although we have not verified the details, we expect that the converse of the corollary holds.

It is immediate from the definition that if ϕ is a quasicharacter of a tame maximal torus of G and if $(T,\phi) \in \mathcal{T}(G)$ and χ is a depth-zero quasicharacter of T, then $(T,\phi) \in \mathcal{T}(G)$ if and only if $(T,\chi\phi) \in \mathcal{T}(G)$.

Proposition 7.4. *Let $(T,\phi) \in \mathcal{T}(G)$ and $M = C_G(A_T)$. If π^G_ϕ is H-distinguished, then, upon replacing (T,ϕ) by a suitable G-conjugate, $\theta(T) = T$ (hence $\theta(M) = M$), and there exists a depth-zero quasicharacter χ of T such that $\pi^M_{(T,\phi\chi)}$ is M^θ-distinguished.*

Proof. By Proposition 7.1(4), after replacing (T,ϕ) by a G-conjugate, we may assume that T is θ-stable and ϕ has an M-factorization $(\vec{M},\vec{\phi})$ that is θ-symmetric on T_{0+}. For each M^i appearing in $\vec{M}$, fix a depth-zero quasicharacter η_i of M^i such that $\eta_i\phi_i$ is θ-symmetric. Set $\phi_i' = \eta_i\phi_i$ and let $\vec{\phi}'$ be the sequence obtained from $\vec{\phi}$ by replacing each ϕ_i by ϕ_i'. Set $\phi' = \phi\eta$, where $\eta = \prod_i \eta_i$. Then $(\vec{M},\vec{\phi}')$ is a θ-symmetric M-factorization of ϕ'.

Fix $y \in \mathcal{B}(T)$ and let χ' be a depth-zero quasicharacter of T. The four-tuple $\Psi := (\vec{M}, y, \chi', \phi')$ is a θ-symmetric cuspidal extended M-datum (in the sense of [4, Definition 3.13]). Moreover, the supercuspidal representation of M associated to Ψ via Yu's construction is (equivalent to) $\pi^M_{(T,\chi'\eta\phi)}$. We may now apply [12, Proposition 6.11] (which is a consequence of [4, Proposition 5.31]) to conclude that there exists a choice of χ' for which $\pi^M_{(T,\chi'\eta,\phi)}$ is M^θ-distinguished. □

Remark 7.5. Suppose that T is a nonelliptic tame maximal torus in G, $M = C_G(A_T)$, $(T,\phi) \in \mathcal{T}_{ell}(M)$, and $P \in \mathcal{P}(M)$. The representation $\operatorname{Ind}_P^G \pi^M_{(T,\phi)}$ is often irreducible, even if ϕ is not G-regular on T_{0^+}. In that case, despite the fact that $\pi^M_{(T,\phi)}$ is a positive regular supercuspidal representation of M, obtaining precise information about distinction of $\operatorname{Ind}_P^G \pi^M_{(T,\phi)}$ is more involved than in the G-regular setting and may involve questions about distinction of depth-zero types. In particular, although there is an analogue of Proposition 7.4 in this case, the methods of this paper are not sufficient to prove it. This, and other more general questions about distinction of tame representations, are treated in [13]. The methods include analysis of distinction and θ-symmetry of types arising from associated G-data (defined in [8]). The types involved are more complicated than the linear characters of the form $\vartheta(\phi)$).

8. Relatively supercuspidal positive regular representations

Proposition 8.1. *Let $(T,\phi) \in \mathcal{T}(G)$ be such that $\theta(T) = T$ and $\phi \,|\, T^\theta_{0^+} = 1$. Let L be a θ-stable proper Levi subgroup of G and let $Q \in \mathcal{P}(L)$. If the Jacquet module $r_Q(\pi^G_{(T,\phi)})$ is L^θ-distinguished, then there exists $g \in G$ such that ${}^gT \subset L$ and $g^{-1}\theta(g) \in T$.*

Proof. Let $M = C_G(A_T)$. Fix $P \in \mathcal{P}(M)$. Because $\pi^M_{(T,\phi)}$ is supercuspidal, the filtration of the Jacquet module $r_Q(\operatorname{Ind}_P^G \pi^M_{(T,\phi)})$ given by the Geometric Lemma ([2, Theorem 5.2]) involves a finite number of quotients of the form $\operatorname{Ind}^L_{L\cap {}^gP}({}^g\pi^M_{(T,\phi)})$, where each g is such that $L \cap {}^gP$ is a parabolic subgroup of L having ${}^gM$ as a Levi factor. Since

$$\operatorname{Ind}^L_{L\cap {}^gP}({}^g\pi^M_{(T,\phi)}) \simeq \operatorname{Ind}_{L\cap {}^gP}(\pi^{{}^gM}_{({}^gT,{}^g\phi)}) \simeq \pi^L_{({}^gT,{}^g\phi)},$$

we see that if $r_Q(\pi^G_{(T,\phi)}) \simeq r_Q(\operatorname{Ind}_P^G \pi^M_{(T,\phi)})$ is L^θ-distinguished, then there exists $g \in G$ such that ${}^gM \subset L$ and $\pi^L_{({}^gT,{}^g\phi)}$, is L^θ-distinguished.

From ${}^gT \subset {}^gM \subset L$, $({}^gT,{}^g\phi) \in \mathcal{T}(G)$, and Lemma 3.6, we obtain $({}^gT,{}^g\phi) \in \mathcal{T}(L)$. Therefore, an application of Proposition 7.1 shows that if $\pi^L_{({}^gT,{}^g\phi)}$ is L^θ-distinguished, then there exists $\ell \in L$ such that ${}^{\ell g}T$ is θ-stable and ${}^{\ell g}\phi \,|\, ({}^{\ell g}T)^\theta_{0^+} = 1$. After replacing ℓg by g, we assume ${}^gT$ is θ-stable and

${}^g\phi \,|\, ({}^gT)^\theta_{0+} = 1$. Because both T and ${}^gT$ are θ-stable, we have $g^{-1}\theta(g) \in N_G(T)$. Note that

$$ {}^g\phi(\theta({}^gt)) = \phi({}^{g^{-1}\theta(g)}\theta(t)), \qquad t \in T. $$

Combining this with θ-symmetry of ${}^g\phi \,|\, {}^gT_{0+}$ and $\phi \,|\, T_{0+}$ yields

$$ {}^g\phi(\theta({}^gt)) = {}^g\phi^{-1}({}^gt) = \phi^{-1}(t) = \phi(\theta(t)), \qquad t \in T_{0+}. $$

Thus $\phi({}^{g^{-1}\theta(g)}t) = \phi(t)$ for all $t \in T_{0+}$. Because ϕ is G-regular on T_{0+}, it follows from Lemma 3.7, that $g^{-1}\theta(g) \in T$. □

Corollary 8.2. *Let (T, ϕ), L and Q be as in Proposition 8.1. If T is θ-elliptic and $r_Q(\pi^G_{(T,\phi)})$ is L^θ-distinguished, then $A^-_L = A^-_G$.*

Proof. Let g be as in Proposition 8.1. Observe that $g^{-1}\theta(g) \in T$ implies that ${}^gT$ is θ-stable and $({}^gT)^- = {}^g(T^-)$. It follows that $({}^gA_T)^- = {}^g(A^-_T) = {}^gA^-_G = A^-_G$. That is, ${}^gT$ is θ-elliptic. Next, $A_{{}^gT} = {}^gA_T \supset A_L$ implies $A^-_G = A^-_{{}^gT} \supset A^-_L$. □

Recall that the definition of H-relatively supercuspidal representation is given in the introduction – see Definition 1.1.

Proposition 8.3. *Let $(T, \phi) \in \mathcal{T}(G)$. If T is θ-elliptic and $\pi^G_{(T,\phi)}$ is H-distinguished, then $\pi^G_{(T,\phi)}$ is H-relatively supercuspidal.*

Proof. Assume that T is θ-elliptic and $\pi := \pi^G_{(T,\phi)}$ is H-distinguished.

Let Q be a proper θ-split proper parabolic subgroup of G and let $L = Q \cap \theta(Q)$. As shown in [7, Propositions 5.5, 5.6] and [9, Theorem 1], there exists a unique linear map $\lambda \mapsto r_Q(\lambda)$ from $\mathrm{Hom}_H(\pi, 1)$ to $\mathrm{Hom}_{L^\theta}(r_Q(\pi), 1)$ exhibiting specific properties (not stated here). According to [7, Theorem 6.9], π is H-relatively supercuspidal if and only if $r_Q(\mathrm{Hom}_H(\pi, 1)) = 0$ for any proper θ-split parabolic subgroup Q of G. Hence it suffices to prove that $\mathrm{Hom}_{L^\theta}(r_Q(\pi), 1) = 0$ for every proper θ-split parabolic subgroup Q of G.

If Q is a proper θ-split parabolic subgroup of G such that $\mathrm{Hom}_{L^\theta}(r_Q(\pi), 1)$ is nonzero, then it follows from the assumption that T is θ-elliptic and Corollary 8.2 that $A^-_L = A^-_G$. Because Q is θ-split, $L = C_G(A^-_L)$. Thus $L = G$, contradicting the fact that Q is a proper parabolic subgroup. □

Proposition 8.4. *Let $(T, \phi) \in \mathcal{T}(G)$ be such that T is θ-elliptic. Let $M = C_G(A_T)$. Then M is θ-stable and $\mathrm{Hom}_{M^\theta}(\pi^M_{(T,\phi)}, 1)$ is isomorphic to a subspace of $\mathrm{Hom}_H(\pi^G_{(T,\phi)}, 1)$. Furthermore, if $\pi^M_{(T,\phi)}$ is M^θ-distinguished, then $\pi^G_{(T,\phi)}$ is H-relatively supercuspidal.*

Proof. Note that T θ-elliptic implies A_T is θ-stable. Hence M is θ-stable.

Assume that $\pi^M_{(T,\phi)}$ is M^θ-distinguished. By Proposition 8.3, since T is θ-elliptic, $\pi^G_{(T,\phi)}$ is H-relatively supercuspidal. Hence it suffices to prove the first assertion of the statement of the proposition.

Fix $P = MN \in \mathcal{P}(M)$. Because T is θ-elliptic, $A_M^- = A_T^- = A_G^-$. Hence for every root α of A_M in G, $\alpha\circ\theta = \alpha$. It follows that $\theta(P) = P$. Consequently (see [5, Proposition 13.3]), PH is closed in G. Let V be the space of $\pi := \mathrm{Ind}_P^G\, \pi^M_{(T,\phi)}$ and let ρ be the right regular representation of H on the image $V(PH)$ of restriction of functions in V to PH. Composition with restriction yields an injection from $\mathrm{Hom}_H(\rho, 1)$ to $\mathrm{Hom}_H(\pi, 1)$.

Restriction of functions in $V(PH)$ to H is an equivalence of ρ and the unnormalized induced representation $\rho_H := \mathrm{ind}_{P^\theta}^H(\delta_P^{1/2}\pi^M_{(T,\phi)}\,|\,P^\theta)$. The identity component of $(\boldsymbol{P}^\theta)^\circ$ of $\boldsymbol{P}^\theta$ is an F-parabolic subgroup of $\boldsymbol{H}^\circ$. Thus $P^\theta\backslash H$ is compact, and the representation ρ_H is compactly induced. Let δ_{P^θ} be the modular function of P^θ. Integration against a positive semi-invariant measure on $P^\theta\backslash H$, yields an isomorphism between $\mathrm{Hom}_H(\rho_H, 1)$ and $\mathrm{Hom}_M^\theta(\delta_P^{1/2}\pi^M_{(T,\phi)}, \delta_{P^\theta})$ ([1, Proposition 2.29]). Combining this with comments in the previous paragraph results in a linear injection of the space $\mathrm{Hom}_{M^\theta}(\delta_P^{1/2}\pi^M_{(T,\phi)}, 1)$ into $\mathrm{Hom}_H(\pi, 1)$.

In general, given a θ-stable parabolic subgroup Q of G with θ-stable Levi factor L, the functions $\delta_Q^{1/2}$ and δ_{Q^θ} need not agree on L^θ. However, in the current setting, M contains θ-split G-regular elements(see the following paragraph). This can be used to show that $\delta_P^{1/2}\,|\,M^\theta = \delta_{P^\theta}$.

To complete the proof of the first assertion of the proposition (which, as discussed above, is sufficient to prove the proposition), we observe that some θ-stable M-conjugate of T contains θ-split G-regular elements. By assumption, $\pi^M_{(T,\phi)}$ is M^θ-distinguished. An application of Proposition 7.1 (to $\pi^M_{(T,\phi)}$) yields existence of $m \in M$ such that ${}^mT$ is θ-stable and ${}^m\phi\,|\,({}^mT)^\theta_{0+} = 1$. Because $({}^mT, {}^m\phi) \in \mathcal{T}(G)$, we may infer from Lemma 3.9 that ${}^mT$ contains θ-split G-regular elements. $\square$

Theorem 8.5. *Let $(T,\phi) \in \mathcal{T}(G)$ and $M = C_G(A_T)$. If T is θ-elliptic and $\phi\,|\,T^\theta_{0+} = 1$, then there exists a depth-zero quasicharacter χ of T such that $\pi^M_{(T,\chi\phi)}$ is M^θ-distinguished. Furthermore,* $\mathrm{Hom}_{M^\theta}(\pi^M_{(T,\chi\phi)}, 1)$ *is isomorphic to a subspace of* $\mathrm{Hom}_H(\pi^G_{(T,\chi\phi)}, 1)$ *and* $\pi^G_{(T,\chi\phi)}$ *is relatively supercuspidal.*

Proof. Let $M = C_G(A_T)$. By Proposition 8.4, it suffices to prove the first assertion of the theorem. As noted in Lemma 3.6, ϕ is M-regular on T_{0+}. This allows us to use Lemma 3.9 and the assumption $\phi\,|\,T^\theta_{0+} = 1$ to conclude that ϕ has an M-factorization that is θ-symmetric on T_{0+}. As shown in the proof of Proposition 7.4, this implies existence of χ as in the statement of the theorem. $\square$

Acknowledgements

We thank Tasho Kaletha for helpful discussions regarding his recent results on quasicharacters and supercuspidal representations ([6]). The author's research is supported in part by an NSERC Discovery Grant.

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