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Author(s):

A. Rastegar

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ON ATKIN-LEHNER CORRESPONDENCES ON SIEGEL SPACES

A. RASTEGAR

This paper is dedicated to Professor Shahidi for the occasion of his 70th birthday

Abstract. We introduce a higher dimensional Atkin-Lehner theory for Siegel-Parahoric congruence subgroups of $GSp(2g)$. Old Siegel forms are induced by geometric correspondences on Siegel moduli spaces which commute with almost all local Hecke algebras. We also introduce an algorithm to get equations for moduli spaces of Siegel-Parahoric level structures, once we have equations for prime levels and square prime levels over the level one Siegel space. This way we give equations for an infinite tower of Siegel spaces after N. Elkies who did the genus one case.

Keywords: Atkin-Lehner theory, Siegel moduli space, old and new Siegel modular forms.
MSC(2010): Primary: 14G35; Secondary: 32N10.

1. Introduction

Classical Atkin-Lehner theory for $GL(2)$ has two conceptual ingredients: The first one is Casselman's theory of local new-forms for $GL(2)$ [4], generalized to $GSp(4)$-case by R. Schmidt for square-free level [24]. But, at the moment there is no general local theory of new-forms available for $GSp(2g)$, not even for $GSp(4)$ in the general case. The second ingredient of Atkin-Lehner theory is strong multiplicity one for cuspidal automorphic representations of $GL(2)$. But, multiplicity one fails to hold in higher $GSp(2g)$'s. The strong multiplicity one holds for "globally generic" cuspidal automorphic representations of GSp(4) over a number field. See Jiang-Soundry [15] for update of this result. Multiplicity one is not expected to hold for the cuspidal automorphic representations of GSp(2n), not even for GSp(4), if you consider all cuspidal representations [9]. However, it does hold for a larger class of cuspidal representations of GSp(4) than just globally generic ones. They are sometimes referred to as those of Ramanujan type. These are the ones that are supposed

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to satisfy the Ramanujan conjecture. A nice classification of cuspidal automorphic representations of GSp(4) can be found in a paper of Arthur in the Shalika volume: Arthur [1]. For the current status of multiplicity one on $GSp(4)$ we refer to work of Schmidt [25].) Therefore, here, we regard all Siegel forms with the same Hecke eigenvalues as old Siegel-forms to be old forms again. Despite this loss, Atkin-Lehner geometric theory can be generalized to Siegel-parahoric congruence subgroups of $GSp(2g)$.

We use elements of the Weyl group of $GSp(2g)$ to construct new congruence subgroups sandwiched between the Siegel-parahoric congruence group $\Gamma^P(n)$ and what we call the diagonal congruence group $\Gamma^D(n)$. These groups are all defined in terms of the mod-n reduction of elements in $Sp(2g, \mathbb{Z})$. We shall use these congruence subgroups to introduce geometric correspondences on Siegel moduli spaces with explicit moduli interpretations. These correspondences induce an injection of a number of copies of Siegel modular forms with Siegel-parahoric n-level structure inside the space of Siegel modular forms of Siegel-parahoric level pn for $(p, n) = 1$. Since our correspondences commute with all q-Hecke correspondences for $(q, pn) = 1$, any satisfactory local definition of p-old Siegel forms will imply that our correspondences introduce a higher dimensional Atkin-Lehner theory for p-old Siegel forms of Siegel parahoric level pn. By a satisfactory local theory of new-forms, we mean that any new eigenform of almost all local Hecke algebras should be an eigenform of all local Hecke algebras prime to the level. Also, eigenforms with eigenvalues repeated in lover levels should be considered as old-forms. Our results are in characteristic zero. We shall discuss the consequences of our results in mod-p embeddings in a forthcoming paper. In brief, we get mod-p embedding for primes $p > g$ which are relatively prime to the level n.

Next, we give algebraic equations for moduli spaces of Siegel-Parahoric level structures, by taking fiber products of the same moduli spaces with prime power level structure, which in turn can be constructed via algebraic equations for Siegel moduli spaces of Siegel-Parahoric prime level and square prime level structures. This way we can give equations for an infinite tower of level structures after Elkies who did the genus one case [7].

2. **Atkin-Lehner theory**

2.1. **The Siegel-parahoric subgroup of** $GSp(2g)$**.** The Chevalley group scheme $GSp(2g)$ is defined as the set of matrices $P \in GL(2g)$ with ${}^tPJP = \lambda(P)J$ where $\lambda(P) \in GL(1)$ and

$$J = \begin{pmatrix} 0 & Id_g \\ -Id_g & 0 \end{pmatrix}.$$

The representation λ is called the similitude. Similar to above, we also use two by two matrices whose entries are $g \times g$ matrices to represent elements of $GSp(2g)$. The symplectic group scheme $Sp(2g)$ is defined to be the kernel of the multiplier representation, which is the space of transformations on the symplectic space $\mathbb{Z}^{2g}$ with its standard alternating form:

$$\langle , \rangle : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z} \qquad \langle (u, v), (z, w) \rangle \mapsto u.^t w - v.^t z$$

Let $T \cong \mathbb{G}_m^{g+1}$ denote the maximal torus in $GSp(2g)$. An element of T is a diagonal matrix $diag(a_1, ..., a_g, d_1, ..., d_g)$ with $a_i d_i$ equal to the multiplier. Let M denote the subgroup of $GSp(2g)$ which respects the standard decomposition $\mathbb{Z}^{2g} \cong \mathbb{Z}^g \oplus \mathbb{Z}^g$. The subgroup M consists of diagonal elements in $GSp(2g)$ in their two by two representation. These elements are of the form $diag(A, D)$ with $A.^t D = \lambda I_g$. The subgroup of $Sp(2g, \mathbb{Z})$ which fixes only the first direct summand $\mathbb{Z}^g \subset \mathbb{Z}^{2g}$ is denoted by U which is the space of all $\mathbb{Z}$-valued bilinear symmetric forms. Elements of U are upper-triangular with I_g on diagonal entries and a symmetric matrix B on the upper right corner. The subgroup $P = M \ltimes U$ is a maximal parabolic subgroup whose elements are zero in the lower left $g \times g$ corner. This parabolic subgroup is also called the Siegel-parahoric subgroup. Fix the Borel subgroup contained in P consisting of the matrices with $A, B, 0, D$ entries with A upper-triangular and D lower-triangular. Weyl groups of $GSp(2g)$ and P with respect to the maximal torus T are denoted by W_G and W_P respectively. $W_G \cong S_g \ltimes (\pm 1)^g$ and $W_P \cong S_g$ act on diagonal matrices $diag(a_1, ..., a_g, d_1, ..., d_g)$ by permutation or exchange of the a_i's and the d_i's.

2.2. **Congruence subgroups of $Sp(2g, \mathbb{Z})$.** A discrete subgroup

$$\Gamma \subset Sp(2g, \mathbb{Q})$$

is called a congruence group, if it contains $\Gamma(n)$ for some positive integer n, where $\Gamma(n)$ is the kernel of reduction map modulo n on $Sp(2g, \mathbb{Z})$:

$$\Gamma(n) = \{\gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv I_{2g} \in Sp(2g, \mathbb{Z}/n\mathbb{Z}) \pmod{n}\}.$$

The Siegel-parahoric congruence group is defined by

$$\Gamma^P(n) = \{\gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, (\text{mod } n)\}$$

and the diagonal congruence group by

$$\Gamma^D(n) = \{\gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv diag(*, ..., *), (\text{mod } n)\}.$$

$\Gamma^P(n)$ and $\Gamma^D(n)$ are examples of congruence groups. The significance of congruence groups is that they carry arithmetic information. In this paper, we are only interested in congruence subgroups of $Sp(2g, \mathbb{Z})$.

By the theory of Tits systems [14], every parabolic subgroup of $GSp(2g)$ is conjugate to a "standard" parabolic subgroup which contains B. Each standard parabolic subgroup P_I corresponds to one of the 2^{g+1} subsets $I \subset S$ where S is a minimal generating set for the Weyl group W_G consisting of involutions

ρ_i which are elements of order 2. S can be taken as the set of simple reflections corresponding to the base of the root system determined by B. In fact, $P_I = BW_IB$ where $W_I \subseteq W_G$ is the subgroup generated by elements in I. In particular, $P_\emptyset = B$ and $P_S = GSp(2g)$.

By Bruhat decomposition theorem, $GSp(2g) = \cup B\sigma B$ is a decomposition to disjoint subsets where σ runs in the Weyl group W_G. Two such double cosets coincide if and only if the middle Weyl elements coincide. One can assume that axioms of Tits systems are satisfied. Namely,

(T1) For $\rho \in S$ and $\sigma \in W_G$ we have $\rho B\sigma \subset B\sigma B \cup B\rho\sigma B$.

(T2) For $\rho \in S$ we have $\rho B\rho \neq B$.

An expression $\sigma = \rho_1...\rho_k$ with $\rho_i \in S$ is called reduced if k is as small as possible. the minimal length k of a reduced expression is denoted by $\ell(\sigma)$. By convention, $\ell(\sigma) = 0$ if and only if $\sigma = id$ and $\ell(\sigma) = 1$ if and only if $\sigma \in S$. Tits axioms imply that for $\rho \in S$ we have $\ell(\rho\sigma) = \ell(\sigma) \pm 1$. This implies that for the reduced form $\sigma = \rho_1...\rho_k$ and $I = \{\rho_1, ..., \rho_k\}$, the parabolic subgroup P_I is generated B and $\sigma B\sigma^{-1}$ or by B and σ. Given a choice of a Borel subgroup S is precisely the set of elements in W_G such that $B \cup B\sigma B$ is a group. P_I is conjugate to P_J implies that $P_I = P_J$ and $P_I \subset P_J$ implies that $I \subset J$.

We define a parabolic congruence subgroup $\Gamma^{P_I}(n) \subset Sp(2g, \mathbb{Z})$ to be the set of elements which reduce to P_I modulo n:

$$\Gamma^{P_I}(n) = \{\gamma \in Sp(2g, \mathbb{Z}) | \gamma \in P_I \subset GSp(2g, \mathbb{Z}/n\mathbb{Z}) \pmod{n}\}.$$

Fix a generating set S for W_G consisting of $g-1$ pairs in $S_g \subset W_g$ and a nonzero element in $(\pm 1)^g$. Now, there exists a generating set $w_1, ..., w_g$ for the weyl group such that w_i have increasing length if represented in reduced form in term of involutions in S.

Assume that $w_{k+1} = \rho_k w_k$ for all k where ρ_k is an involution. Also assume w_1,...,w_{g-1} generate S_g. We have $\Gamma^{P_{I_{g-1}}}(n) = \Gamma^P(n)$ where P is the maximal parabolic we fixed in notations. The fact that B together with $I_k = \{w_1, ..., w_k\}$ can not generate any of w_{k+1},...,w_g implies that each P_{I_k} has exactly $\frac{g!}{(k+1)!}2^g$ conjugates of the form $\sigma P_{I_k}\sigma^{-1}$ for σ in W_G. To these we associate $\frac{g!}{(k+1)!}2^g$ parabolic congruence subgroups of the form $\sigma\Gamma^{P_{I_k}}(n)\sigma^{-1}$ for σ in W_G. In particular, $\Gamma^B(n)$ has $g!2^g$ conjugates in $Sp(2g, \mathbb{Z})$ and each $\Gamma^{P_{I_k}}(n)$ has $k+2$ conjugates in $\Gamma^{P_{I_{k+1}}}(n)$.

We have made a nested family of parabolic congruence groups contained in the maximal parabolic congruence group $\Gamma^P(n)$ consisting of levels 0 to $g-1$. The k-th level is formed by $\frac{g!}{(k+1)!}2^g$ congruence groups. Each group in level k contains $k+1$ groups in level $k-1$ for $k = 1$ to $g-1$. There are 2^g congruence groups in level $g-1$ and the congruence groups in level 0 are the $g!2^g$ conjugates by elements in W_g of $\Gamma^B(p)$ which are lying inside $Sp(2g, \mathbb{Z})$.

2.3. **Siegel moduli spaces.** A general reference for the arithmetic of Siegel moduli spaces is [8]. Let A_g denote the moduli stack of principally polarized abelian schemes of relative dimension g. By a symplectic principal level-n structure, we mean a symplectic isomorphism $\alpha : A[n] \to (\mathbb{Z}/n\mathbb{Z})^{2g}$, where $(\mathbb{Z}/n\mathbb{Z})^{2g}$ is equipped with the standard non-degenerate skew-symmetric pairing.

Let ζ_n denote an n-th root of unity for $n \geq 3$. The moduli scheme classifying the principally polarized abelian schemes over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$ together with a symplectic principal level-n structure is a scheme over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$ and will be denoted by $\mathcal{A}_g(n)$. The symplectic group $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{A}_g(n)$ as a group of symmetries by acting on level structures. We will recognize these moduli spaces and their etale quotients under the action of subgroups of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ as Siegel spaces.

A $\Gamma^P(n)$-level structure of type I on (A, λ) is choice of a subgroup $H \subset A[n]$ of order n^g which is totally isotropic with respect to the Weil pairing induced by λ. A $\Gamma^P(n)$-level structure of type II on (A, λ) is choice of a principally polarized isogeny $(A_1, \lambda_1) \to (A_2, \lambda_2)$ of degree n^g. By a principally polarized isogeny, we mean an isogeny $\sigma : A_1 \to A_2$ such that $\sigma \circ \lambda_2 \circ \sigma^t \circ \lambda_1^{-1}$ is multiplication by an integer. For $n \geq 3$ type I and type II $\Gamma^P(n)$-level structures induce isomorphic moduli schemes over $Spec(\mathbb{Z}[1/n])$ [16]. We denote this moduli scheme by $\mathcal{A}_g^P(n)$. There exists a natural involution

$$w_n^P : \mathcal{A}_g^P(n) \to \mathcal{A}_g^P(n)$$

taking $(\sigma : (A_1, \lambda_1) \to (A_2, \lambda_2))$ to $(\sigma^t : (A_2, (\lambda_2^t)^{-1}) \to (A_1, (\lambda_1^t)^{-1}))$ which we call the Atkin-Lehner involution.

A $\Gamma^B(n)$-level structure of type I on (A, λ) is choice of g subgroups $H_i \subset A[n]$ of order n^i with $H_1 \subset ... \subset H_g$ where H_g is totally isotropic. A $\Gamma^B(n)$-level structure of type II on (A, λ) is choice of a chain of g isogenies $(A_0, \lambda_0) \stackrel{\alpha}{\to} ... \stackrel{\alpha}{\to} (A_g, \lambda_g)$ each of degree n which satisfy $n.id_{A_i} = \alpha^i \circ \lambda_0^{-1} \circ (\alpha^t)^g \circ \lambda_g \circ \alpha^{g-i}$ for all $i = 1, ..., g$. In case $n \geq 3$ type I and type II $\Gamma^B(n)$-level structures induce isomorphic moduli schemes over $Spec(\mathbb{Z}[1/n])$ [16]. We denote this moduli scheme $\mathcal{A}_g^B(n)$. There also exists a natural involution

$$w_n^B : \mathcal{A}_g^B(n) \to \mathcal{A}_g^B(n)$$

taking $((A_0, \lambda_0) \stackrel{\alpha}{\to} ... \stackrel{\alpha}{\to} (A_g, \lambda_g))$ to $((A_g, (\lambda_g^t)^{-1}) \stackrel{\alpha^t}{\to} ... \stackrel{\alpha^t}{\to} (A_0, (\lambda_0^t)^{-1}))$ which commutes with the Atkin-Lehner involution under the natural projection between the Siegel spaces

$$\begin{array}{ccc} \mathcal{A}_g^B(n) & \stackrel{w_n^B}{\longrightarrow} & \mathcal{A}_g^B(n) \\ \downarrow & & \downarrow \\ \mathcal{A}_g^P(n) & \stackrel{w_n^P}{\longrightarrow} & \mathcal{A}_g^P(n). \end{array}$$

A $\Gamma^T(n)$-level structure on (A,λ) is choice of $2g$ subgroups $H_i \subset A[n]$ for $i=1$ to $2g$, each isomorphic to $(\mathbb{Z}/n\mathbb{Z})$ such that $H_1 \oplus ... \oplus H_g$ and $H_{g+1} \oplus ... \oplus H_{2g}$ are totally isotropic subgroups of order n^g which do not intersect with $H_i \oplus H_{g+i}$ hyperbolic for $i=1$ to g. For A and A' abelian schemes over the schemes S and S' respectively, we define a morphism from $(S,A,\lambda,H_1,...,H_{2g})$ to $(S',A',\lambda',H'_1,...,H'_{2g})$ to be a pair of morphisms (f,g) where $f: S \to S'$ and $g: A \to A'$ satisfy $g^*(\lambda') = \lambda$ and $g(H_i) = H'_i$ for all $1 \leq i \leq 2g$. Also we want the pair (f,g) to induce an isomorphism $A \simeq S \times_{S'} A'$. Having these morphisms defined, we have formed a category $\mathbf{A}_g^T(n)$. The functor $\pi: \mathbf{A}_g^T(n) \to Sch$ defined by $\pi(S,A,\lambda,H_1,...,H_{2g}) = S$ makes $\mathbf{A}_g^T(n)$ into a stack in groupoids over S. The 1-morphism of stacks $\pi': \mathbf{A}_g^T(n) \to \mathbf{A}_g$ defined by $\pi'(S,A,\lambda,H_1,...,H_{2g}) = (S,A,\lambda)$ is representable and is a proper surjective morphism. For $n \geq 3$ we get a separated scheme of finite type $A_g^T(n)$ which is smooth over $Spec(\mathbb{Z}[1/n])$.

Let $\mathbb{H}_g$ denote the Siegel upper half-space, which consists of the set of complex symmetric $g \times g$ matrices Ω with $\Im(\Omega)$ positive definite. As a complex analytic stack $\mathcal{A}_{g/\mathbb{C}}$ is the quotient of Siegel upper half-space $\mathbb{H}_g$ by the action of $Sp(2g,\mathbb{Z})$ via Möbius transformations. The family of principally polarized abelian varieties over $\mathbb{H}_g$ is given by $A(\Omega) = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega.\mathbb{Z}^g)$. To any congruence subgroup $\Gamma \subset Sp(2g,\mathbb{Z})$ one can associate the quotient $\Gamma \backslash \mathbb{H}_g$ which is a Siegel moduli-space with some extra level structure. The corresponding level structure can be made explicit. Indeed, $\Gamma(n)\backslash\mathbb{H}_g$ corresponds to $\mathcal{A}_g(n)_{/\mathbb{C}}$ whose quotient under the action of the symplectic group $Sp(2g,\mathbb{Z}/n\mathbb{Z})$ is $\mathcal{A}_{g/\mathbb{C}}$. Any congruence subgroup $\Gamma(n) \subset \Gamma \subset Sp(2g,\mathbb{Z})$ corresponds to the quotient of $\mathcal{A}_g(n)_{/\mathbb{C}}$ under the action of the finite group $\Gamma/\Gamma(n)$. This helps to associate explicit level structures to the space $\Gamma\backslash\mathbb{H}_g$ which makes it a moduli space.

2.4. **The $\Gamma^{P_I}(p)$-level structure.** The Atkin-Lehner involution can be generalized from $GL(2)$ [3] to the higher dimensional case $GSp(2g)$. This generalization involves Siegel-parahoric and Borel congruence groups. On the other hand, local considerations show that p-old Siegel modular forms with respect to the Siegel-parahoric or Borel congruence groups of level pn contain several copies of Siegel forms of level n. This implies that a single Atkin-Lehner involution would not do the job of geometrically generating the p-old part. In this section, we intend to introduce geometric correspondences which complement the role of Atkin-Lehner involution.

Let p be a prime not dividing the positive integer n and let $\mathcal{A}_g^T(p)_{/\mathbb{C}}$ and $\mathcal{A}_g^{T,P}(p,n)_{/\mathbb{C}}$ denote the Siegel spaces associated to congruence groups $\Gamma^T(p)$ and $\Gamma^{T,P}(p,n) = \Gamma^T(p) \cap \Gamma^P(n)$, respectively. The group $\Gamma^T(p)$ remains invariant under conjugation by elements in $(\pm 1)^g \subset W_g$. So $(\pm 1)^g$ acts on $\mathcal{A}_g^T(p)_{/\mathbb{C}}$ and $\mathcal{A}_g^{T,P}(p,n)_{/\mathbb{C}}$ by 2^g involutions. Let $\mathcal{A}_g^{P_{I_k}}(p)_{/\mathbb{C}}$ and $\mathcal{A}_g^{P_{I_k},P}(p,n)_{/\mathbb{C}}$ denote the Siegel spaces associated to the congruence groups $\Gamma^{P_{I_k}}(p)$ and

$\Gamma^{P_{I_k},P}(p,n) = \Gamma^{P_{I_k}}(p) \cap \Gamma^P(n)$ respectively. We have a chain of etale maps

$$\mathcal{A}_g^{B,P}(p,n) \to ... \to \mathcal{A}_g^{P_{I_k},P}(p,n) \to \mathcal{A}_g^{P_{I_{k+1}},P}(p,n) \to ... \to \mathcal{A}_g^P(pn).$$

Since each congruence group $\Gamma^{P_{I_{k+1}}}(p)$ on the $(k+1)$-th level contains $k+2$ conjugates (by Weyl elements) of $\Gamma^{P_{I_k}}(p)$ on the k-th level, we expect that for each k we get $k+2$ copies of forms on $\mathcal{A}_g^{P_{I_{k+1}},P}(p,n)$ injecting in forms on $\mathcal{A}_g^{P_{I_k},P}(p,n)$. We will use the geometry of $\mathcal{A}_g^{B,P}(p,n)$ to give a geometric construction of these $k+2$ copies.

In order to simplify the notations, let us forget the $\Gamma^P(n)$-level structure which is auxiliary. We get a chain of etale maps

$$\mathcal{A}_g^B(p) \to ... \to \mathcal{A}_g^{P_{I_k}}(p) \to \mathcal{A}_g^{P_{I_{k+1}}}(p) \to ... \to \mathcal{A}_g^P(p).$$

which corresponds to a chain of congruence groups

$$\Gamma^B(p) \hookrightarrow ... \hookrightarrow \Gamma^{P_{I_k}}(p) \hookrightarrow \Gamma^{P_{I_{k+1}}}(p) \hookrightarrow ... \hookrightarrow \Gamma^P(p).$$

Each $\Gamma^{P_{I_k}}(p)$ maps to $\Gamma^{P_{I_{k+1}}}(p)$ by $k+1$ maps: natural inclusion and conjugation by representatives $\sigma_{k+1} \in W_k \backslash W_{k+1}$ followed by inclusion, where W_k is the subgroup of the Weyl group generated by $w_1, ..., w_k$. Inclusion induces the natural projection map $\pi_{P_{I_k},P_{I_{k+1}}} : \mathcal{A}_g^{P_{I_k}}(p) \to \mathcal{A}_g^{P_{I_{k+1}}}(p)$. Conjugation by σ_{k+1} induces an inclusion

$$\sigma_{k+1}\Gamma^{P_{I_k}}(p)\sigma_{k+1}^{-1} = \Gamma^{\sigma_{k+1}P_{I_k}\sigma_{k+1}^{-1}}(p) \hookrightarrow \Gamma^{P_{I_{k+1}}}(p).$$

This inclusion corresponds to another projection from a different moduli-space

$$\mathcal{A}_g^{\sigma_{k+1}P_{I_k}\sigma_{k+1}^{-1}}(p) \to \mathcal{A}_g^{P_{I_{k+1}}}(p).$$

Conjugation by σ_{k+1} identifies $\mathcal{A}_g^{P_{I_k}}(p)$ with $\mathcal{A}_g^{\sigma_{k+1}P_{I_k}\sigma_{k+1}^{-1}}(p)$. The moduli-space $\mathcal{A}_g^T(p)$ is the appropriate moduli space to geometrically realize all the endomorphisms

$$v_p^\sigma : \mathcal{A}_g^T(p) \to \mathcal{A}_g^T(p)$$

induced by conjugation via elements σ in the Weyl group W_G. In fact, the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{A}_g^T(p) & \xrightarrow{v_p^\sigma} & \mathcal{A}_g^T(p) \\ \downarrow & & \downarrow \\ \mathcal{A}_g^{P_I}(p) & \xrightarrow{\sigma.} & \mathcal{A}_g^{\sigma P_I \sigma^{-1}}(p) \\ \downarrow & & \downarrow \\ \mathcal{A}_g^{P_J}(p) & \xrightarrow{\sigma.} & \mathcal{A}_g^{\sigma P_J \sigma^{-1}}(p). \end{array}$$

2.5. **The geometry of Siegel spaces $\mathcal{A}_g^B(p)$ and $\mathcal{A}_g^{P_I,P}(p)$.** In this section we try to geometrically characterize fibers of natural maps between moduli spaces, we have already introduced.

Definition 2.1. We say that two points x and y on $\mathcal{A}_g^{P_I,P}(p,n)$ are σ-connected, for an element σ in the Weyl group, if there exists a chain of points $x = x_1, ..., x_t = y$ on $\mathcal{A}_g^{P_I,P}(p,n)$ such that for each i there are points x_i' and x_{i+1}' on $\mathcal{A}_g^{T,P}(p,n)$ mapping to x_i and x_{i+1} respectively, with $x_{i+1} = v_p^\sigma(x_i)$ where

$$v_p^\sigma : \mathcal{A}_g^{T,P}(p,n) \to \mathcal{A}_g^{T,P}(p,n)$$

is the endomorphism induced by the action of σ on p-level structure.

Proposition 2.2. *Every fiber of the map $\pi_k : \mathcal{A}_g^{P_{I_k},P}(p,n) \to \mathcal{A}_g^{P_{I_{k+1}},P}(p,n)$ is an equivalence class of ρ_{k+1}-connected points.*

Proof. The n-level structure is auxiliary. Let x be a point on the Siegel upper half-plane $\mathbb{H}_g$ and let $[x]^T$ denote the equivalence class containing x defined by left quotient of the Siegel upper space by $\Gamma^T(p)$. We have $\sigma.[y]^T = [\sigma.y]^T$. The group $\Gamma^{P_{I_{k+1}}}(p)$ is generated $\Gamma^{P_{I_k}}(p)$ and ρ_{k+1}. So every element in $\Gamma^{P_{I_{k+1}}}(p)$ can be written as a product of elements of the form $\gamma_i\rho_{k+1}$ with $\gamma_i \in \Gamma^{P_{I_k}}(p)$. Define the equivalence class $[x]^{P_{I_k}}$ similarly. The classes $[x]^{P_{I_k}}$ and $[\gamma\rho_{k+1}.x]^{P_{I_k}}$ are ρ_{k+1}-connected. So the equivalence class $[x]^{P_{I_{k+1}}}$ is obtained by joining the ρ_{k+1}-connected points. □

Definition 2.3. For a subset $W \subset W_g$, we say that two points x and y on $\mathcal{A}_g^{P_I,P}(p,n)$ are W-connected, if there exists a chain of points $x = x_1, ..., x_t = y$ on $\mathcal{A}_g^{P_I,P}(p,n)$, such that for each i, x_i and x_{i+1} are σ-connected for some $\sigma \in W$.

Proposition 2.4. *Every fiber of the map $\mathcal{A}_g^{B,P}(p,n) \to \mathcal{A}_g^{P_{I_k},P}(p,n)$ is an equivalence class of W_k-connected points.*

Proof. This is a direct consequence of definition. □

Proposition 2.5. *Every fiber of the map $\mathcal{A}_g^P(pn) \to \mathcal{A}_g^P(n)$ is an equivalence class of σ-connected points for some nonzero representative σ of $W_P\backslash W_G$.*

Proof. The Bruhat-Tits decomposition modulo p implies that the congruence groups $\Gamma^P(p)$ and $\sigma\Gamma^P(p)\sigma^{-1}$ generate $Sp(2g,\mathbb{Z})$. This is a consequence of the fact that any two conjugates of a maximal parabolic subgroup over $\mathbb{F}_p$ generate the whole algebraic group $GSp(2g,\mathbb{F}_p)$. □

2.6. **p-old Siegel modular forms on $\mathcal{A}_g^P(p)$.** Let $\mathcal{B}_g^P(n)$ denote the universal abelian variety over $\mathcal{A}_g^P(n)$. The Hodge bundle ω is defined to be the pull back via the zero section $i_0 : \mathcal{A}_g^P(n) \to \mathcal{B}_g^P(n)$ of the line bundle $\wedge^{top}\Omega^1_{\mathcal{B}_g(n)/\mathcal{A}_g(n)}$.

The Hodge bundle is an ample invertible sheaf on $\mathcal{A}_g^P(n)$. Let R be a $\mathbb{Z}[1/n]$-module. By a $\Gamma^P(n)$-Siegel modular form of weight k with coefficients in R, we mean an element in $H^0(\mathcal{A}_g^P(n)/R, \omega^{\otimes k}/R)$. The same notation is used for other congruence subgroups, but in this paper we focus on Siegel modular forms with respect to Siegel parahoric congruence subgroup P and Borel congruence subgroup B.

If we pull back the Hodge bundle ω to the Siegel upper half-space, the pull back is canonically isomorphic to $\mathcal{O}_{\mathbb{H}_g} \otimes_{\mathbb{C}} \wedge^g(\mathbb{C}^g)$. A complex modular form of weight k becomes an expression of the form $f(\Omega).(dz_1 \wedge ... \wedge dz_g)^{\otimes k}$ where f is an $\Gamma^P(n)$-invariant complex holomorphic function on $\mathbb{H}_g$ which is holomorphic at ∞. For genus ≥ 2 the condition, holomorphic at infinity, is automatically satisfied by Koecher principle. Trivializing ω on $\mathbb{H}_g$, complex modular forms of weight k are identified with holomorphic functions $f(\Omega)$ on $\mathbb{H}_g$, satisfying the transformation rule $f|[\gamma]_k = f$ for all $\gamma \in \Gamma^P(n)$, where

$$f|[\gamma]_k(\Omega) = \eta(\gamma)^{gk/2} det(C.\Omega + D)^{-k} f(\gamma.\Omega).$$

Let l be a prime not dividing pn. To any Siegel modular form f of weight k and level n, one associates an irreducible admissible representation $\pi = \bigotimes \pi_\ell$ of $GSp(2g, \mathbb{A}_f)$ over $\bar{\mathbb{Q}}_l$ [2]. This association is not unique, but we use it as a motivation to understand the notion of p-old form. Let U denote the open subgroup of $GSp(2g, \mathbb{A}_f)$ associated to the congruence group $\Gamma^P(n)$. If $(p, n) = 1$ and $\pi^U \neq 0$, then π_p is spherical, and it is the unique unramified irreducible subquotient of some unramified principal series representation π_χ with respect to Borel subgroup $B(\mathbb{Q}_p)$. One can show that $\pi_\chi^{GSp(2g,\mathbb{Z}_p)}$ is one-dimensional. So the number of copies of modular forms with respect to $\Gamma^P(n)$ inside modular forms with respect to $\Gamma^P(pn)$ is equal to the dimension of $\pi_\chi^{\Gamma^P(p)}$. The mod-p Bruhat-Tits decomposition implies that, we have the following decomposition

$$GSp(2g, \mathbb{Q}_p) = \coprod_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g \subset W_g} B(\mathbb{Q}_p)\sigma\Gamma^P(p).$$

So to specify $f \in \pi_\chi^{\Gamma^P(p)}$ it is enough to specify it on elements $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$. Because $W_P \simeq S_g$ and the subgroup $(\mathbb{Z}/2\mathbb{Z})^g \subset W_G$ is a complete set of representatives for $W_P \backslash W_G$. Therefore, the space of p-old forms on $\mathcal{A}_g^P(pn)$ consists of 2^g copies of forms on $\mathcal{A}_g^P(n)$. The vector space $\pi_\chi^{\Gamma^P(p)}$ has a basis consisting of functions $f_1, ..., f_{2^g}$ where f_i is supported on $B(\mathbb{Q}_p)\sigma_i\Gamma^P(p)$ and $f_i(\sigma_i) = 1$. The group of involutions $(\mathbb{Z}/2\mathbb{Z})^g \subset W_G$ acts on the space of p-old forms by $f(z) \mapsto f^\sigma(z) := f(\sigma.z)$ for $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$.

Similar considerations show that we expect $g!2^g$ copies of forms on $\mathcal{A}_g^P(n)$ inside the space of p-old forms on $\mathcal{A}_g^{B,P}(p, n)$. Following Atkin-Lehner theory, we need a geometric characterization of the space of p-old forms.

2.7. **Atkin-Lehner correspondences.** Let $\pi_{T,P_I} : \mathcal{A}_g^T(p) \to \mathcal{A}_g^{P_I}(p)$ and $\pi_{P_I,P_J} : \mathcal{A}_g^{P_I}(p) \to \mathcal{A}_g^{P_J}(p)$ denote the natural projection maps induced by inclusions of the corresponding congruence groups. Pulling back forms from level n to level np using the natural projection map

$$\pi_n : \mathcal{A}_g^{B,P}(p,n) \to \mathcal{A}_g^P(n)$$

induces the first copy of p-old forms in $H^0(\mathcal{A}_g^{B,P}(p,n), \omega^{\otimes k})$. For simplicity, let us forget the auxiliary level structure and consider the projection

$$\pi_1 : \mathcal{A}_g^B(p) \to \mathcal{A}_g,$$

and p-old forms in $H^0(\mathcal{A}_g^B(p), \omega^{\otimes k})$. At first glance, it seems that geometric correspondences of the form $D_B^\sigma(p) = \pi_{T,B*} v_p^{\sigma *} \pi_{T,B}^*$ should induce more copies of p-old Siegel forms on $\mathcal{A}_g^B(p)$ out of the pull-back copy.

$$\begin{array}{ccc} \mathcal{A}_g^T(p) & \xrightarrow{v_p^\sigma} & \mathcal{A}_g^T(p) \\ \downarrow & & \downarrow \\ \mathcal{A}_g^B(p) & & \mathcal{A}_g^B(p). \end{array}$$

But $\pi_{T,B} \circ \pi_1$ commutes with v_p^σ for all $\sigma \in W_G$. Therefore, correspondences of above type generate the same copy as the pull back copy. To disturb the symmetry of the picture, we use Atkin-Lehner involution. $\pi_{T,B} \circ w_p^B \circ \pi_1$ no longer commutes with v_p^σ and we can hope that correspondences of the form $C_P^\sigma(p) = \pi_{T,B*} v_p^{\sigma *} \pi_{T,B}^* w_p^{B*}$ could generate more copies of p-old forms

$$\begin{array}{ccccc} \mathcal{A}_g^T(p) & \xrightarrow{v_p^\sigma} & \mathcal{A}_g^T(p) & & \\ \downarrow & & \downarrow & & \\ \mathcal{A}_g^B(p) & & \mathcal{A}_g^B(p) & \xrightarrow{w_p^B} & \mathcal{A}_g^B(p). \end{array}$$

To generate p-old forms in $H^0(\mathcal{A}_g^B(p), \omega^{\otimes k})$ we should use $g!2^g$ correspondences $C_B^\sigma(p)$ for $\sigma \in W_G$.

Theorem 2.6 (**Main Theorem**). *The linear subspaces of* $H^0(\mathcal{A}_g^{B,P}(p,n), \omega^{\otimes k})$ *generated by Atkin-Lehner correspondences* $C_B^\sigma(p)\pi_n^*$ *where* $C_B^\sigma(p)$ *is defined by* $\pi_{T,B*} v_p^{\sigma *} \pi_{T,B}^* w_p^{P*}$

$$\begin{array}{ccccc} \mathcal{A}_g^{T,P}(p,n) & \xrightarrow{v_p^\sigma} & \mathcal{A}_g^{T,P}(p,n) & & \\ \downarrow & & \downarrow & & \\ \mathcal{A}_g^{B,P}(p,n) & & \mathcal{A}_g^{B,P}(p,n) & \xrightarrow{w_p^B} & \mathcal{A}_g^{B,P}(p,n) \end{array}$$

for σ *varying in* W_G *give* $g!2^g$ *linearly independent copies of* $H^0(\mathcal{A}_g^P(n), \omega^{\otimes k})$ *inside* p*-old forms of level* pn *living on* $\mathcal{A}_g^{B,P}(p,n)$.

The corresponding theorem for $H^0(\mathcal{A}_g^P(p), \omega^{\otimes k})$ can also be proved. However, in order to generate p-old forms in $H^0(\mathcal{A}_g^P(p), \omega^{\otimes k})$ we should divide the space of p-old forms on $\mathcal{A}_g^B(p)$ by the action of $(\mathbb{Z}/2\mathbb{Z})^{2g} \subset W_G$.

Main theorem is proved in a few stages. In the first stage, we prove that Atkin-Lehner involution induces a second copy of level-n forms inside p-old part of level-np forms which has trivial intersection with the pull-back copy.

Proposition 2.7. *The subspaces*

$$\pi_{B,P_I}^* H^0(\mathcal{A}_g^{P_I,P}(p,n), \omega^{\otimes k}) \text{ and } w_p^{B*} \pi_{B,P_I}^* H^0(\mathcal{A}_g^{P_I,P}(p,n), \omega^{\otimes k}),$$

as subspaces of $H^0(\mathcal{A}_g^{B,P}(p,n), \omega^{\otimes k})$, have trivial intersection.

Proof. Let f and g be nonzero Siegel modular forms in

$$\pi_{B,P_I}^* H^0(\mathcal{A}_g^{P_I,P}(p,n), \omega^{\otimes k})$$

with $w_p^{B*} f = g$. Since w_p^{B*} is an involution, $f \pm g$ are eigenforms of w_p^{B*} and the proposition follows from the following □

Lemma 2.8. *Any Siegel form which is eigenform of w_p^{B*} on $\mathcal{A}_g^{B,P}(p,n)$ and also pull back of a Siegel form on $\mathcal{A}_g^{P_I,P}(p,n)$ vanishes if $P_I \subsetneq B$.*

Proof. The zero locus of such an eigenform is pull back of the zero locus of a form living on $\mathcal{A}_g^{P_I,P}(p,n)$ and also w_p^{B*}-invariant. This contradicts density of Hecke orbit [5]. □

In the second stage, we show that Atkin-Lehner correspondences induce $g!2^g$ non-intersecting copies of level-n forms inside p-old part of level-np forms.

Lemma 2.9. *Let $D_B^\sigma(p) = \pi_{T,B_*} v_p^{\sigma *} \pi_{T,B}^*$, where*

$$\pi_{T,B} : \mathcal{A}_g^{T,P}(p,n) \to \mathcal{A}_g^{B,P}(p,n)$$

is the natural projection. For $\sigma, \sigma' \in W_G$ the correspondences $D_B^\sigma(p) D_B^{\sigma'}(p)$ and $D_B^{\sigma\sigma'}(p)$ acting on any linear subspace of $H^0(\mathcal{A}_g^{B,P}(p,n), \omega^{\otimes k})$ generate the same image subspaces.

Proof. For simplicity, let us forget the auxiliary n-level structure. Then, lemma follows from commutativity of the following diagram

$$\begin{array}{ccccc} \mathcal{A}_g^T(p) & \xrightarrow{v_p^\sigma} & \mathcal{A}_g^T(p) & \xrightarrow{v_p^{\sigma'}} & \mathcal{A}_g^T(p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_g^{\sigma\sigma' B \sigma'^{-1} \sigma^{-1}}(p) & \xrightarrow{\sigma} & \mathcal{A}_g^{\sigma' B \sigma'^{-1}}(p) & \xrightarrow{\sigma'} & \mathcal{A}_g^B(p) \end{array}$$

and $v_p^\sigma \circ v_p^{\sigma'} = v_p^{\sigma\sigma'}$, and $\sigma \circ \sigma' = \sigma\sigma'$ hold for $\sigma, \sigma' \in W_G$. □

Proposition 2.10. *Every two linear subspaces of $H^0(\mathcal{A}_g^{B,P}(p,n),\omega^{\otimes k})$ generated by correspondences $C_B^\sigma(p)\pi_n^*$ acting on $H^0(\mathcal{A}_g^P(n),\omega^{\otimes k})$ for $\sigma \in W_G$ have trivial intersection.*

Proof. By the previous lemma, it is enough to show that the linear subspaces generated by correspondences $\pi_{T,B_*}v_p^{\sigma *}\pi_{T,B}^* w_p^{B*}\pi_1^*$ and $w_p^{B*}\pi_1^*$ have such intersection. Suppose $\pi_{T,B_*}v_p^{\sigma *}\pi_{T,B}^* f = g$ for nonzero Siegel modular forms on $\mathcal{A}_g^B(p)$ which are elements of $w_p^{B*}\pi_n^* H^0(\mathcal{A}_g^P(n),\omega^{\otimes k})$. Let $d = deg(\pi_{T,B})$. Then $df \pm g$ are eigenforms of $\pi_{T,B_*}v_p^{\sigma *}\pi_{T,B}^*$ with eigenvalue $\pm d$. Now, the truth of proposition is a consequence of the first stage and the following □

Lemma 2.11. *Any nonzero Siegel modular form on $\mathcal{A}_g^{B,P}(p,n)$ which is an eigenform of $\pi_{T,B_*}v_p^{\sigma *}\pi_{T,B}^*$ for a non-zero σ in W_G is pull back of a Siegel form on $\mathcal{A}_g^{P',P}(p,n)$ where P' is the parabolic subgroup generated by B and $\sigma B \sigma^{-1}$.*

Proof. This is a consequence of Propositions 2.2 and 2.4. □

In the final stage we show that the above-mentioned $g!2^g$ copies of level-n Siegel forms inside the space of Siegel forms of level pn are indeed p-old forms of level-pn and they are linearly independent.

Proposition 2.12. *The Atkin-Lehner correspondences $C_B^\sigma(p)\pi_n^*$ commute with all Hecke correspondences which generate the local Hecke algebras H_q for q relatively prime to pn.*

Proof. The action of Atkin-Lehner correspondences can be interpreted in terms of the p-torsion of abelian varieties representing points of the moduli-space. By geometric base-change one can see that such an action commutes with those interpreted in terms of the q-torsion points. □

Now, we have enough tools to prove the 2.6.

Proof of the main theorem 2.6. The $g!2^g$ copies of level-n Siegel forms induced by Atkin-Lehner correspondences are contained in the space of p-old forms by previous proposition. Recall that in this paper Siegel modular forms with eigenvalues repeated from lover levels are considered to be old. The space of Siegel forms on $\mathcal{A}_g^{B,P}(p,n)$ is finite-dimensional and has a basis of all prime to pn Hecke eigenforms. So is the case for any of the $g!2^g$ Atkin-Lehner copies, by previous proposition. Fix a basis for the space of Siegel forms of level pn whose elements are eigenforms of all prime to pn local Hecke algebras. Suppose the $g!2^g$ Atkin-Lehner copies are linearly dependent. It means that a nonzero eigenform f of all prime to pn local Hecke algebras is generated by some basis elements of the Atkin-Lehner copies which have the same eigenvalues. Consider the vector space V of all Siegel forms with the same Hecke eigenvalues f and consider the $g!2^g$ Atkin-Lehner copies in it. V is invariant under the action of all the $g!2^g$ correspondences $D^{\sigma}_B(p) = \pi_{T,B_*} v_p^{\sigma *} \pi^*_{T,B}$ for $\sigma \in W_G$, because these correspondences commute with the Atkin-Lehner copies. By this symmetry, $df + \pi_{T,P_*} v_p^{\sigma *} \pi^*_{T,P} f$ is also a Hecke eigenform in V with the same Hecke eigenvalues as f. There exists a $\sigma \in W_G$ which gives a nonzero Siegel form $df + \pi_{T,B_*} v_p^{\sigma *} \pi^*_{T,B} f$ generated by the Atkin-Lehner copies. But such a vector is a Siegel form which can be pulled back from a lower Siegel space by Proposition 2.4. This contradicts linear dependency. □

Having constructed the Atkin-Lehner copies of p-old forms on $\mathcal{A}_g^{B,P}(p,n)$ one can get the 2^g copies of p-old forms on $\mathcal{A}_g^P(pn)$ by pushing forward all the $g!2^g$ p-old copies down to $\mathcal{A}_g^P(pn)$.

Theorem 2.13. *The linear subspaces of $H^0(\mathcal{A}_g^P(pn), \omega^{\otimes k})$ generated by Atkin-Lehner correspondences $C^{\sigma}_P(p)\pi^*_n$ for σ varying in W_G where $C^{\sigma}_P(p)$ is defined by*

$$\sum \pi_{T,P*} v_p^{\eta\sigma *} \pi^*_{T,B} w_p^{B*},$$

where the sum ranges over $\eta \in S_g \subset W_G$ give 2^g linearly independent copies of $H^0(\mathcal{A}_g^P(n), \omega^{\otimes k})$ inside p-old forms of level pn living on $\mathcal{A}_g^P(pn)$.

Proof. This is a direct consequence of the 2.6 and Proposition 2.7. We just have to consider $\mathcal{A}_g^P(n)$ as an equivarient quotient of $\mathcal{A}_g^{B,P}(p,n)$. □

It is worth to note that, this theorem is a direct generalization of Atkin-lehner theory. It reduces to a statement similar to their statement in genus one case. We will review the genus one case after we mention the special case of the above theorem in genus 2.

It may seem curious that we don't use the pull back copy to generate all the 2^g copies, like what we did in classical Atkin-Lehner theory. The reason is that, the pull back copy does generate a new copy applying w_p, but not 2^g copies, because it is invariant under the action of the corresponding subgroup of the Weyl group. In fact, we apply w_p to break this symmetry. This is how we are able to apply correspondences generated by elements of the Weyl group to generate new copies. Here we shall reformulate the special gase of genus 2 for the nonepert to have a better feeling of what is going on.

In genus two case, there are four correspondences which are used to generate the four copies. The mod-p Bruhat decomposition implies that, we have the following decomposition

$$GSp(4, \mathbb{Q}_p) = \coprod B(\mathbb{Q}_p) w_i \Gamma_0(p),$$

where B is the Borel subgroup and w_i for $i = 1$ to 4 are running over the representatives of $W_{Sp(4,\mathbb{Z}_p)}/W_{\Gamma_0(p)}$. We choose the following representatives

$$w_1 = id, w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is suggested in [23] that because of this picture, there should be 4 copies of modular forms of level n inside the p-old part which is analytically characterized by the conjugates $w_i\Gamma_0(p)w_i^{-1}$ for $i = 1$ to 4. All of these conjugates lie in $Sp(4,\mathbb{Z}_p)$. We get the copy associated to $w_1 = id$ simply by pulling back level-n forms to level np via the projection map between moduli spaces. Following the genus one case, we can get a second copy using the action of universal p-isogeny w_p which is realized on $\Gamma_0(p)$. Now we have candidates for two of the copies of level-n forms inside the p-old part.

we are interested in finding 4 algebraic correspondences on $A_2^0(np)$ such that the image of the pull back copy under the action of these correspondences gives us all of the p-old copies. The most natural way to look for these correspondences is to pull back all the 4 copies of modular forms of level n to a congruence subgroup which is of a richer geometric structure. For example, we can pull back to $\Gamma'(n,p) = \Gamma_0(n) \cap \Gamma'(p)$ where

$$\Gamma'(p) = \{\gamma \in Sp(4,\mathbb{Z}) | \gamma \equiv diag(*,*,*,*) \ (mod\ p)\}.$$

The 4 specified elements of the Weyl group induce 4 involutions on the moduli space $A_2'(n,p)$ corresponding to $\Gamma'(n,p)$. Indeed, the conjugations $w_i\Gamma'(p)w_i^{-1}$ stabilize the congruence group. These involutions have a nice simple interpretation in terms of the moduli property. We could also work with $\Gamma'(np)$ and the associated Siegel space $A_2'(np)$.

The Siegel space associated to $\Gamma'(p)$ is the quotient of the level-p Siegel space by the subgroup of $Sp(4,\mathbb{Z}/p\mathbb{Z})$ which stabilizes all of the 4 copies of $\mathbb{Z}/p\mathbb{Z}$ in $(\mathbb{Z}/p\mathbb{Z})^{2g}$. The conjugations correspond to symplectic automorphisms which are well defined on the kind of level structure we are considering here. It is essential to note that we can not obtain the 4 copies of p-old modular forms by applying the above 4 involutions on the direct pull back of modular forms via the natural map $\pi' : A_2'(n,p) \to A_2^0(n)$. Because forms in the pull-back are already invariant under all w_i's. Instead, we shall apply w_p after pulling these forms back as far as $A_2^0(np)$ and then pull them back to $A_2'(n,p)$. The second copy we get in this manner, generates a new copy of p-old forms on $A'_2(n,p)$ by applying involutions by Weyl elements, to the pull back of the second copy generated by w_p which we can push forward down to $A_2^0(np)$.

Theorem 2.14. *The space of p-old Siegel modular forms of level np is generated by the images of correspondences $\pi'_* \circ w_i \circ \pi'^* \circ w_p$ acting on the pull-back copy of Siegel modular forms of level n inside those of level np.*

These are called the Atkin-Lehner correspondences. The space of new forms is defined to be the orthogonal complement of the space of p-old forms with respect to the Petterson inner product. The careful considerations of R. Schmidt shows that this is a well-defined notion of old-form for square-free n (Look at table one in [25]). Moreover, one can prove that these correspondences produce all of the 4 copies we are expecting inside the space of Siegel modular forms of level np.

In genus one, we choose the representatives

$$w_1 = id, w_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we get the following version of Atkin-Lehner theory

Theorem 2.15. *The space of p-old elliptic modular forms of level np is generated by the images of correspondences w_p and $\pi'_* \circ w_2 \circ \pi'^* \circ w_p$ acting on the pull-back copy of elliptic modular forms of level n inside those of level np.*

Note that, Atkin-Lehner consider the corresponce w_p and the pull back copy in their original work [3].

3. Algebraic equations for towers of Siegel spaces

3.1. **Construction of $\mathcal{A}_g^P(p^k)$ from $\mathcal{A}_g^P(p)$'s and $\mathcal{A}_g^P(p^2)$'s.** In this section, we follow Elkies, who did the genus one case [7]. Fix a prime $p > 1$. For positive k, the Siegel moduli space $\mathcal{A}_g^P(p^k)$ parametrizes principally polarized abelian varieties with a cyclic p^k-isogeny, or equivalently sequences of p-isogenies

$$A_0 \to A_1 \to A_2 \to \cdots \to A_k$$

such that the composite isogeny $A_{j-1} \to A_{j+1}$ of degree p^{2g} is cyclic for each j with $0 < j < k$. Thus for each $m = 0, 1, \ldots, k$ there are $k + 1 - m$ maps $\pi_j : \mathcal{A}_g^P(p^k) \to \mathcal{A}_g^P(p^m)$ obtained by extracting for some $j = 0, 1, \ldots, k-m$ the cyclic p^m-isogeny $A_j \to A_{j+m}$ from the above sequence. In particular we have a tower of maps

$$\mathcal{A}_g^P(p^k) \to \mathcal{A}_g^P(p^{k-1}) \to \mathcal{A}_g^P(p^{k-2}) \to \cdots \to \mathcal{A}_g^P(p^2) \to \mathcal{A}_g^P(p),$$

each map being of degree p^g. Each $\mathcal{A}_g^P(p^k)$ also has an Atkin-Lehner involution $w_p = w_p^{(k)}$, taking a cyclic p^k-isogeny to its dual isogeny, and the above sequence to the sequence

$$A_k \to \cdots \to A_2 \to A_1 \to A_0$$

of dual isogenies. We thus have

$$w_p^{(m)} \circ \pi_j = \pi_{k-m-j} \circ w_p^{(k)},$$

where π_j, π_{k-m-j} are our jth and $(k-m-j)$th maps from $\mathcal{A}_g^P(p^k)$ to $\mathcal{A}_g^P(p^m)$.

Now the explicit formulas for $\mathcal{A}_g^P(p), \mathcal{A}_g^P(p^2)$, together with the involutions $w_p^{(1)}, w_p^{(2)}$ of these moduli spaces and the map $\pi_0 : \mathcal{A}_g^P(p^2) \to \mathcal{A}_g^P(p)$ between them, suffice to exhibit the entire tower explicitly:

Proposition 3.1. *For $k \geq 2$ the product map*

$$\pi = \pi_0 \times \pi_1 \times \pi_2 \times \cdots \times \pi_{k-2} : \mathcal{A}_g^P(p^k) \to (\mathcal{A}_g^P(p^2))^{k-1}$$

is a 1:1 map from $\mathcal{A}_g^P(p^k)$ to the set of $(P_1, P_2, \ldots, P_{k-1}) \in \left(\mathcal{A}_g^P(p^2)\right)^{k-1}$ such that

$$\pi_0\big(w_p^{(2)}(P_j)\big) = w_p^{(1)}\big(\pi_0(P_{j+1})\big)$$

for each $j = 1, 2, \ldots, k-2$.

Informally, we get from $\mathcal{A}_g^P(p^2)$ up to $\mathcal{A}_g^P(p^k)$ by iterating $k-2$ times the involution $w_p^{(2)}$ composed with the "p-valued involution" $\pi_0^{-1} w_p^{(1)} \pi_0$. Of course the maps $\pi_j : \mathcal{A}_g^P(p^k) \to \mathcal{A}_g^P(p^m)$ (for $m \geq 2$) are then simply

$$(P_1, \ldots, P_{k-1}) \mapsto (P_{j+1}, \ldots, P_{j+m-1}),$$

and the involution $w_p^{(k)}$ is

$$(P_1, P_2, \ldots, P_{k-2}, P_{k-1}) \leftrightarrow (w_p^{(2)} P_{k-1}, w_p^{(2)} P_{k-2}, \ldots, w_p^{(2)} P_2, w_p^{(2)} P_1),$$

i.e. reversing the order of $P_1, \ldots, P_{k-1}$ and applying $w_p^{(2)}$ to each coordinate.

Proof. It is clear that the map is 1:1 to its image, because a sequence of p-isogenies is determined by the p^2-isogenies $A_{j-1} \to A_{j+1}$ parametrized by the jth coordinate of π $(0 < j < k)$. Now $(P_1, \ldots, P_{k-1})$ is in the image of π if and only if the p^2-isogenies parametrized by $P_1, \ldots, P_{k-1}$, regarded as sequences $A_0^j \to A_1^j \to A_2^j$ of p-isogenies, fit together to form a sequence with $A_i^j = A_{i+j}$, i.e. if and only if the isogenies $A_1^j \to A_2^j$ and $A_0^{j+1} \to A_1^{j+1}$ coincide for each $j = 1, 2, \ldots, k-2$. But these isogenies are represented by the points $\pi_1(P_j)$ and $\pi_0(P_{j+1})$ on $\mathcal{A}_g^P(p)$. Thus the necessary and sufficient condition is that

$$\pi_1(P_j) = \pi_0(P_{j+1})$$

for each $j = 1, 2, \ldots, k-2$; applying $w_p^{(1)}$ to both sides, and then commutativity of the Atkin-Lehner involutions to $w_p^{(1)}\big(\pi_1(P_j)\big)$, then yields the equivalent form of what we seek. □

3.2. **Construction of $\mathcal{A}_g^P(n)$ from $\mathcal{A}_g^P(p^k)$'s.**

Proposition 3.2. *Let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime decomposition of an integer $n \geq 2$ then if $\pi_j : \mathcal{A}_g^P(p_j^{\alpha_j}) \to \mathcal{A}_g(1)$ denotes the natural projection forgetting the level structure for $j = 1, ..., k$, then $\mathcal{A}_g^P(n)$ is nothing but the fiber product of $\mathcal{A}_g^P(p_j^{\alpha_j})$'s over $\mathcal{A}_g(1)$ via π_j.*

In fact, the following is true:

Proposition 3.3. *Let n and m relatively prime natural numbers. Let $\pi' : \mathcal{A}_g^P(n) \to \mathcal{A}_g(1)$ and $\pi'' : \mathcal{A}_g^P(m) \to \mathcal{A}_g(1)$ denote the natural projections forgetting the level structures, then*

$$\mathcal{A}_g^P(nm) = \mathcal{A}_g^P(n) \times_{\mathcal{A}_g(1)} \mathcal{A}_g^P(m)$$

where $\times$ is the fiber product over $\mathcal{A}_g(1)$ via π' and π''.

Proof. Considering the moduli interpretation of the above mentioned moduli spaces, the fact that n-level atructures are independent of m-level structures for $(n, m) = 1$ and fixing these two we can get an mn-level structure gives a one-to-one map from the right hand side to the left. □

By the previous section, once we have algebraic equations for $\mathcal{A}_g^P(p)$ and $\mathcal{A}_g^P(p^2)$ and morphisms over $\mathcal{A}_g(1)$ we get algebraic equations for $\mathcal{A}_g^P(p^k)$ over $\mathcal{A}_g(1)$ for all primes p and this will suffice to get algebraic equations for all $\mathcal{A}_g^P(n)$ as we desire.

3.3. **Compactification of Siegel moduli spaces.** The space of Siegel modular forms can also be formulated in the language of schemes. Let S be a base scheme. A modular form f of weight k is a rule which assigns to each principally polarized abelian variety $(A/S, \lambda)$ a section $f(A/S, \lambda)$ of $\omega_{A/S}^{\otimes k}$ over S depending only on the isomorphism class of $(A/S, \lambda)$ commuting with arbitrary base change. Here $\omega_{A/S}$ is the top wedge of tangent bundle at origin of A over S.

To define Siegel modular forms of higher level, one should equip principally polarized abelian varieties with level structures. Let ζ_n denote an n-th root of unity where $n \geq 3$. On a principally polarized abelian scheme (A, λ) over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$ of relative dimension g we define a symplectic principal level-n structure to be a symplectic isomorphism $\alpha : A[n] \to (\mathbb{Z}/n\mathbb{Z})^{2g}$ where $(\mathbb{Z}/n\mathbb{Z})^{2g}$ is equipped with the standard non-degenerate skew-symmetric pairing

$$\langle , \rangle : (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to \mathbb{Z}/n\mathbb{Z}$$

$$\langle (u, v), (z, w) \rangle \mapsto u.w^t - v.z^t$$

Let S be a scheme over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$. The moduli scheme classifying the principally polarized abelian schemes over S together with a symplectic principal level-n structure is a scheme over S and will be denoted by $\mathcal{A}_g(n)$. The moduli scheme $\mathcal{A}_g(n)$ over S can be constructed from $\mathcal{A}_g(n)$ over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$ by base change.

$Sp(2g, \mathbb{Z}/n\mathbb{Z})$ acts as a group of symmetries on $\mathcal{A}_g(n)$ by acting on level structures. We will recognize these moduli spaces and their equivariant quotients under the action of subgroups of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ as Siegel spaces. We restrict our attention to Siegel spaces over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$. $\mathcal{A}_g(n)$ is connected and smooth over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$. The condition $n \geq 3$ is to guarantee that we get a moduli scheme, instead of getting only a moduli stack. The natural morphism $\mathcal{A}_g(n) \to \mathcal{A}_g(m)$ where m,n are positive integers ≥ 3 and $m|n$ is a finite and etale morphism over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$.

Let $\mathcal{B}_g(n)$ denote the universal abelian variety over $\mathcal{A}_g(n)$. The Hodge bundle ω is defined to be the pull back via the zero section $i_0 : \mathcal{A}_g(n) \to \mathcal{B}_g(n)$ of the line bundle $\wedge^{top}\Omega_{\mathcal{B}_g(n)/\mathcal{A}_g(n)}$. The Hodge bundle is an ample invertible sheaf on $\mathcal{A}_g(n)$ and can be naturally extended to a bundle ω on $\mathcal{A}_g^*(n)$. We could define the minimal compactification $\mathcal{A}_g^*(n)$ by the formula

$$\mathcal{A}_g^*(n) = proj(\oplus_{k\geq 0} H^0(\mathcal{A}_g(n), \omega^{\otimes k})).$$

The graded ring above is regarded as a $\mathbb{Z}[\zeta_n, 1/n]$-algebra. The scheme $\mathcal{A}_g^*(n)$ is equipped with a stratification by locally closed subschemes which are geometrically normal and flat over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$. Each of these strata is canonically isomorphic to a moduli space $A_i(n)$ for some i between 0 and g. The map $\mathcal{A}_g(n) \to \mathcal{A}_g(m)$ can be extended uniquely to $\mathcal{A}_g^*(n) \to \mathcal{A}_g^*(m)$ for $m|n$. These maps when restricted to strata, induce the corresponding natural maps between lower genera Siegel spaces $A_i(n) \to A_i(m)$. The action $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ on $\mathcal{A}_g(n)$ naturally extends to an action on the compactified Siegel space $\mathcal{A}_g^*(n)$. This action is compatible with the maps $\mathcal{A}_g^*(n) \to \mathcal{A}_g^*(m)$ for $m|n$.

Let $K(n)$ denote the subgroup of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ fixing the g first $(\mathbb{Z}/n\mathbb{Z})$-basis elements of $(\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$ on which $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ acts. Since $\mathcal{A}_g^*(n)$ is a projective scheme, we can define the quotient projective schemes $\mathcal{A}_g^{P*}(n)$ to be the geometric quotient of $\mathcal{A}_g^*(n)$ by $K(n)$. This quotient provides us with a compactification of $\mathcal{A}_g^P(n)$ which is the moduli scheme of principally polarized abelian schemes (A, λ) over $Spec(\mathbb{Z}[\zeta_n, \frac{1}{n}])$, together with g elements in $A[n]$ generating a symplectic subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})^g$. Again we have natural maps $\mathcal{A}_g^{P*}(n) \to \mathcal{A}_g^{P*}(m)$ for $m|n$.

We define the Hodge bundle ω on $\mathcal{A}_g^{P*}(n)$ to be the quotient of the Hodge bundle ω on $\mathcal{A}_g^*(n)$ under the action of the corresponding subgroup $K(n)$ of $Sp(2g, \mathbb{Z}/n\mathbb{Z})$. This is possible because the line bundle ω on the space $\mathcal{A}_g^*(n)$ is $Sp(2g, \mathbb{Z}/n\mathbb{Z})$-linearizable. A Siegel modular form of weight k and full level n is a global section of ω^k on $\mathcal{A}_g^*(n)$. Over the complex numbers, this corresponds to a Siegel modular form of weight k with respect to $\Gamma(n)$. In this paper, by a Siegel modular form of weight k and level n we mean a global section of ω^k on $\mathcal{A}_g^{P*}(n)$. This corresponds to the congruence subgroup $\Gamma_0(n)$.

3.4. **Algebraic equations for $\mathcal{A}_2^*(2)$.** In this section, we follow Lee and Weintraub [21]. For construction of compactifications of $\mathcal{A}_2(1)$ look at [19]. The compactification we have built in the previous section is called the Satake compactification which is a projective algebraic variety with severe singularities. It would be also handy to introduce algebraic equations for smooth compactifications as was constructed by Igusa for $g = 2$ and generalized by Mumford and his collaborators to general genus and extended to schemes by Falting and Chai [8] which is called the toroidal compactification.

The Siegel moduli space $\mathcal{A}_2^*(2)$ is related another variety which appeared in the work of Deligne and Mostow [6], constructed by means of Mumfords geometric invariant theory.

Let S denote the set $\{1, 2, 3, 4, 5, 6\}$, and let $\mathbb{P}^5$ denote the space of functions of S to $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. There is a natural action of $PGL_2(\mathbb{C})$ on the space $\mathbb{P}^1$ induced by the linear fractional transformation of $PGL_2(\mathbb{C})$ on $\mathbb{P}^1$. The subspace of injective functions can be identified with $(\mathbb{P}^1)^6 - \Delta$, and its quotient with the moduli space $\mathcal{M}$ of nonsingular curves with level 2 structure. By a stable point (resp. semi-stable point) of $\mathbb{P}^5$, we mean a point with the property that no more than two (resp. three) elements in S have the same image. The group PGL_2 operates freely on the subspace of stable points, and its quotient space $\mathcal{Q}_{st}$, is a quasi-projective variety. To compactify $\mathcal{Q}_{st}$, we consider the space of semi-stable points. Define an equivalence relation for which two stable points are equivalent if and only if they have the same PGL_2-orbit and if two points are semi-stable but not stable they are equvalent if they induce the same partition of S into two sets of three elements S_1 and S_2 such that each functaion separates them and on constant on one of them. The quotient space $\mathcal{Q}_{sst}$ of semi-stable points module this relation is a projective variety, and contains $\mathcal{Q}_{st}$ as a Zariski open set. In fact, $\mathcal{Q}_{cusp} = \mathcal{Q}_{sst} - \mathcal{Q}_{st}$, consists of ten isolated singular points. To desingularize this variety, we blow up these points and obtain a nonsingular variety $\mathcal{Q}_{sst}^{\sim}$. In fact, $\mathcal{Q}_{sst}^{\sim}$ is isomorphic to the Igusa compactification $\mathcal{A}_2^{\sim}(2)$.

The Igusa compactification $\mathcal{A}_2^{\sim}(2)$ of $\mathcal{A}_2(2)$ may be constructed by desingularizing, or blowing up, the Satake compactificationx of$\mathcal{A}_2^*(2)$. Lee and Weintraub construct a birational transformation $f : \mathcal{A}_2^*(2) \to \mathcal{Q}_{sst}$. To begin, they identify $\mathcal{Q}_{sst}$, with a classical object, Segres cubic threefold. From this it follows that $\mathcal{Q}_{sst}$ is isomorphic to the threefold in $\mathbb{P}^5$ defined by the homogeneous equations

$$\sum_{i=1}^{6} x_i = 0,$$

$$\sum_{i=1}^{6} x_i^3 = 0.$$

known as Segres cubic threefold. Since Segres time it has been known that considering the dual hypersurface to the symmetric quartic threefold defined by the equations

$$\sum_{i=1}^{6} x_i = 0,$$

$$\left(\sum_{i=1}^{6} x_i^4\right) - \left(\sum_{i=1}^{6} x_i^2\right)^2 = 0,$$

yields Segres cubic threefold. van der Geer [vdG] has shown that the quartic threefold defined above can be identified with the Satake compactification $\mathcal{A}_2^*(2)$, so one obtains a birational transformation

$$\tilde{f} : \mathcal{A}_2^*(2) \dashrightarrow \mathcal{Q}_{sst}.$$

Alternately, we may consider the Igusa compactification $\mathcal{A}_2^{\sim}(2)$. Note that the Satake compactification $\mathcal{A}_2^*(2)$ is a hypersurface in $\mathbb{P}^5$ defined by a single function $F(x_1, ..., x_5) = 0$. The derivatives $\partial F/\partial x_i$, on the one hand define the coordinate functions of the projective dual, and on the other hand generate the ideal $I = (\partial F/\partial x_1, ..., \partial F/\partial x_5)$ that defines the boundary components ∂. Since $\mathcal{A}_2^{\sim}(2)$ is defined by blowing up $\mathcal{A}_2^*(2)$ along ∂, it follows that $\tilde{f}$ lifts to a morphism from $\mathcal{A}_2^{\sim}(1)$ to the projective dual of $\mathcal{Q}_{sst}$. one can blow down the ten components of the Humbert surface in $\mathcal{A}_2^{\sim}(1)$ to points to get a complex analytic space $\mathcal{A}_2^*(2)$. Then, from the definitions, we have the mapping $\hat{f}$ in the following diagram:

$$\begin{array}{ccc} \hat{f} : \mathcal{A}_2^{\sim}(2) & \longrightarrow & \mathcal{Q}_{sst}^{\sim} \\ \downarrow & & \downarrow \\ \tilde{f} : \mathcal{A}_2^*(2) & \longrightarrow & \mathcal{Q}_{sst}^*. \end{array}$$

It can be shown that $\hat{f}$ and $f^{\sim}$ are isomorphisms.

By division by the two-level structure on $\mathcal{Q}_{sst}$, we can find the ring of invariants of $\mathcal{A}_2^*(1)$ and the relative morphism of $\mathcal{A}_2^*(2)$ over $\mathcal{A}_2^*(1)$.

3.5. **Algebraic equations for $\mathcal{A}_2^*(3)$.** In this section, we follow Hoffman and Weintraub [13]. Here, one uses a variety $\mathcal{B}$ defined over $\mathbb{Q}(\sqrt{-3})$ such that $\mathcal{B}$ over $\mathbb{C}$ is isomorphic with $\mathcal{A}_2^*(3)$. Felix Klein initiated the study of the moduli spaces of genus 2 curves and the coverings defined by "Stufe". Two of his students, H. Burkhardt and H. Maschke, took up the case where Stufe = 3. Burkhardt managed to write down an explicit equation for this moduli space. The general idea is this: Consider the 9 thetanullwerte

$$X_{\alpha,\beta} = \theta \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} (\tau, 0), \ \alpha \in 1/3\mathbb{Z}/\mathbb{Z} \ \beta \in 1/3\mathbb{Z}/\mathbb{Z}.$$

These 9 values have the property that as $\tau \mapsto \gamma.\tau$ with $\gamma \in P\Gamma_2(1)$ they undergo a linear transformation, which is the identity up to scalar multiples for $\gamma \in P\Gamma_2(3)$. In other words, we have a projective representation of the finite simple group of order 25920, $G = P\Gamma_2(1)/P\Gamma_2(3) = PSp(4;\mathbb{F}_3)$. This representation splits into two invariant subspaces of dimension 4 and 5 respectively, the spaces of

$$Z_{\alpha,\beta} = (X_{\alpha,\beta} - X_{-\alpha,-\beta})/2 \; Y_{\alpha,\beta} = (X_{\alpha,\beta} + X_{-\alpha,-\beta})/2.$$

Maschke studied the action of G on the Z's, Burkhardt studied the action on the Y's, and both managed to find the ring of G-invariant forms in their respective cases. Let

$$Y_0 = Y_{0,0},\; 2Y_1 = Y_{1/3,0},\; 2Y_2 = Y_{0,1/3},\; 2Y_3 = Y_{1/3,1/3},\; 2Y_4 = Y_{1/3,2/3}.$$

Burkhardt found the invariant form of degree 4:

$$J_4 = Y_0^4 - Y_0(Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3) + 3Y_1Y_2Y_3Y_4.$$

Proposition 3.4. *Let $\mathcal{B}_0 \subset \mathbb{P}^4$ be the quartic hypersurface defined by $J_4 = 0$ There is an isomorphism between a Zariski open subset of $\mathcal{A}_2(3)$ and a Zariski open subset of $\mathcal{B}_0$. Let $\mathcal{B}$ be the variety obtained by resolving the 45 nodes on $\mathcal{B}_0$. The map above extends to an isomorphism with the Igusa compactification: $\mathcal{B}_0 \simeq \mathcal{A}_2^*(3)$.*

In this form the proposition was first proved by van der Geer [vdG2], who asserted some thing stronger, namely that these results were true for the corresponding schemes over $\mathbb{Z}[1/3;\epsilon]$, where ϵ is a primitive cube root of unity (the existence of a model of $\mathcal{A}_2^*(3)$ over that ring being a consequence of Faltings' theory [Cha-Fal]).

Since we have the group action explicitly, we can find the ring of invariants of $\mathcal{A}_2^*(1)$ and the relative morphism of $\mathcal{A}_2^{P*}(3)$ over $\mathcal{A}_2^*(1)$.

3.6. **Algebraic equations for $\mathcal{A}_2^*(4)$.** In this section, we follow van Geeman and Nygaard [10] and its exposition by Okazaki and Yamauchi [22]. Let $\mathcal{A}_2(2,4,8)$ be the moduli space of abelian surfaces with some level structure which has been studied by van Geemen and Nygaard. It is the quotient space of the Siegel upper half plane of degree 2 by the arithmetic subgroup $\Gamma(2,4,8)$ of the symplectic group $Sp_4(\mathbb{Z})$. This congruence subgroup $\Gamma(2,4,8)$ is contained in the principal congruence subgroup $\Gamma(4) := \{\gamma \in Sp_4(\mathbb{Z}) \mid \gamma \equiv 1_4 \bmod 4\}$. $\mathcal{A}(2,4,8)$ is a quasi-projective smooth threefold. By [10] we have the projective model $\mathcal{A}_2^*(2,4,8)$ the Satake compactification of $\mathcal{A}_2(2,4,8)$ which is defined over $\mathbb{Q}$ in $\mathbb{P}^{13}$ as follows:

$$\begin{aligned}
Y_0^2 &= Q_0(X_0, X_1, X_2, X_3) := X_0^2 + X_1^2 + X_2^2 + X_3^2 \\
Y_1^2 &= Q_1(X_0, X_1, X_2, X_3) := X_0^2 - X_1^2 + X_2^2 - X_3^2 \\
Y_2^2 &= Q_2(X_0, X_1, X_2, X_3) := X_0^2 + X_1^2 - X_2^2 - X_3^2 \\
Y_3^2 &= Q_3(X_0, X_1, X_2, X_3) := X_0^2 - X_1^2 - X_2^2 + X_3^2 \\
Y_4^2 &= Q_4(X_0, X_1, X_2, X_3) := 2(X_0X_1 + X_2X_3) \\
Y_5^2 &= Q_5(X_0, X_1, X_2, X_3) := 2(X_0X_2 + X_1X_3) \\
Y_6^2 &= Q_6(X_0, X_1, X_2, X_3) := 2(X_0X_3 + X_1X_2) \\
Y_7^2 &= Q_7(X_0, X_1, X_2, X_3) := 2(X_0X_1 - X_2X_3) \\
Y_8^2 &= Q_8(X_0, X_1, X_2, X_3) := 2(X_0X_2 - X_1X_3) \\
Y_9^2 &= Q_9(X_0, X_1, X_2, X_3) := 2(X_0X_3 - X_1X_2).
\end{aligned}$$

Since we have the group action of $Sp_4(\mathbb{Z})$ explicitly, we can find the ring of invariants of $\mathcal{A}_2^*(4)$ and $\mathcal{A}_2^*(1)$ and the relative morphism of $\mathcal{A}_2^{P*}(4)$ over $\mathcal{A}_2^*(1)$.

3.7. **Algebraic equations for $\mathcal{A}_2^{P*}(2^k)$ and $\mathcal{A}_2^{P*}(3.2^k)$.** By Section 2.1. we can get algebraic equations for $\mathcal{A}_2^{P*}(2^k)$ using algebraic equations for $\mathcal{A}_2^{P*}(4)$ and $\mathcal{A}_2^{P*}(2)$ over $\mathcal{A}_2^{P*}(1)$. By Section 2.2. using algebraic equations for $\mathcal{A}_2^{P*}(2^k)$ and $\mathcal{A}_2^{P*}(3)$ over $\mathcal{A}_2^{P*}(1)$ one can get algebraic equations for $\mathcal{A}_2^{P*}(3.2^k)$.

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References

[1] J. Arthur, The Endoscopic Classification of Representations, Orthogonal and Symplectic Groups, Amer. Math. Soc. Colloq. Publ. **61**. Amer. Math. Soc. Providence, RI, 2013.
[2] M. Asgari and R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104** (2001) 173–200.
[3] A.O.L. Atkin and J. Lehner, Hecke Operators on $\Gamma_0(m)$, *Math. Ann.* **185** (1970) 134–160.
[4] W. Casselman, On Some Results of Atkin and Lehner, *Math. Ann.* **201** (1973) 301–314.
[5] C.-L. Chai, Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli, *Invent. Math.* **121** (1995), no. 3, 439–479.
[6] P. Deligne and G.D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, *Publ. Math. Inst. Hautes Études Sci.*, **63** (1986) 5–89.
[7] N.D. Elkies, Explicit modular towers, Arxiv:math/0103107 [math.NT].
[8] G. Faltings and C. Chai, Degeneration of Abelian Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete 22, Springer, Berlin-Heidelberg, 1990.

[9] D. File and R. Takloo-Bighash, A remark on the failure of multiplicity one for GSp(4), *Manuscripta Math.* **140** (2013), no. 1-2, 263-272.

[10] B. van Geemen and N.O. Nygaard, On the geometry and arithmetic of some Siegel modular threefolds, *J. Number Theory* **53** (1995) 45–87.

[11] G. van der Geer, On the geometry of a Siegel modular threefold, *Math. Ann.* **260** (1982) 317–350.

[12] G. van der Geer, Note on abelian schemes of level three, *Math. Ann.* **278** (1987) 401–408.

[13] J.W. Hoffman and S.H. Weintraub, The Siegel modular variety of degree two and level three, *Trans. Amer. Math. Soc.* **353** (2001), no. 8, 3267–3305.

[14] J.E. Humphreys, Linear Algebraic Groups, Grad. Texts in Math. 21, Springer-Verlag, New York-Heidelberg, 1975.

[15] D. Jiang and D. Soudry, The multiplicity-one theorem for generic automorphic forms of $GSp(4)$, *Pacific J. Math.* **229** (2007), no. 2, 381388.

[16] A.J. de Jong, The moduli space of principally polarized abelian varieties with $\Gamma_0(p)$-level structure, *J. Algebraic Geom.* **2** (1993), no. 4, 667–688.

[17] R. Lee and S.H. Weintraub, Cohomology of a Siegel modular variety of degree 2, in: R. Schultz (ed.), Groups Acting on Manifolds (Boulder Colo. 1983), pp. 433-488, Contemp. Math. 36, Amer. Math. Soc. Providence, RI, 1985.

[18] R. Lee and S.H. Weintraub, Cohomology of $Sp_4(\mathbb{Z})$ and related groups and spaces, *Topology* **24** (1985) 391–410.

[19] R. Lee and S.H. Weintraub, Topology of the Siegel spaces of degree two and their compactifications, *Topology Proc.* **11** (1986) 115–175.

[20] R. Lee and S.H. Weintraub, The Siegel modular variety of degree two and level four: a report, in: W.- P. Barth and H. Lange (eds.), Arithmetic of Complex Manifolds, Lecture Notes in Math. 1399, Springer Verlag, 1989.

[21] R. Lee and S.H. Weintraub, The Siegel modular variety of degree two and level four, *Mem. Amer. Math. Soc.* **133** (1998), no. 631, 58 pages.

[22] T. Okazaki and T. Yamauchi, On some Siegel threefold related to the tangent cone of the Fermat quartic surface, arXiv:1310.1662 [math.NT].

[23] A. Rastegar, On a theorem of Ihara, *Sci. Iran,* **12** (2005), no. 1, 1–9.

[24] B. Roberts and R. Schmidt, Local Newforms For $GSp(4)$, Lecture Notes in Math. 1918, Springer, Berlin, 2007.

[25] R. Schmidt, Packet structure and paramodular forms, *Trans. Amer. Math. Soc.* to appear.

(Arash Rastegar)
E-mail address: `rastegar1352@gmail.com`