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ENDOSCOPY AND THE COHOMOLOGY OF $GL(n)$

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Dedicated to Freydoon Shahidi on the occasion of his 70th birthday

Abstract. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$ where F is a number field. Let $S^G_{K_f}$ denote an adèlic locally symmetric space for some level structure K_f. Let $\mathcal{M}_{\mu,\mathbb{C}}$ be an algebraic irreducible representation of $G(\mathbb{R})$ and we let $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ denote the associated sheaf on $S^G_{K_f}$. The aim of this paper is to classify the data (F, n, μ) for which cuspidal cohomology of G with μ-coefficients, denoted $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}})$, is nonzero for some K_f. We prove nonvanishing of cuspidal cohomology when F is a totally real field or a totally imaginary quadratic extension of a totally real field, and also for a general number field but when μ is a parallel weight.
Keywords: Locally symmetric spaces, cuspidal cohomology.
MSC(2010): Primary: 11F75; Secondary: 11F70, 22E55.

1. **Introduction**

Let F be a totally real field. In Grobner–Raghuram [14] the special values of the standard L-function of a cuspidal automorphic representation π of $\mathrm{GL_{2n}/F}$ of cohomological type and admitting a Shalika model were studied. As explained via examples in *loc. cit.,* the geometric condition of being cohomological and the analytic condition of admitting a Shalika model are entirely independent of each other, and there is no *a priori* reason why such a π should even exist. The purpose of this article is to address such existence questions; indeed, we prove that such a π exists. More generally, let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$ where F now is any number field, and let $S^G_{K_f}$ denote an adèlic locally symmetric space for some level structure K_f. Let $\mathcal{M}_{\mu,\mathbb{C}}$ be an algebraic irreducible representation of $G(\mathbb{R})$ and we let $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ denote the associated sheaf on $S^G_{K_f}$. The basic problem addressed in this paper is to classify the data (F, n, μ) for which cuspidal cohomology of G with μ-coefficients, denoted $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}})$, is nonzero for some K_f. We prove nonvanishing of cuspidal cohomology when F

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is a totally real field or a totally imaginary quadratic extension of a totally real field, and also for a general number field but when μ is a parallel weight. The proof in the totally real case involves Arthur's endoscopic classification [1] of the discrete spectrum for classical groups, together with Clozel's results on globalizing discrete series representations via limit multiplicity arguments [8]; in the CM case we use Mok's classification [22] of the discrete spectrum for unitary groups; and for the parallel weight case it is an easy generalization of the constructive proof in Clozel [9]. The above fits into the general discussion of functoriality and cohomology as in Raghuram–Shahidi [25]. Finally, we add some remarks on an endoscopic stratification of inner cohomology suggested by results on arithmeticity as in Gan–Raghuram [13].

2. Statements of the main results

2.1. Cuspidal cohomology of GL_n.

We need a somewhat lengthy notational preparation; the reader familiar with such objects can directly jump to (2.3). Let F be a number field of degree $d = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}$. For any place v we write F_v for the topological completion of F at v. Let $S_\infty(F) = S_\infty$ be the set of archimedean places of F. Let $S_\infty = S_r \cup S_c$, where S_r (resp., S_c) is the set of real (resp., complex) places. Let $\mathcal{E}_F = \mathrm{Hom}(F, \mathbb{C})$ be the set of all embeddings of F as a field into $\mathbb{C}$. There is a canonical surjective map $\mathcal{E}_F \to S_\infty$, which is a bijection on the real embeddings and real places, and identifies a pair of complex conjugate embeddings $\{\iota_v, \bar{\iota}_v\}$ with the complex place v. For each $v \in S_r$, we fix an isomorphism $F_v \cong \mathbb{R}$ which is canonical. Similarly for $v \in S_c$, we fix $F_v \cong \mathbb{C}$ given by (say) ι_v; this choice is not canonical. Let $r_1 = |S_r|$ and $r_2 = |S_c|$; hence $d = r_1 + 2r_2$. If $v \notin S_\infty$, we let $\mathcal{O}_v$ be the ring of integers of F_v, and $\wp_v$ it's unique maximal ideal. Moreover, $\mathbb{A}_F$ denotes the ring of adèles of F and $\mathbb{A}_{F,f}$ its finite part. The group of idèles of F will be denoted $\mathbb{A}_F^\times$ and similarly, $\mathbb{A}_{F,f}^\times$ is the group of finite idèles. We will drop the subscript F when talking about $\mathbb{Q}$. Hence, $\mathbb{A}$ is $\mathbb{A}_\mathbb{Q}$, etc.

The algebraic group GL_n/F will be denoted as $\underline{G}_n$, and we put $G_n = \mathrm{Res}_{F/\mathbb{Q}}(\underline{G}_n)$; an F-group will be denoted by an underline and the corresponding $\mathbb{Q}$-group via Weil restriction of scalars will be denoted without the underline; hence for any $\mathbb{Q}$-algebra A, the group of A-points of G_n is $G_n(A) = \underline{G}_n(A \otimes_\mathbb{Q} F)$. Let $\underline{B}_n = \underline{T}_n\underline{U}_n$ stand for the standard Borel subgroup of $\underline{G}_n$ of all upper triangular matrices, where $\underline{U}_n$ is the unipotent radical of $\underline{B}_n$, and $\underline{T}_n$ the diagonal torus. The center of $\underline{G}_n$ will be denoted by $\underline{Z}_n$. These groups define the corresponding $\mathbb{Q}$-groups $G_n \supset B_n = T_nU_n \supset Z_n$. (Observe that Z_n is not $\mathbb{Q}$-split if $\mathbb{Q} \neq F$.) Let S_n be the maximal $\mathbb{Q}$-split torus in Z_n; we have $S_n \cong \mathbb{G}_m$ over $\mathbb{Q}$.

Note that

$$G_{n,\infty} := G_n(\mathbb{R}) = \underline{G}_n(F \otimes_\mathbb{Q} \mathbb{R}) = \prod_{v \in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v \in S_r} \mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in S_c} \mathrm{GL}_n(\mathbb{C}).$$

We have $Z_n(\mathbb{R}) \simeq \prod_{v\in S_r} \mathbb{R}^\times \times \prod_{v\in S_c} \mathbb{C}^\times$. The subgroup $S_n(\mathbb{R})$ is $\mathbb{R}^\times$ diagonally embedded in $Z_n(\mathbb{R})$. Let $C_{n,\infty} = \prod_{v\in S_r} \mathrm{O}(n) \times \prod_{v\in S_c} \mathrm{U}(n)$ be the maximal compact subgroup of $G_n(\mathbb{R})$, and let $K_{n,\infty} = Z_n(\mathbb{R})C_{n,\infty} = Z_n(\mathbb{R})^0 C_{n,\infty}$. Let $K^0_{n,\infty}$ be the topological connected component of $K_{n,\infty}$. For a real Lie group G, we denote its Lie algebra by $\mathfrak{g}^0$ and the complexified Lie algebra by $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{g}^0 \otimes_{\mathbb{R}} \mathbb{C}$. If $G = \mathrm{GL}_n(\mathbb{R})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, on the other hand, if G stands for the real Lie group $\mathrm{GL}_n(\mathbb{C})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{C})$ as a Lie algebra over $\mathbb{R}$, and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. With this notational scheme, we have $\mathfrak{g}_n$, $\mathfrak{b}_n$, $\mathfrak{t}_n$ and $\mathfrak{k}_n$ denoting the complexified Lie algebras of $G_n(\mathbb{R})$, $B_n(\mathbb{R})$, $T_n(\mathbb{R})$ and $K^0_{n,\infty}$ respectively. For example, $\mathfrak{g}_n = \prod_{v\in S_r} \mathfrak{gl}_n(\mathbb{C}) \times \prod_{v\in S_c} (\mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$.

Consider $T_n(\mathbb{R}) = \underline{T}_n(F \otimes \mathbb{R}) \cong \prod_{v\in S_\infty} \underline{T}_n(F_v)$. We let $X^*(T_n \times \mathbb{C})$ stand for the group of all algebraic characters of $T_n \times \mathbb{C}$, and let $X^+(T_n \times \mathbb{C})$ stand for all those characters in $X^*(T_n \times \mathbb{C})$ which are dominant with respect to B_n. A weight $\mu \in X^+(T_n \times \mathbb{C})$ may be described as follows: $\mu = (\mu^\iota)_{\iota\in\mathcal{E}_F}$, also sometimes written as $\mu = (\mu^v)_{v\in S_\infty}$, where $\mu^v = (\mu^{\iota_v}, \mu^{\bar\iota_v})$ for $v \in S_c$, and furthermore,

- for $v \in S_r$, we have $\mu^v = (\mu^v_1, \dots, \mu^v_n)$, $\mu^v_i \in \mathbb{Z}$, $\mu^v_1 \geq \cdots \geq \mu^v_n$, and the character μ^v sends $t = \mathrm{diag}(t_1, \dots, t_n) \in \underline{T}_n(F_v)$ to $\prod_i t_i^{\mu^v_i}$, and
- if $v \in S_c$ then μ^v is a pair $(\mu^{\iota_v}, \mu^{\bar\iota_v})$, with $\mu^{\iota_v} = (\mu^{\iota_v}_1, \dots, \mu^{\iota_v}_n)$, $\mu^{\iota_v}_i \in \mathbb{Z}$, $\mu^{\iota_v}_1 \geq \cdots \geq \mu^{\iota_v}_n$; likewise $\mu^{\bar\iota_v} = (\mu^{\bar\iota_v}_1, \dots, \mu^{\bar\iota_v}_n)$ and $\mu^{\bar\iota_v}_1 \geq \cdots \geq \mu^{\bar\iota_v}_n$; the character μ^v is given by sending $t = \mathrm{diag}(z_1, \dots, z_n) \in \underline{T}_n(F_v)$ to $\prod_{i=1}^n z_i^{\mu^{\iota_v}_i} \bar{z}_i^{\mu^{\bar\iota_v}_i}$, where $\bar{z}_i$ is the conjugate of z_i.

For $\mu \in X^+(T_n \times \mathbb{C})$, define a finite-dimensional complex representation $(\rho_\mu, \mathcal{M}_{\mu,\mathbb{C}})$ of $G_n(\mathbb{R})$ as follows: For $v \in S_r$, let $(\rho_{\mu^v}, \mathcal{M}_{\mu^v,\mathbb{C}})$ be the irreducible complex representation of $G_n(F_v) = \mathrm{GL}_n(\mathbb{R})$ with highest weight μ_v. For $v \in S_c$, let $(\rho_{\mu^v}, \mathcal{M}_{\mu^v,\mathbb{C}})$ be the complex representation of the real algebraic group $G(F_v) = \mathrm{GL}_n(\mathbb{C})$ defined as $\rho_{\mu^v}(g) = \rho_{\mu^{\iota_v}}(g) \otimes \rho_{\mu^{\bar\iota_v}}(\bar{g})$; here $\rho_{\mu^{\iota_v}}$ (resp., $\rho_{\mu^{\bar\iota_v}}$) is the irreducible representation of the complex group $\mathrm{GL}_n(\mathbb{C})$ with highest weight μ^{ι_v} (resp., $\mu^{\bar\iota_v}$). Now we let $\rho_\mu = \otimes_{v\in S_\infty} \rho_{\mu^v}$ which acts on $\mathcal{M}_{\mu,\mathbb{C}} = \otimes_{v\in S_\infty} \mathcal{M}_{\mu^v,\mathbb{C}}$.

Let K_f be an open compact subgroup of $G_n(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{A}_{F,f})$. Let us write $K_f = \prod_p K_p$ where each K_p is an open compact subgroup of $G_n(\mathbb{Q}_p)$ and for almost all p we have $K_p = \prod_{v|p} \mathrm{GL}_n(\mathcal{O}_v)$. Define the double-coset space

$$S^{G_n}_{K_f} = G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K^0_{n,\infty}K_f = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/K^0_{n,\infty}K_f.$$

Given a dominant-integral weight $\mu \in X^+(T_n \times \mathbb{C})$ we get a sheaf $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ of $\mathbb{C}$-vector spaces on $S^{G_n}_{K_f}$. (See, for example, [24, Section 2.3.3].) We are interested in the sheaf cohomology groups $H^\bullet(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}})$. Here $\widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}$ is the sheaf attached to the contragredient representation $\mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}$ of $\mathcal{M}_{\mu,\mathbb{C}}$. This dualizing is only for convenience. Let ω_{ρ_μ} be the central character of ρ_μ. For the sheaf $\widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}$ or $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ to be nonzero we need the condition:

(2.1) "The central character ω_{ρ_μ} is trivial on $Z_n(\mathbb{Q}) \cap K^{\circ}_{n,\infty} K_f$."

Henceforth, we will assume this condition on μ and K_f. (This can be a nontrivial condition even in simple situations: take, for example, $F = \mathbb{Q}$, $n = 2$, $K_f = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$, and $\mu = (a, b)$ with integers $a \geq b$; then this condition boils down to $(-1)^{a+b} = 1$; but by taking K_f slightly deep enough we can ensure $Z_2(\mathbb{Q}) \cap K^{\circ}_{2,\infty} K_f$ is trivial and so the condition vacuously holds.) For more details, the reader is referred to Harder [16, (1.1.3)]. Passing to the limit over all open compact subgroups K_f and let $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) := \varinjlim_{K_f} H^\bullet(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}})$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,E})$, and the cohomology of $S^{G_n}_{K_f}$ is obtained by taking invariants under K_f, i.e., $H^\bullet(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,E}) = H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,E})^{K_f}$. We can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \ \simeq \ H^\bullet(\mathfrak{g}_n, K^0_{n,\infty};\ CCC^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}).$$

The inclusion $C^\infty_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [5], an injection in cohomology; this defines cuspidal cohomology:

(2.2)
$$\begin{array}{ccc}
H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) & \longrightarrow & H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}) \\
\uparrow & & \uparrow \\
H^\bullet_{\mathrm{cusp}}(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) & \longrightarrow & H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}).
\end{array}$$

With level structure K_f we have:

(2.3) $$H^\bullet_{\mathrm{cusp}}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \simeq H^\bullet(\mathfrak{g}_n, K^0_{n,\infty};\ C^\infty_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}).$$

A fundamental problem concerning cuspidal cohomology for GL_n/F is:

Problem 2.4. *Classify all (F, n, μ, K_f), subject to (2.1), for which $H^\bullet_{\mathrm{cusp}}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0$.*

This problem includes classical situations such as: take $F = \mathbb{Q}$ and $n = 2$, and given integers $N \geq 1$ and $k \geq 2$ is there a holomorphic cusp form of weight k for $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$? One may relax the dependence on an explicit level structure K_f and ask for a solution of the weaker

Problem 2.5. *Classify all (F, n, μ) for which $H^{\bullet}_{\rm cusp}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0$ for some K_f.*

The purpose of this article is to provide a solution of the weaker problem for F a totally real field, or a totally imaginary quadratic extension of a totally real field, or for a general number field F and for a parallel weight μ; see below for a definition of a parallel weight.

2.2. **Necessary conditions for nonvanishing of cuspidal cohomology.** Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of $\pi_0(G_n(\mathbb{R})) \times G_n(\mathbb{A}_f)$-modules:

$$H^{\bullet}_{\rm cusp}(S^{G_n}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \ = \ \bigoplus_{\Pi} H^{\bullet}(\mathfrak{g}_n, K^0_{n,\infty}; \Pi_\infty \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}) \otimes \Pi_f. \tag{2.6}$$

We say that Π *contributes to the cuspidal cohomology of G_n with coefficients in $\mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}$*, and we write $\Pi \in {\rm Coh}(G_n, \mu^{\mathsf{v}})$, if Π has a nonzero contribution to the above decomposition; equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by $\mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}$ has nontrivial relative Lie algebra cohomology. With a level structure K_f, (2.6) takes the form:

$$H^{\bullet}_{\rm cusp}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \ = \ \bigoplus_{\Pi} H^{\bullet}(\mathfrak{g}_n, K^0_{n,\infty}; \Pi_\infty \otimes \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}) \otimes \Pi_f^{K_f}. \tag{2.7}$$

We write $\Pi \in {\rm Coh}(G_n, \mu^{\mathsf{v}}, K_f)$ if a cuspidal automorphic representation Π contributes nontrivially to (2.7). Note that each ${\rm Coh}(G_n, \mu^{\mathsf{v}}, K_f)$ is a finite set, and clearly, ${\rm Coh}(G_n, \mu^{\mathsf{v}}) = \cup_{K_f} {\rm Coh}(G_n, \mu^{\mathsf{v}}, K_f)$.

Suppose $\Pi \in {\rm Coh}(G_n, \mu^{\mathsf{v}})$. The fact that μ^{v} supports cuspidal cohomology places some restrictions on μ. First of all, essential unitarity of Π, and in particular of Π_∞ gives, via Wigner's Lemma, essential self-duality of μ: there is an integer $\mathsf{w}(\mu)$ such that

(1) For $v \in S_r$ and $1 \leq i \leq n$ we have $\mu_i^{\iota_v} + \mu_{n-i+1}^{\iota_v} = \mathsf{w}(\mu)$;

(2) For $v \in S_c$ and $1 \leq i \leq n$ we have $\mu_i^{\bar{\iota}_v} + \mu_{n-i+1}^{\iota_v} = \mathsf{w}(\mu)$.

We will call such a weight μ as a *pure weight* and call $\mathsf{w}(\mu)$ the *purity weight of* μ. Let $X_0^+(T_n)$ denote the set of dominant integral pure weights. Applying Clozel [10, Theorem 3.13], we get

$$\Pi \in {\rm Coh}(G_n, \mu^{\mathsf{v}}) \implies {}^\sigma\Pi \in {\rm Coh}(G_n, {}^\sigma\mu^{\mathsf{v}}), \quad \forall \sigma \in {\rm Aut}(\mathbb{C}),$$

where, ${}^\sigma\mu \in X^*(T_n \times \mathbb{C})$ is defined as: ${}^\sigma\mu \ = \ ({}^\sigma\mu^\iota)_{\iota \in \mathcal{E}_F}$ with ${}^\sigma\mu^\iota := \mu^{\sigma^{-1} \circ \iota}$. The reader is referred to [10] for a definition of ${}^\sigma\Pi$. In particular, ${}^\sigma\mu$ also satisfies the purity conditions (1) and (2) above. Note that $\mathsf{w}(\mu) = \mathsf{w}({}^\sigma\mu)$. As in [24], we make the following

Definition 2.8. *Let $\mu \in X_0^+(T_n)$ be a pure dominant integral weight. We say μ is strongly pure if ${}^\sigma\mu$ is pure for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Let $X_{00}^+(T_n)$ stand for the set of dominant integral strongly pure weights.*

For any F, we have the following inclusions $X_{00}^+(T_n) \subset X_0^+(T_n) \subset X^+(T_n) \subset X^*(T_n)$ and in general they are all strict inclusions. If F is a totally real field or a CM field (totally imaginary quadratic extension of a totally real field) then μ is pure if and only if μ is strongly pure. For any number field, one may see that there are strongly pure weights: take an integer b and integers $a_1 \geq a_2 \geq \cdots \geq a_n$ such that $a_j + a_{n-j+1} = b$; now for each $\iota \in \mathcal{E}_F$ put $\mu^\iota = (a_1, \dots, a_n)$; then μ is strongly pure with $\mathsf{w}(\mu) = b$; such a weight may be called a *parallel weight*. We formulate the following conjecture as a possible answer to Problem 2.5.

Conjecture 2.9. *Let $\mu \in X^+(T_n \times \mathbb{C})$ be a dominant integral weight. Then*

$$H^\bullet_{\mathrm{cusp}}(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0 \text{ for some } K_f \quad \Longleftrightarrow \quad \mu \in X^+_{00}(T_n \times \mathbb{C}).$$

We give a brief glimpse, without pretending to be exhaustive, into available results in the literature. Borel, Labesse and Schwermer [6, Theorem 11.3] proved nonvanishing of cuspidal cohomology for $G = \mathrm{SL}_n$ over any number field which is an extension via cyclic prime degree extensions of a totally real number field, and for the trivial coefficient system, i.e., when $\mu = 0$; their proof involved studying Lefschetz numbers for rational automorphisms of G. Barbasch and Speh [3, Section XI] considered $\mathrm{GL}_n/\mathbb{Q}$ and a coefficient system with certain technical restrictions on the weight μ (which in fact exclude many pure weights μ) and proved nonvanishing of cuspidal cohomology via an application of trace formula for certain Lefschetz functions. Clozel [9, Theorem 4] gave a constructive proof of the nonvanishing of cuspidal cohomology for GL_{2n} over any number field for the trivial coefficient system using automorphic induction.

2.3. The main results of this article on cuspidal cohomology of $\mathrm{GL}(n)$.

Theorem 2.10. *Take an integer $N \geq 2$. Let F_0 be a totally real field extension of $\mathbb{Q}$, and we take an extension F/F_0 to be either*

1. *$F = F_0$, i.e., F itself is a totally real field; or*
2. *F is a totally imaginary quadratic extension over F_0.*

Let $G = G_N = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_N/F)$. Let $\mu \in X_0^+(T_N \times \mathbb{C})$ be a pure dominant integral weight with purity $\mathsf{w}(\mu)$. In case (2), assume furthermore that μ is trivial on the center Z_N, i.e., for all $\iota \in \mathcal{E}_F$ suppose that $\mu_1^\iota + \cdots + \mu_N^\iota = 0$. Then

$$H^\bullet_{\mathrm{cusp}}(S^G, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0.$$

The proof involves endoscopic transfer from certain classical groups and breaks up into the following sub-cases:

(1a) F is totally real, and $N = 2n + 1$ is odd;
(1b) F is totally real, $N = 2n$ is even, and $\mathsf{w}(\mu)$ is even;
(1c) F is totally real, $N = 2n$ is even, and $\mathsf{w}(\mu)$ is odd;
(2) F is a totally imaginary quadratic extension of a totally real F_0.

Here we make a crucial use of the following observation about the integral Tate twists of dominant integral weights:

Observation 2.11. *For any $m \in \mathbb{Z}$, $\pi \in \mathrm{Coh}(G_N, \mu)$ if and only if $\pi \otimes |\ |^m \in \mathrm{Coh}(G_N, \mu - m)$.*

We also observe that if μ is a pure weight, so is $\mu - m$ and the purity of $\mathsf{w}(\mu - m) = \mathsf{w}(\mu) - 2m$. Thus we can reduce to one of the situations $\mathsf{w}(\mu) = 0$ or $\mathsf{w}(\mu) = 1$ depending on the parity of $\mathsf{w}(\mu)$.

In case (1a), we transfer from the group $G' = \mathrm{Sp}(2n)/F$. To begin, we transfer the weight μ to a weight μ' on G'. Then, using results on limit multiplicities due to Clozel [8], we produce a cuspidal representation π' of G' with a discrete series representation at infinity which is cohomological with respect to μ'. Furthermore, one can arrange for π' to have the Steinberg representation at some finite place. Arthur's results ensure then that π' corresponds to an Arthur parameter π on G which is cuspidal; the cuspidality of π uses an observation of Magaard and Savin [21], and then one checks that indeed π contributes to the above cohomology group. For the proof in case (1b) (resp., (1c)) we transfer from the (split) odd orthogonal group $G' = \mathrm{SO}(2n+1)$ (resp., a suitable form–depending on the parity of n–of the split $\mathrm{SO}(2n)$). In case (2) we transfer from a unitary group in N-variables. In case (1a) it might also be possible to use the results of Weselmann's papers [32] and [31].

We may draw several inferences from the above proof. As summarized in [14, Section 3.1], a cuspidal representation π of GL_{2n}/F is a transfer from $\mathrm{SO}(2n + 1)$ if and only if a partial exterior-square L-function $L^S(s, \wedge^2, \pi)$ has a pole at $s = 1$ and this is so if and only if π admits a Shalika model, which gives us

Corollary 2.12. *Let F be a totally real field, and take $G = \mathrm{GL}(2n)/F$. Then for any pure weight μ with $\mathsf{w}(\mu) = 0$, there exists a cuspidal automorphic representation π of G such that*

(1) *$\pi \in \mathrm{Coh}(G, \mu)$, i.e., it is cohomological with respect to μ, and*
(2) *π has a Shalika model, or equivalently, a partial exterior-square L-function $L^S(s, \wedge^2, \pi)$ has a pole at $s = 1$.*

A similar characterization for a π on GL_{2n+1}/F being a transfer from $\mathrm{Sp}(2n)$ if and only if a partial symmetric-square L-function $L^S(s, \mathrm{Sym}^2, \pi)$ having a pole at $s = 1$ gives us

Corollary 2.13. *Let F be a totally real field, and take $G = \mathrm{GL}(2n+1)/F$. Then for any pure weight μ with $\mathsf{w}(\mu) = 0$, there exists a cuspidal automorphic representation π of G such that*

(1) *$\pi \in \mathrm{Coh}(G, \mu)$, i.e., it is cohomological with respect to μ, and*
(2) *a partial symmetric-square L-function $L^S(s, \mathrm{Sym}^2, \pi)$ has a pole at $s = 1$.*

Using an idea in Labesse and Schwermer [20], that at a finite place the Steinberg representation retains the property of being Steinberg upon base change, and at archimedean places the property of being cohomological is preserved under base-change, we get the following

Corollary 2.14. *Let F be a totally real field and suppose that $\tilde{F}/F$ is a finite extension that is filtered by cyclic extensions of prime degrees. Let μ be a pure weight for $G = \mathrm{GL}(N)/F$. Define a weight $\tilde{\mu}$ for $\tilde{G} = \mathrm{GL}(N)/\tilde{F}$ as for any $\tilde{\iota} : \tilde{F} \to \mathbb{C}$, we let $\tilde{\mu}^{\tilde{\iota}} = \mu^{\iota}$ where $\iota = \tilde{\iota}|_F$. (If μ is a parallel weight then so is $\tilde{\mu}$.) Then*

$$H^{\bullet}_{\mathrm{cusp}}(S^{\tilde{G}}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\tilde{\mu},\mathbb{C}}) \neq 0.$$

The conditions on $\tilde{F}/F$ are dictated by the main theorem on base change for GL_N due to Arthur and Clozel [2].

Next, we consider a general number field F. The following theorem is a generalization of the main theorem of Clozel [9] which is proved by constructing cohomological cuspidal representations via automorphic induction. See also Ramakrishnan–Wang [28, Appendix].

Theorem 2.15. *Let F be any number field, and take $G = \mathrm{GL}(2n)/F$. Let μ be a parallel weight with purity $\mathsf{w}(\mu) = 0$. Then*

$$H^{\bullet}_{\mathrm{cusp}}(S^{G}, \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0.$$

Remark 2.16. Using a similar 'Tate twist trick' as earlier, it is possible to prove the above theorem for any pure weight μ such that $\mathrm{w}(\mu)$ is even.

Let $Y = \Gamma\backslash \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ be a compact hyperbolic 3-manifold, where Γ is a cocompact discrete subgroup of $SL(2,\mathbb{C})$. Let μ and $\mathcal{M}_{\mu,\mathbb{C}}$ be as in Theorem 2.15 above. As a corollary, we obtain the following generalization of Clozel's result in [9].

Corollary 2.17. *There exists a finite index subgroup Γ' of Γ such that $\mathrm{H}^1(\Gamma', \mathcal{M}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0$. In particular, there exists a finite covering $\pi : Y' \to Y$ such that*

$$\mathrm{H}^1(\mathrm{Y}', \pi^* \widetilde{\mathcal{M}}^{\mathsf{v}}_{\mu,\mathbb{C}}) \neq 0.$$

3. Proof of Theorem 2.10

3.1. **Archimedean preliminaries. Case (1a) (F is totally real, and $N = 2n+1$ is odd.)** Fix a real place v of F. Since μ is pure weight with purity

$\mathsf{w}(\mu)$ and N is odd, it follows that $\mathsf{w}(\mu)$ must be even. Hence without loss of generality, we can assume that μ is a pure weight with purity 0. (Recall the comments after Observation 2.11.) Thus,

$$\mu^v \;=\; \mu_1^v \geq \mu_2^v \geq \cdots \geq \mu_n^v \geq 0 \geq -\mu_n^v \geq \cdots \geq \mu_2^v \geq -\mu_1^v.$$

The place v of F is fixed, and for brevity, we will drop v from the notation for μ^v.

Consider the endoscopy group $G' = \mathrm{Sp}(2n)/F$ defined so that upper-triangular subgroup B' in G' is a Borel subgroup. The connected component of the L-group of G' is ${}^LG'^\circ = \mathrm{SO}(2n+1,\mathbb{C})$. The maximal compact subgroup K' of $G'(\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R})$ is isomorphic to $\mathrm{U}(n)$. Define a dominant integral weight μ' for G' which at the place v (dropped from the notation) is given by:

$$\mu' := (\mu_1, \mu_2, \ldots, \mu_n) \;=\; \sum_{i=1}^{n} \mu_i e_i,$$

where e_i gives the i-th coordinate of a diagonal matrix. Let ρ' be the half-sum of positive roots for G', which is written as $\rho' = \sum_{j=1}^{n}(n+1-j)e_j = (n, n-1, \ldots, 1)$. Let

$$\Lambda' \;=\; \mu' + \rho' \;=\; (\mu_1 + n,\ \mu_2 + n - 1, \ldots,\ \mu_{n-1} + 1,\ \mu_n).$$

Thus Λ' is a regular weight and using Harish-Chandra's classification theorem of discrete series representations, there exists a discrete series representation $\pi' = \pi_{\Lambda'}$ of $G'(\mathbb{R})$ whose infinitesimal character is $\chi_{\Lambda'}$. Let $\mathcal{M}_{\mu',\mathbb{C}}$ be the algebraic irreducible representation of $G'(\mathbb{C})$ with highest weight μ'. From the well-known results on the cohomology of discrete series representations, one knows that π' is cohomological with respect to the coefficient system $\mathcal{M}_{\mu',\mathbb{C}}$ of G', i.e., the relative Lie algebra cohomology $H^\bullet(\mathfrak{g}'_\infty, K', \pi' \otimes \mathcal{M}_{\mu'}^{\mathsf{v}})$ is nonzero; in fact it is nonzero only in the middle degree $\bullet = \frac{1}{2}\dim(G'(\mathbb{R}))/\dim(K')$. (For the above assertions made about discrete series representations, see Borel–Wallach [7, Chapter II, Section 5].)

The shape of the Langlands parameter, denoted $\tau_{\Lambda'}$, of the discrete series representation $\pi_{\Lambda'}$ of $G'(\mathbb{R})$ is well-known; we can deduce the following from [4, Example 10.5]:

$$\tau_{\Lambda'} \;=\; \mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}}(\chi_{\ell_1}) \oplus \mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}}(\chi_{\ell_2}) \oplus \cdots \oplus \mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}}(\chi_{\ell_n}) \oplus \mathrm{sgn}^\epsilon,$$

where $\ell_1, \ldots, \ell_n$ are positive integers and the first n-summands are irreducible 2-dimensional representations of the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ induced from characters of $\mathbb{C}^\times$ with the character χ_{ℓ_j} sending $z = re^{i\theta} \in \mathbb{C}^\times$ to $e^{i\ell_j\theta} = (z/\bar{z})^{\ell_j/2}$ with the proviso that each summand be of orthogonal type forcing each ℓ_j to be even, and the determinant of the parameter is 1 so as to have image inside ${}^LG'^\circ$ forcing the last summand to be sgn^n. Furthermore, the relation between the integers ℓ_j and the weight Λ' is captured by

$$\tau_{\Lambda'}(z) \; = \; z^{\Lambda'}\bar{z}^{-\Lambda'}, \quad \forall z \in \mathbb{C}^\times \subset W_{\mathbb{R}}.$$

(Here we have tacitly used that Λ', which is a character of a maximal torus T' of G', is also, by definition, a co-character of the dual ${}^LT'^\circ \subset {}^LG'^\circ$ justifying the above notation.) If we put $\ell = (\ell_1, \dots, \ell_n)$ then we get $\ell \; = \; 2\Lambda'$, i.e.,

$$(\ell_1, \dots, \ell_n) \; = \; (2\mu_1 + 2n, \; 2\mu_2 + 2n - 2, \dots, \; 2\mu_{n-1} + 2, \; 2\mu_n).$$

Let π_μ denote the Langlands transfer of π' to an irreducible representation of $\mathrm{GL}_{2n+1}(\mathbb{R})$, where the transfer is mitigated by the Langlands parameter of π' being that of π_μ via the standard embedding ${}^LG'^\circ = \mathrm{SO}(2n+1, \mathbb{C}) \subset \mathrm{GL}(2n+1, \mathbb{C}) = {}^LG^\circ$. Using the local Langlands correspondence for $\mathrm{GL}_N(\mathbb{R})$ (see, for example, Knapp [19]), we can deduce

$$\pi_\mu \; = \; \mathrm{Ind}^G_{P(2,2,\dots,2,1)} \left(D_{\ell_1} \otimes D_{\ell_2} \otimes \cdots \otimes D_{\ell_n} \otimes \mathrm{sgn}^n \right),$$

where, for any integer l, we denote by D_l the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ as normalized in [26, 3.1.3]. It is well-known ([10, Lemme 3.14]) that

$$H^\bullet(\mathfrak{gl}_N, \mathbb{R}_+^\times \mathrm{SO}(N); \pi_\mu \otimes \mathcal{M}_{\mu,\mathbb{C}}) \; \neq \; 0.$$

Case (1b) (F is totally real, $N = 2n$ is even and $\mathsf{w}(\mu)$ is even.) In this case we take $G' = \mathrm{SO}(2n+1)/F$ which is the split orthogonal group in $2n+1$ variables. We can assume $\mathsf{w}(\mu) = 0$ as in case (1a). We have $G'(\mathbb{R}) = \mathrm{SO}(n, n+1)$. The maximal compact subgroup K' of $G'(\mathbb{R})$ is isomorphic to $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n+1))$. The connected component of the L-group is ${}^LG'^\circ = \mathrm{Sp}(2n, \mathbb{C})$. Fix a real place v of F and we drop it from the notations. We have

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq -\mu_n \geq \cdots \geq -\mu_2 \geq -\mu_1)$$

is the given dominant integral weight with purity weight 0. Following the argument as in Case (1) above, we put

$$\mu' \quad = \quad (\mu_1, \mu_2, \dots, \mu_n), \quad \text{and}$$

$$\Lambda' \quad = \quad \mu' + \rho' \; = \; (\mu_1 + n - \tfrac{1}{2}, \; \mu_2 + n - \tfrac{3}{2}, \; \cdots, \; \mu_{n-1} + \tfrac{3}{2}, \; \mu_n + \tfrac{1}{2}).$$

Consider the discrete series representation $\pi' = \pi_{\Lambda'}$ with infinitesimal character given by Λ'. The Langlands parameter $\tau_{\Lambda'}$ of $\pi_{\Lambda'}$ has the form

$$\tau_{\Lambda'} \; = \; \mathrm{Ind}^{W_\mathbb{R}}_{\mathbb{C}^\times}(\chi_{\ell_1}) \oplus \mathrm{Ind}^{W_\mathbb{R}}_{\mathbb{C}^\times}(\chi_{\ell_2}) \oplus \cdots \oplus \mathrm{Ind}^{W_\mathbb{R}}_{\mathbb{C}^\times}(\chi_{\ell_n}),$$

with all the ℓ_j being odd positive integers. The infinitesimal character of the discrete series is seen in terms of the exponents of the parameter restricted to $\mathbb{C}^\times$ giving us $\ell \; = \; 2\Lambda'$, or that

$$(\ell_1, \dots, \ell_n) \; = \; (2\mu_1 + 2n - 1, \; 2\mu_2 + 2n - 3, \; \cdots, 2\mu_{n-1} + 3, \; 2\mu_n + 1).$$

Via the local Langlands correspondence for $\mathrm{GL}_{2n}(\mathbb{R})$ we get that π' transfers to π_μ given by

$$\pi_\mu \;=\; \mathrm{Ind}_{P(2,2,\ldots,2)}^{G}\left(D_{\ell_1}\otimes D_{\ell_2}\otimes\cdots\otimes D_{\ell_n}\right),$$

which has the property that $H^\bullet(\mathfrak{gl}_N, \mathbb{R}_+^\times\mathrm{SO}(N); \pi_\mu\otimes\mathcal{M}_{\mu,\mathbb{C}}) \neq 0$.

Case (1c) (F is totally real, $N = 2n$ is even and w is odd.)

Let $N = 2n$, and for simplicity take $F = \mathbb{Q}$. Here want to construct a representation as in Case (1b) of the proof of Theorem 2.10 using a suitable even orthogonal group as our endoscopy group. In this case, we can assume that $\mathsf{w}(\mu) = 1$ by applying a Tate twist if necessary.

If n is even, we take $G' = \mathrm{SO}(2n)/\mathbb{Q}$ as the (split) special orthogonal group defined by the quadratic form $x_1x_{n+1}+\cdots+x_nx_{2n}$. If n is odd we take $G' = \mathrm{SO}'(2n)/\mathbb{Q}$ as the special orthogonal group defined by $x_1x_n+\cdots+x_{n-1}x_{2n-2}+N_{E/F}$ for a quadratic extension E/F. Then $G'(\mathbb{R}) = \mathrm{SO}(n,n)$ if n is even, and $G'(\mathbb{R}) = \mathrm{SO}(n-1,n+1)$ if n is odd, see [15]. In both cases $G'(\mathbb{R})$ has discrete series representations. Let

$$\mu \;=\; (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 1-\mu_n \geq \cdots \geq 1-\mu_2 \geq 1-\mu_1)$$

be the given dominant integral weight for GL_N with purity weight 1. (It follows that $\mu_n \geq 1$.) Let μ', ρ', ℓ' and ℓ be defined as follows:

$$\mu' \;=\; (\mu_1,\mu_2,\ldots\mu_n);$$
$$\rho' \;=\; (n-1,n-2,\ldots,1,0);$$
$$\ell' \;=\; (\ell'_i)_{i=1}^n :=\; (2\mu_1+2n-2,\ 2\mu_2+2n-4,\ \ldots,\ 2\mu_{n-1}+2,\ 2\mu_n);$$
$$\ell := (\ell_1^{'},\ \cdots,\ell'_n,\ -\ell'_n,\ \cdots,-\ell'_1).$$

If we use a similar argument, we get a representation π_μ of $\mathrm{GL}_{2n}(\mathbb{R})$ given by:

$$\pi_\mu = \mathrm{Ind}_{P(2,2,\ldots,2)}^{\mathrm{GL}_N(\mathbb{R})}\left(D_{\ell'_1}\otimes D_{\ell'_2}\otimes\cdots\otimes D_{\ell'_n}\right).$$

Note that the representation $\pi_\mu\otimes|\;|^{\mathsf{w}/2}$ is cohomological with respect to a pure weight λ with purity weight w if and only if $\lambda = \frac{1}{2}(\mathsf{w}+\ell-2\rho)$ where ρ is the half sum of positive roots for $\mathrm{GL_N}$ given by $\rho = \left(n-\frac{1}{2}, n-\frac{3}{2},\cdots,-n+\frac{1}{2}\right)$.

In our situation, $\mathsf{w}=1$, hence $\lambda = \frac{1}{2}(1+\ell-2\rho) = (\lambda_i)_{1\leq i\leq 2n}$, where:

$\lambda_1 = \frac{1}{2}(1+2\mu_1+2n-2-2n+1) = \mu_1$;

$\lambda_2 = \frac{1}{2}(1+2\mu_2+2n-4-2n+3) = \mu_2$;

$\cdots$

$\lambda_n = \frac{1}{2}(1+2\mu_n-1) = \mu_n$;

$\lambda_{n+1} = \frac{1}{2}(1-2\mu_n+1) = 1-\mu_n$;

$\cdots$

$\lambda_{2n} = \frac{1}{2}(1-(2\mu_1+2n-2)+2n-1) = 1-\mu_1$.

Thus $\lambda = \mu$, and hence $\pi_\mu \otimes |\ |^{1/2}$ is cohomological with respect to the weight μ.

Case (2) (F_0 is totally real and F a totally imaginary quadratic extension of F_0.) Let μ be a pure dominant integral weight for G, which we recall is assumed to be trivial on the center Z_N, and hence it's purity weight is 0. For a (complex) place v of F, we have $\mu^v = (\mu^{\iota_v}, \mu^{\bar{\iota}_v})$. The place v is fixed and we drop it from the notations. We have $\mu^\iota = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N)$ and $\mu^{\bar{\iota}} = (\mu_1^* \geq \mu_2^* \geq \cdots \geq \mu_N^*)$ where the purity-weight being 0 implies $\mu_j^* = -\mu_{N-j+1}$. In this case, we let G' be the quasi-split unitary group over F; more precisely, define the matrix

$$[\Phi] \;=\; \Phi_{i,j}, \quad \text{where} \quad \Phi_{i,j} = (-1)^{i+1}\delta(j, N-i+1).$$

This gives a Hermitian form Φ on an N-dimensional F-vector space (with respect to a fixed basis) via $\Phi(x,y) = {}^t x \cdot [\Phi] \cdot \bar{y}$, for $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in F^N$ and $\bar{y} = (\bar{y_1}, \ldots, \bar{y_N})$ being induced by the nontrivial Galois automorphism of F/F_0. Denote $\mathrm{U}(\Phi)$ the corresponding unitary group over F_0, whose F_0-points consists of all $g \in \mathrm{GL}_N(F)$ such that ${}^t g \cdot [\Phi] \cdot \bar{g} = [\Phi]$. Let $G'/F_0 := \mathrm{U}(\Phi)$. Note that $G'(F_0 \otimes \mathbb{R})$ is a product of copies of $\mathrm{U}(\frac{N}{2}, \frac{N}{2})$ if N is even, and is a product of copies of $\mathrm{U}(\frac{N+1}{2}, \frac{N-1}{2})$ if N is odd. At v (dropped from the notation) we put

$$\mu' \;=\; (\mu_1, \mu_2, \ldots, \mu_N).$$

The half sum ρ' of positive roots is given by

$$\rho' = \frac{1}{2} \sum_{1 \leq i < j \leq N} (e_i - e_j) = \left(\frac{N-1}{2},\ \frac{N-3}{2},\ \cdots,\ \frac{N-2i+1}{2},\ \cdots \frac{1-N}{2} \right).$$

Once again define

$$\Lambda' \;:=\; \mu' + \rho' \;=\; \left(\mu_1 + \frac{N-1}{2},\ \mu_2 + \frac{N-3}{2},\ \ldots,\ \mu_N + \frac{1-N}{2} \right),$$

which parametrizes a discrete series representation $\pi' = \pi_{\Lambda'}$ of $\mathrm{U}(\lceil \frac{N}{2} \rceil, \lfloor \frac{N}{2} \rfloor)$. One may see, either from [12, Section 2] or from [4, Example 10.5], that the Langlands parameter $\tau_{\Lambda'}$ of $\pi_{\Lambda'}$ has the form

$$\tau_{\Lambda'} \;=\; z^{a_1} \bar{z}^{-a_1} \;\oplus\; z^{a_2} \bar{z}^{-a_2} \;\oplus \cdots \oplus\; z^{a_N} \bar{z}^{-a_N},$$

where, once again, comparing the infinitesimal character with the exponents of the parameter we get $(a_1, \ldots, a_N) = 2\Lambda'$, i.e.,

$$(a_1, \ldots, a_N) \;=\; (2\mu_1 + N - 1,\ 2\mu_2 + N - 3,\ \ldots,\ 2\mu_N + 1 - N).$$

The transfer π_μ of π' to a representation $\mathrm{GL}_N(\mathbb{C})$ is given by:

$$\pi_\mu \;=\; \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_N(\mathbb{C})} (z^{a_1} \bar{z}^{-a_1} \otimes \cdots \otimes\ z^{a_N} \bar{z}^{-a_N}),$$

which has the property that $H^\bullet(\mathfrak{gl}_N, \mathbb{C}^\times \mathrm{U}(N);\, \pi_\mu \otimes \mathcal{M}_{\mu,\mathbb{C}}) \neq 0$.

The above discussion in all the three cases, together with Künneth theorem for relative Lie algebra cohomology, gives the following

Proposition 3.1. *Let F and G be as in Theorem 2.10. Let μ be a pure dominant integral weight. Define G' as:*

- (1a) *if $F = F_0$ is totally real, and $N = 2n+1$ is odd; take $G' = \mathrm{Sp}(2n)/F$;*
- (1b) *if $F = F_0$ is totally real, $N = 2n$ is even and purity weight of μ is even; take $G' = \mathrm{SO}(2n+1)/F$ (the split group);*
- (1c) *if $F = F_0$ is totally real, $N = 2n$ is even and purity weight of μ is odd; take $G' = \mathrm{SO}(2n)/F$ if n is even, and $G' = \mathrm{SO}'(2n)/F$ if n is odd.*
- (2) *if F is a totally imaginary quadratic extension of a totally real F_0; take $G' = \mathrm{U}(\Phi)/F_0$. Here we assume that μ is trivial on the center $Z(N)$ (hence, $\mathrm{w}(\mu) = 0$).*

In each case, we define a dominant integral weight μ' for G', and put $\Lambda' = \mu' + \rho'$. The discrete series representation $\pi_{\Lambda'}$ of $G'(F_0 \otimes \mathbb{R})$ with infinitesimal character given by Λ' has the property that it transfers to a representation π_μ of $G(\mathbb{R}) = \mathrm{GL}_N(F \otimes \mathbb{R})$ that has nontrivial relative Lie algebra cohomology after twisting by $\mathcal{M}_{\mu,\mathbb{C}}$.

3.2. Consequences of embedding theorems and global transfer.

3.2.1. **Clozel's result on globalizing local discrete series representations.** In this section we describe a result of Clozel ([8, Section 4.3]) on limit multiplicity of discrete series. Let $\mathcal{G}$ be a semi-simple connected group defined over a number field F. Let v_0 be a place such that $\mathcal{G}_{v_0}$ has supercuspidal representations. Let S be a finite set of places containing the archimedean ones and such that $v_0 \notin S$. Let S' be a finite set of finite places disjoint from S. Let $K_{S'}$ be a compact open subgroup of $\mathcal{G}_{S'} = \prod_{v \in S'} \mathcal{G}_v$. Fix K, a compact open subgroup of $\prod_{v \in S \cup S' \cup p_0} \mathcal{G}_v$ and let $\mathcal{L}^{K_0 \times K_{S'} \times K}$ be the $K_0 \times K_{S'} \times K$-invariant functions in the space $L^2_{\mathrm{cusp}}\ (\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}))$ of cusp forms on $\mathcal{G}$. Let δ_S be a discrete series representation of the group $\mathcal{G}_S$.

Theorem 3.2.

$$\liminf_{K_{S'} \to 1} v(K_{S'})\ \mathit{mult}\ (\delta_S, \mathcal{L}^{K_0 \times K_{S'} \times K})\ \geq\ c,$$

where c is a positive real number.

The above theorem is applicable to the simple groups $\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(2n)$, however, it is not directly applicable to $\mathrm{U}(\Phi)$. It is for this reason, that in Theorem 2.10, in Case (2), we take μ to be trivial on the center, and so the discrete series representation $\pi_{\Lambda'}$ has trivial central character and so gives a representation of $\mathrm{PU}(\Phi)_\infty$; and the above theorem is applicable to $\mathrm{PU}(\Phi)$. (One expects, although we have not carried this out, that Clozel's theorem should apply to reductive groups as well and so in particular to $\mathrm{U}(\Phi)$.) Having gotten a representation of $\mathrm{PU}(\Phi)$ we will think of it as a representation of $\mathrm{U}(\Phi)$ with trivial central character. We get the following consequence of Clozel's theorem:

Proposition 3.3. *Let F, G and μ be as in Theorem 2.10. Fix two distinct finite places v and w of F_0. Then there exists a cuspidal automorphic representation π' of G'/F_0 such that*

- *$\pi'_\infty = \pi_{\Lambda'}$, the discrete series representation of $G'_\infty = G'(F_0 \otimes \mathbb{R})$ as in Proposition 3.1,*
- *π'_v is a supercuspidal representation of $G'(F_{0,v})$, and furthermore*
-
  - *in Cases (1a), (1b) and (1c): π'_w is the Steinberg representation of $G'(F_{0,w})$, whereas,*
  - *in Case (2): v is chosen such that $\mathrm{U}(\Phi) \times F_{0,v} \simeq \mathrm{GL}(n)/F_{0,v}$, and we impose no condition on w.*

In Case (2)*, the representation π' has trivial central character.*

3.2.2. **Consequences of the classification of the discrete spectrum of classical groups.** Let's begin by recalling the following result due to Magaard and Savin [21, Proposition 8.2]:

Proposition 3.4. *Let σ be a cuspidal automorphic representation on* $\mathrm{Sp}(2n)$ *over a number field F, such that for some finite place v of F the local component σ_v is the Steinberg representation. Let π be the automorphic representation of* $\mathrm{GL}(2n+1)/F$ *which is the lift of σ as in* [1, Theorem 1.5.2]*. Then π_v is the Steinberg representation and π is cuspidal.*

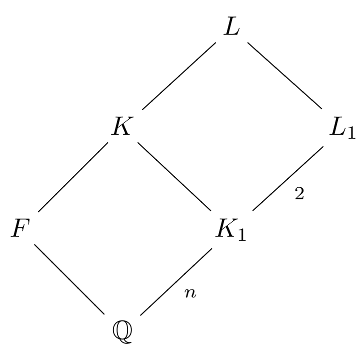
The key point in the proof of the above proposition in *loc. cit.,* is to show that π_v is the Steinberg representation (cuspidality of π then easily follows). But, it is not *a priori* guaranteed from Arthur's work whether the Steinberg representation σ_v transfers to the Steinberg representation. However, Arthur transfer is well-behaved with respect to the Aubert involution, and this involution switches the Steinberg representation with the trivial representation, and one observes using strong approximation for $\mathrm{Sp}(2n)$ that an automorphic representation containing the trivial representation as some local component is itself the trivial representation. Let's remark that this proof goes through *mutatis mutandis* for the orthogonal groups after one makes the same observation concerning the trivial representation: let τ be an automorphic representation of G' as in (1b) or (1c) of Proposition 3.1; assume that τ has a trivial local component, say at v; now inflate τ to an automorphic representation $\tilde{\tau}$ of $\tilde{G}$, where $\tilde{G}$ is the appropriate Spin group–the simply-connected cover of G; then $\tilde{\tau}$ also has a trivial component at v; applying strong approximation which is known for the simply-connected group $\tilde{G}$ (see, for example, Platonov–Rapinchuk [23, Theorem 7.12]) we conclude that $\tilde{\tau}$ is trivial, *a fortiori*, τ is trivial. We record the analogue of the above proposition as:

Proposition 3.5. *Let σ be a cuspidal automorphic representation of G' as above, such that for some finite place v of F the local component σ_v is the Steinberg representation. Let π be the automorphic representation of* $\mathrm{GL}(N)/F$ *which is the lift of σ as in* [1, Theorem 1.5.2]*. Then π_v is the Steinberg representation and π is cuspidal.*

3.2.3. **Conclusion of the proof.** Now consider the situation of Proposition 3.3. The cuspidal automorphic representation π' of G' transfers to an automorphic representation of $G = \mathrm{GL}_N/F$; this is via Arthur [1, Theorem 1.5.2] in cases (1a) and (1b) and by Mok [22, Theorem 2.5.2] in case (2). In cases (1a) and (1b), Propositions 3.4 and 3.5 ensure that π is cuspidal. In case (2) it is clear since the place v is taken so that the unitary group splits and has a supercuspidal local component guaranteeing cuspidality of π. Now by Proposition 3.1 we know that π is cohomological with respect to the given weight μ.

4. **Proof of Theorem 2.15**

We give a proof of Theorem 2.15 generalizing Clozel's construction in [9] using automorphic induction. Let F be the given number field and μ be the given parallel weight assumed to have purity weight $\mathsf{w} = 0$. So μ is of the form $\mu = (\mu_1 \geq \cdots \geq \mu_{2n})$ with $\mu_j \in \mathbb{Z}$ and such that $\mu_j = -\mu_{2n+1-j}$, $\forall\ 1 \leq j \leq n$. Let $\rho = \left(n - \frac{1}{2}, n - \frac{3}{2}, \cdots, -n + \frac{1}{2}\right)$ be the half sum of positive roots for $\mathfrak{gl}(2n, \mathbb{C})$. Put $\ell = (\ell_1, \ldots, \ell_{2n}) = 2\mu + 2\rho$. Note that $\ell_{2n-i+1} = -\ell_i$. Choose a totally real number field K_1, which is cyclic over $\mathbb{Q}$ and linearly disjoint with F over $\mathbb{Q}$; let $\mathrm{Gal}(K_1|\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}/n\mathbb{Z}$. We also choose a totally imaginary quadratic extension L_1 of K_1 which is also linearly disjoint with F over $\mathbb{Q}$. Let $L = FL_1$ and $K = FK_1$ be the corresponding compositum fields as shown in the diagram below.



Let $\mathbb{I}_{L_1} = L_1^\times \backslash \mathbb{A}_{L_1}^\times$ be the idèle class group of the number field L_1. Let $\chi_1 \in \mathrm{Hom}(\mathbb{I}_{L_1}, S^1)$ be a unitary algebraic Hecke character of $\mathbb{I}_{L_1}$. From [30], one knows that the infinity type of χ_1, i.e., its restriction to the group

$\prod_{v \in S_\infty(L_1)} L_{1,v}^\times \cong (\mathbb{C}^\times)^n$, is of the form:

$$\chi_1((a_v)) = \prod_{v \in S_\infty(L_1)} (a_v/|a_v|)^{-f_v},$$

for integers f_v. Conversely, from [30], given any set of integers $\{f_v\}$, there is a Hecke character with infinity type as above; this is because L_1 is a totally imaginary quadratic extension of a totally real number field K_1. Fix an ordering on $S_\infty(L_1)$ and take $\{f_v\} = (\ell_1, \dots, \ell_n)$, to get a character χ_1 with infinity type:

$$(\chi_1)(z_1, z_2, \dots, z_n) = \prod_{j=1}^{n} (z_j/\overline{z_j})^{\ell_j/2} = \prod_{j=1}^{n} (z_j/|z_j|)^{\ell_j},$$

where, $(z_1, \dots, z_n) = (z_v)_{v \in S_\infty(L_1)}$.

Now we briefly explain the construction of a representation $\pi(\chi)$ of $\mathrm{GL}(2n, \mathbb{A}_F)$ obtained by automorphic induction (in two steps) from $\chi = \chi_1 \circ N(L|L_1)$. Following Jacquet and Langlands [18], one constructs a cuspidal representation $\pi_K(\chi)$ of $\mathrm{GL}(2, \mathbb{A}_K)$ via automorphic induction across L/K. Next, we let Π_K be the representation of $\mathrm{GL}(2n, \mathbb{A}_K)$ obtained by inducing

$$\pi_K(\chi) \times \pi_K(\chi)^\sigma \times \cdots \times \pi_K(\chi)^{\sigma^{n-1}}$$

from the Levi component $\mathrm{GL}(2, \mathbb{A}_K) \times \mathrm{GL}(2, \mathbb{A}_K) \times \cdots \times \mathrm{GL}(2, \mathbb{A}_K)$ of the parabolic subgroup $\mathrm{P}(2, 2, \dots, 2)$. Since Π_K is stable under $\mathrm{Gal}(K_1|\mathbb{Q})$ action, it descends to a unique cuspidal representation $\pi(\chi)$ of $\mathrm{GL}(2n, \mathbb{A}_F)$ such that for every archimedean place v of F, the local component π_v is described as follows:

- For a real place v of F,

$$\pi_v = J_v(\mu) := \mathrm{Ind}_{\mathrm{P}(2,2,\cdots,2)}^{\mathrm{GL(2n)}} \left(D(\ell_1) \otimes D(\ell_2) \otimes \cdots D(\ell_n) \right).$$

- For a complex place v of F,

$$\pi_v = J_v(\mu) := \mathrm{Ind}_{\mathrm{B}(\mathbb{C})}^{\mathrm{GL}_{2n}(\mathbb{C})} \left(z^{a_1} \overline{z}^{b_1} \otimes \cdots \otimes z^{a_{2n}} \overline{z}^{b_{2n}} \right).$$

  where $a := \mu + \rho$, $b := -\mu - \rho$.

As in [9], from the results of Speh [29] and Enright [11], one knows that in both the above cases, $J_v(\mu)$ is the only *generic* representation of $\mathrm{GL}(2n, F_v)$ which is cohomological with respect to the given parallel weight μ. The proof of Theorem 2.15 follows as in [9] using the above observations and (2.6).

5. An endoscopic stratification of inner cohomology

Let's recall the definition of inner or interior cohomology. Take a field E that is Galois over $\mathbb{Q}$ and containing a copy of F. Refining the notations as in Section 2.1, consider a dominant integral weight $\mu \in X^+(T \times E)$, giving us a rational finite-dimensional representation $\mathcal{M}_{\mu,E}$ of $G \times E$, which in turn gives a sheaf $\widetilde{\mathcal{M}}_{\mu,E}$ of E-vector spaces on $S^G_{K_f}$. Inner cohomology is the image of cohomology with compact supports in global cohomology:

$$H^\bullet_!(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,E}) \ := \ \mathrm{Image}\left(H^\bullet_c(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,E}) \ \to \ H^\bullet(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,E})\right).$$

If we pass to a transcendental situation via any embedding $\iota : E \to \mathbb{C}$, then it is well-known

$$H^\bullet_{\rm cusp}(S^{G_n}, \widetilde{\mathcal{M}}^\vee_{\iota\mu,\mathbb{C}}) \ \subset \ H^\bullet_!(S^{G_n}_{K_f}, \widetilde{\mathcal{M}}^\vee_{\mu,E}) \otimes_{E,\iota} \mathbb{C}.$$

(See, for example, [10] or [17].) Next, let's recall arithmeticity for Shalika models ([14, Appendix]): if Π is a cuspidal representation of GL_{2n} over a totally real F, and suppose that Π is cohomological and has a Shalika model, then for $\sigma \in \mathrm{Aut}(\mathbb{C})$, the representation ${}^\sigma\Pi$ also has a Shalika model. In other words, if Π is cohomological and is a transfer from $\mathrm{SO}(2n+1)$ then so is any conjugate of Π. The above considerations gives the following

Corollary 5.1. *Let F be a totally real field, and take $G = \mathrm{GL}(2n)/F$. Let μ be a parallel weight with purity 0. Let E be a Galois extension of $\mathbb{Q}$ that contains a copy of F. There exists a nontrivial E-subspace*

$$H^\bullet_{\rm symp}(S^G, \mathcal{M}^\vee_{\mu,E}) \ \subset \ H^\bullet_!(S^G, \mathcal{M}^\vee_{\mu,E})$$

stable under all Hecke operators, such that $H^\bullet_{\rm symp}(S^G, \mathcal{M}^\vee_{\mu,E}) \otimes_{E,\iota} \mathbb{C}$ is spanned by cuspidal representations of G that are all transfers from $\mathrm{SO}(2n+1)$.

Similarly, arithmeticity for a cuspidal representation Π of $\mathrm{GL}(2n+1)$ over a totally real field for which a partial symmetric square L-function has a pole at $s = 1$ (cf. [13, Remark 5.5]) gives:

Corollary 5.2. *Let F be a totally real field, and take $G = \mathrm{GL}(2n+1)/F$. Let μ be a parallel weight with purity 0. Let E be a Galois extension of $\mathbb{Q}$ that contains a copy of F. There exists a nontrivial E-subspace*

$$H^\bullet_{\rm orth}(S^G, \mathcal{M}^\vee_{\mu,E}) \ \subset \ H^\bullet_!(S^G, \mathcal{M}^\vee_{\mu,E})$$

stable under all Hecke operators, such that $H^\bullet_{\rm orth}(S^G, \mathcal{M}^\vee_{\mu,E}) \otimes_{E,\iota} \mathbb{C}$ is spanned by cuspidal representations of G that are all transfers from $\mathrm{Sp}(2n)$.

The notation $H^\bullet_{\rm symp}$ (resp., $H^\bullet_{\rm orth}$) is to suggest that we are looking at the contribution of cuspidal representations with 'symplectic' (resp., 'orthogonal') parameters.

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