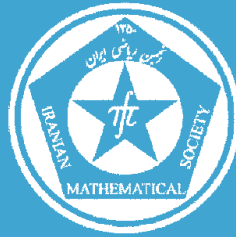


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**Spatial statistics for lattice points on the sphere I: Individual results**

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## SPATIAL STATISTICS FOR LATTICE POINTS ON THE SPHERE I: INDIVIDUAL RESULTS

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*Dedicated to Freydoon Shahidi on the occasion of his 70-th birthday*

**ABSTRACT.** We study the spatial distribution of point sets on the sphere obtained from the representation of a large integer as a sum of three integer squares. We examine several statistics of these point sets, such as the electrostatic potential, Ripley's function, the variance of the number of points in random spherical caps, and the covering radius. Some of the results are conditional on the Generalized Riemann Hypothesis.

**Keywords:** Sums of three squares, spatial statistics, Ripley's functions.

**MSC(2010):** Primary: 11K36; Secondary: 11E12, 11E25.

### 1. Introduction

The goal of this paper is to study the spatial distribution of point sets on the sphere  $S^2$  obtained from the representation of a large integer as a sum of three squares. Some of the results were announced in [4].

Let  $\mathcal{E}(n)$  be the set of integer solutions of the equation  $x_1^2 + x_2^2 + x_3^2 = n$ :

$$\mathcal{E}(n) = \{\mathbf{x} \in \mathbb{Z}^3 : |\mathbf{x}|^2 = n\}.$$

This set might be empty; a necessary and sufficient condition for  $\mathcal{E}(n) \neq \emptyset$  that is for  $n$  to be a sum of three squares, is that  $n \neq 4^a(8b-1)$ . We denote by

$$N = N_n := \#\mathcal{E}(n).$$

It is known that  $N_n \ll n^{1/2+o(1)}$  and if there are *primitive* lattice points, that is  $\mathbf{x} = (x_1, x_2, x_3)$  with  $\gcd(x_1, x_2, x_3) = 1$  (which happens if and only if  $n \not\equiv 0, 4, 7 \pmod{8}$ ) then there is a lower bound of  $N_n \gg n^{1/2-o(1)}$ . For more details concerning  $N_n$  see Section 2.

Once there are many points in  $\mathcal{E}(n)$ , one can ask how they distribute on the sphere. Linnik conjectured, and proved assuming the Generalized Riemann

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Hypothesis (GRH), that for  $n \neq 0, 4, 7 \pmod 8$ , the projected lattice points

$$\widehat{\mathcal{E}}(n) := \frac{1}{\sqrt{n}}\mathcal{E}(n) \subset S^2$$

become uniformly distributed on the unit sphere  $S^2$  as  $n \rightarrow \infty$  along this sequence. That is, for a nice subset  $\Omega \subset S^2$  let

$$Z(n; \Omega) := \#(\widehat{\mathcal{E}}(n) \cap \Omega).$$

Then as  $n \rightarrow \infty$  along this sequence

$$(1.1) \quad \frac{1}{N_n}Z(n, \Omega) \sim \sigma(\Omega),$$

where  $\sigma$  is the normalized area measure on  $S^2$  ( $\sigma(S^2) = 1$ ). This was proved unconditionally by Duke [6, 7] and Golubeva and Fomenko [10].

We will consider various statistics of the point sets  $\widehat{\mathcal{E}}(n) \subset S^2$ , with the aim of comparing these statistics to those of random points, that is  $N$  points chosen independently and uniformly, and contrast them with those of “rigid” point sets, by which we mean points on a planar lattice, such as the honeycomb lattice. See [4] for a detailed discussion and proofs of the statements below concerning random points.

**1.1. Electrostatic energy.** The electrostatic energy of  $N$  points  $P_1, \dots, P_N$  on  $S^2$  is given by

$$E(P_1, \dots, P_N) := \sum_{i \neq j} \frac{1}{|P_i - P_j|}.$$

This energy  $E$  depends on both the global distribution of the points as well as a moderate penalty for putting the points too close to each other. The configurations with minimal energy are rigid in various senses [5] and we will see below in Corollary 1.6 that our points  $\widehat{\mathcal{E}}(n)$  are far from being rigid.

More generally, the Riesz  $s$ -energy is defined as

$$E_s(P_1, \dots, P_N) := \sum_{i \neq j} \frac{1}{|P_i - P_j|^s}.$$

The minimum energy configuration is known to satisfy [30, 31]

$$I(s)N^2 - \beta N^{1-\frac{s}{2}} \leq \min_{P_1, \dots, P_N} E_s(P_1, \dots, P_N) \leq I(s)N^2 - \alpha N^{1-\frac{s}{2}},$$

when  $0 < s < 2$ , for some  $0 < \alpha \leq \beta < \infty$  (depending on  $s$ ), where

$$I(s) = \iint_{S^2 \times S^2} \frac{1}{\|x - y\|^s} d\sigma(x) d\sigma(y) = \frac{2^{1-s}}{2-s}.$$

We will show that for  $0 < s < 2$ ,  $\widehat{\mathcal{E}}(n)$  give points with asymptotically optimal  $s$ -energy:

**Theorem 1.1.** Fix  $0 < s < 2$ . Suppose  $n \rightarrow \infty$  such that  $n \not\equiv 0, 4, 7 \pmod{8}$ . Then there is some  $\delta > 0$  so that

$$E_s(\widehat{\mathcal{E}}(n)) = I(s)N^2 + O(N^{2-\delta}).$$

For a recent application of this result, see [25].

**1.2. Point pair statistics: Ripley’s function.** The point pair statistic and its variants is at the heart of our investigation. It is a robust statistic as for as testing the randomness hypothesis and it is called Ripley’s function in the statistics literature [27]. For  $P_1, \dots, P_N \in S^2$  and  $0 < r < 2$ , set

$$K_r(P_1, \dots, P_N) := \sum_{\substack{i \neq j \\ |P_i - P_j| < r}} 1$$

to be the number of ordered pairs of distinct points at (Euclidean) distance at most  $r$  apart. For fixed  $\epsilon > 0$ , uniformly for  $N^{-1+\epsilon} \leq r \leq 2$ , one has that for  $N$  random points (the binomial process)

$$K_r(P_1, \dots, P_N) \sim \frac{1}{4}N(N-1)r^2.$$

Based on the results below as well as some numerical experimentation, we conjecture that for  $n$  square-free the points  $\widehat{\mathcal{E}}(n)$  behave randomly with respect to Ripley’s statistic at scales  $N_n^{-1+\epsilon} \leq r \leq 2$ ; that is

*Conjecture 1.2.* For squarefree  $n \not\equiv 7 \pmod{8}$ ,

$$K_r(\widehat{\mathcal{E}}(n)) \sim \frac{N^2 r^2}{4}, \quad \text{as } n \rightarrow \infty.$$

We show that Conjecture 1.2 is true at least in terms of an upper bound which is off only by a multiplicative constant.

**Theorem 1.3.** Assume the Generalized Riemann Hypothesis (GRH). Then for fixed  $\epsilon > 0$  and  $N^{-1+\epsilon} \leq r \leq 2$ ,

$$K_r(\widehat{\mathcal{E}}(n)) \ll_\epsilon N^2 r^2$$

for square-free  $n \not\equiv 7 \pmod{8}$ , where the implied constant depends only on  $\epsilon$ .

*Remark 1.4.* We do not need the full force of GRH here, but rather that there are no “Siegel zeros”.

**1.3. Nearest neighbour statistics.** Our study of Ripley’s function allows us to investigate the distribution of nearest neighbour distances in  $\mathcal{E}(n)$ : For  $N$  points  $P_1, \dots, P_N \in S^2$  let  $d_j$  denote the distance from  $P_j$  to the remaining points. Since the balls about the  $P_j$ ’s of radius  $d_j/2$  are disjoint, it follows from considerations of area that  $\sum_{j=1}^N d_j^2 \leq 4^2$ . Hence the mean value of the  $d_j$ ’s is at most  $4/\sqrt{N}$ . For rigid configurations as well as the ones that minimize the electrostatic energy, each one of the  $d_j$ ’s is of this size [5]. This is not true for

random points, however it is still true for these that almost all the points are of order  $N^{-1/2}$  apart.

It is more convenient to work with the squares of these distances. In order to space these numbers at a scale for which they have a limiting distribution in the random case (see [4]), we rescale them by their mean for the random case, i.e. replace  $d_j^2$  by  $\frac{N}{4}d_j^2$ . Thus for  $P_1, \dots, P_N \in S^2$  define the nearest neighbour spacing measure  $\mu(P_1, \dots, P_N)$  on  $[0, \infty)$  by

$$\mu(P_1, \dots, P_N) := \frac{1}{N} \sum_{j=1}^N \delta_{\frac{N}{4}d_j^2},$$

where  $\delta_\xi$  is a delta mass at  $\xi \in \mathbb{R}$ . Note that the mean of  $\mu$  is at most 1 and that for random points we have

$$\mu(P_1, \dots, P_N) \rightarrow e^{-x} dx, \quad \text{as } N \rightarrow \infty.$$

Based on this and on numerical experiments we conjecture:

*Conjecture 1.5.* As  $n \rightarrow \infty$  along square-free integers,  $n \not\equiv 7 \pmod{8}$ ,

$$\mu(\widehat{\mathcal{E}}(n)) \rightarrow e^{-x} dx.$$

Using Theorem 1.3 and its proof we deduce the following basic result about the nearest neighbour measures  $\mu(\widehat{\mathcal{E}}(n))$ :

**Corollary 1.6.** *Assume GRH. If  $\nu$  is a weak limit of the  $\mu(\widehat{\mathcal{E}}(n))$ ,  $n \not\equiv 7 \pmod{8}$  squarefree, then  $\nu$  is absolutely continuous, in fact there is an absolute constant  $c_4 > 0$  such that*

$$\nu \leq c_4 dx.$$

Corollary 1.6 implies that the  $\widehat{\mathcal{E}}(n)$ 's are not rigid for large  $n$  since for rigid configurations,  $\mu_{P_1, \dots, P_N} \rightarrow \delta_{\pi/\sqrt{12}}$ . Moreover, since Corollary 1.6 implies that such a weak limit  $\nu$  cannot charge  $\{0\}$  positively, it follows that almost all the points of  $\widehat{\mathcal{E}}(n)$  are essentially separated with balls of radius approximately  $N^{-1/2}$  from the rest.

**1.4. The number variance in shrinking sets.** We consider families of sets  $\Omega_n$  which shrink as  $n \rightarrow \infty$ , say spherical caps  $\text{Cap}(\xi, r_n) = \{x \in S^2 : \text{dist}(x, \xi) \leq r_n\}$  of radius  $r_n$ , or more generally annuli  $A_{r_n, R_n}(\xi) = \{x \in S^2 : r_n \leq \text{dist}(x, \xi) \leq R_n\}$ . Uniform distribution (1.1) remains true if the sets are allowed to shrink with  $n$  provided  $\text{area}(\Omega_n) \gg n^{-\alpha}$  for some small  $\alpha > 0$ , but one expects this to be true as long as the expected number  $N_n \cdot \text{area}(\Omega_n) \gg n^\epsilon$ . This conjecture (stated by Linnik [18, Chapter XI]) has some profound implications. For instance, applied to annuli centered at the north pole, it implies another conjecture of Linnik, that every integer  $n$  ( $n \not\equiv 0, 4, 7 \pmod{8}$ ) can be written as a sum of two square and a mini-square:  $n = x^2 + y^2 + z^2$ , with

$z = O(n^\epsilon)$  for all  $\epsilon > 0$ . It also implies an old conjecture about the gaps between sums of two squares, see Section 6.

We can ask for a version for “random” sets, meaning we fix a nice set  $\Omega_n \subset S^2$  and investigate the statistics of the number of points  $Z(n; g\Omega_n)$  where  $g \in \text{SO}(3)$  is a random rotation. Examples of such sets would be spherical caps  $\text{Cap}(\xi, r_n)$ , or annuli  $A_{r_n, R_n}(\xi)$ , when the center  $\xi$  is chosen uniformly on  $S^2$  (which is equivalent to choosing a random rotation).

The mean value is tautologically equal to the total number of lattice points times the area, that is

$$\int_{\text{SO}(3)} Z(n; g\Omega_n) dg = N_n \sigma(\Omega_n).$$

where  $dg$  is the Haar probability measure on  $\text{SO}(3)$ .

We turn to study the variance. Note that for “random” points, the variance of the number of points is the expected number of points, so one expects that

*Conjecture 1.7.* Let  $\Omega_n$  be a sequence of spherical caps, or annuli. If  $N_n^{-1+\epsilon} \ll \sigma(\Omega_n) \ll N_n^{-\epsilon}$  as  $n \rightarrow \infty$ ,  $n \neq 0, 4, 7 \pmod 8$ , then

$$(1.2) \quad \int_{\text{SO}(3)} \left| Z(n; g\Omega_n) - N_n \sigma(\Omega_n) \right|^2 dg \sim N_n \sigma(\Omega_n).$$

**Theorem 1.8.** *Let  $\Omega_n$  be a sequence of spherical caps, or annuli. Assume the Lindelöf Hypothesis for standard  $GL(2)/\mathbb{Q}$  L-functions. Then for squarefree  $n \neq 7 \pmod 8$ , we have*

$$(1.3) \quad \int_{\text{SO}(3)} \left| Z(n; g\Omega_n) - N_n \sigma(\Omega_n) \right|^2 dg \ll_\epsilon n^\epsilon N_n \sigma(\Omega_n), \quad \forall \epsilon > 0.$$

**1.5. The covering radius.** Given  $P_1, \dots, P_N \in S^2$ , the covering radius  $M(P_1, \dots, P_N)$  is the least  $r > 0$  so that every point of  $S^2$  is within distance at most  $r$  of some  $P_j$ . An area covering argument shows that for any configuration

$$M(P_1, \dots, P_N) \geq \frac{4}{\sqrt{N}}.$$

For random points,  $M \leq N^{-1/2+o(1)}$ . An effective version of the equidistribution of  $\widehat{\mathcal{E}}(n)$  [10, 7] yields some  $\alpha > 0$  such that  $M(\widehat{\mathcal{E}}(n)) \ll N_n^{-\alpha}$ . Linnik’s conjecture in particular gives

*Conjecture 1.9.*  $M(\widehat{\mathcal{E}}(n)) = N_n^{-1/2+o(1)}$  as  $n \rightarrow \infty$ .

We will show (Section 5.2) that (1.3) implies a quantitative upper bound on the covering radius towards Conjecture 1.9:

**Corollary 1.10.** *For  $n \neq 0, 4, 7 \pmod 8$ , if (1.3) holds then*

$$M(\widehat{\mathcal{E}}(n)) \ll N_n^{-1/4-o(1)}.$$

Under the same assumptions, Theorem 1.8 implies that for a sequence of spherical caps  $\text{Cap}(x, r_n)$ , of area  $A_n$ ,

$$(1.4) \quad \sigma \left\{ x \in S^2 : \widehat{\mathcal{E}}(n) \cap \text{Cap}(x; r_n) = \emptyset \right\} \ll_{\epsilon} \frac{n^{\epsilon}}{N_n A_n}, \quad \forall \epsilon > 0.$$

Thus almost all caps with area  $\gg N_n^{-1+o(1)}$  contain points from  $\widehat{\mathcal{E}}(n)$ . Put another way, the almost all covering exponent

$$(1.5) \quad -\sup \left( \delta : \overline{\lim}_{n \rightarrow \infty} \sigma \left\{ x \in S^2 : \widehat{\mathcal{E}}(n) \cap \text{Cap}(x, N^{-\delta}) \neq \emptyset \right\} = 1 \right)$$

is equal to  $-1/2$  (which is optimally small).

The ergodic method developed by Linnik [18] that was mentioned in the first paragraph allowed him to prove (1.1) for  $n$ 's in special arithmetic progressions, such as those  $n$ 's for which a fixed auxiliary prime  $p$  splits in  $\mathbb{Q}(\sqrt{-n})$ . In [9], Ellenberg, Michel and Venakatesh outline an argument combining Linnik's method with the spectral gap property for an associated Hecke operator  $T_p$  on  $L^2(S^2)$  [20], to show that for  $n$ 's restricted to such a sequence, the almost all covering exponent is equal to  $-1/2$  (they carry the details of the argument for the congruence analogue of the problem in [8, 9]).

In the sequel to this paper we examine Conjectures 1.2 and 1.7 for all  $n$ 's. In particular we establish (1.3) for  $n$ 's of the form  $n = dm^2$  with  $d$  fixed and squarefree, while if  $d$  is varying and  $m \gg n^{\epsilon}$  ( $\epsilon > 0$  arbitrary) then the almost all covering radius is shown to be  $-1/2$ . The main result in part II will be the proof of Conjectures 1.2 and 1.7 for almost all  $n$ .

## 2. Arithmetic background

2.1. **The number of lattice points  $N_n$ .** We first recall what is known about the number of lattice points  $N_n = \#\mathcal{E}(n)$ , that is the number of representations of  $n$  as a sum of three squares. Gauss' formula expresses  $N_n$  in terms of class numbers. For  $n$  square-free,  $n > 3$ , it says that

$$N_n = \begin{cases} 12h(d_n), & n = 1, 2, 5, 6 \pmod{8} \\ 24h(d_n), & n = 3 \pmod{8}, \end{cases}$$

where if  $n$  is square-free,  $d_n$  is the discriminant of the imaginary quadratic field  $\mathbb{Q}(\sqrt{-n})$ , that is  $d_n = -4n$  if  $-n = 2, 3 \pmod{4}$  and  $d_n = -n$  if  $-n = 1 \pmod{4}$ , and  $h(d_n)$  is the class number of  $\mathbb{Q}(\sqrt{-n})$ .

Using Dirichlet's class number formula, one may then express  $N_n$  by means of the special value  $L(1, \chi_{-n})$  of the associated quadratic L-function, where  $\chi_{-n}$  is the corresponding quadratic character

$$\chi_{-n}(m) = \left( \frac{d_n}{m} \right)$$

defined in terms of the Kronecker symbol. It is a Dirichlet character modulo  $|d_n|$ . The resulting formula, for  $n \not\equiv 7 \pmod 8$  square-free, is

$$(2.1) \quad N_n = \frac{24}{\pi} \sqrt{n} L(1, \chi_{d_n}).$$

For any  $n$  we have an upper bound on the number of such points of

$$N_n \ll n^{1/2+\epsilon}$$

for all  $\epsilon > 0$ .

In order that there be *primitive* lattice points (that is  $\mathbf{x} = (x_1, x_2, x_3)$  with  $\gcd(x_1, x_2, x_3) = 1$ ) it is necessary and sufficient that  $n = b^2 m$  with  $b$  odd and  $m \not\equiv 0, 4, 7 \pmod 8$  square-free, equivalently that  $n \not\equiv 0, 4, 7 \pmod 8$ . If there are primitive lattice points then by Siegel's theorem we get a lower bound

$$N_n \gg n^{1/2-\epsilon}.$$

**2.2. The arithmetic function  $A(n, t)$ .** Let  $A(n, t)$  be the number of (ordered) pairs  $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}(n) \times \mathcal{E}(n)$  with inner product  $\mathbf{x} \cdot \mathbf{y} = t$ , equivalently  $|\mathbf{x} - \mathbf{y}|^2 = 2(n - t)$ :

$$A(n, t) = \#\{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |\mathbf{x}|^2 = |\mathbf{y}|^2 = n, \mathbf{x} \cdot \mathbf{y} = t\},$$

which is the number of representations of the binary form  $nu^2 + 2tuv + nv^2$  as a sum of three squares:

$$(2.2) \quad \sum_{j=1}^3 (x_j u + y_j v)^2 = nu^2 + 2tuv + nv^2.$$

The arithmetic function  $A(n, t)$  was studied by Venkov [28] [29, Chaper 4.16], Pall [23, 24] and others, who gave an exact formula for it as a product of local densities. The formulas in [24, Theorem 4] imply that

$$A(n, t) = 24\alpha_2(n, t) \prod_{\substack{p|n^2-t^2 \\ p \neq 2}} \alpha_p(n, t),$$

the product over odd primes dividing the discriminant  $n^2 - t^2$ , where the factors  $\alpha_p(n, t)$  are given as follows:

The 2-adic density  $\alpha_2(n, t)$  equals either one or zero (we will not need to specify when either happens).

To specify  $\alpha_p(n, t)$  for odd primes  $p$ , we need some notations: For a prime  $p$  and an integer  $m$  we denote by  $\text{ord}_p(m)$  the largest integer  $k$  so that  $p^k \mid m$  (when  $t = 0$  we use the convention  $\text{ord}_p(0) = \infty$ ). If  $p$  is an odd prime then  $\left(\frac{m}{p}\right)$  is the Legendre symbol.

Assume now that  $p$  is odd. Then the quadratic form (2.2) is equivalent over the  $p$ -adic integers  $\mathbb{Z}_p$  to a diagonal one

$$\epsilon_1 p^{a_1} u^2 + \epsilon_2 p^{a_2} v^2$$



with  $\epsilon_i$  being  $p$ -adic units and  $0 \leq a_1 \leq a_2$  are given by

$$a_1 = \min(\text{ord}_p(n), \text{ord}_p(t)) = \text{ord}_p(\gcd(n, t))$$

$$a_1 + a_2 = \text{ord}_p(n^2 - t^2)$$

Then

- If  $a_1, a_2 = 1 \pmod 2$ , then

$$(2.3) \quad \alpha_p(n, t) = p^{\frac{a_1-1}{2}} \frac{1 - \frac{1}{p^{(a_1+1)/2}}}{1 - \frac{1}{p}} \left(1 + \left(\frac{-\epsilon_1 \epsilon_2}{p}\right)\right).$$

- If  $a_1 = 1 \pmod 2, a_2 = 0 \pmod 2$ , then

$$(2.4) \quad \alpha_p(n, t) = p^{\frac{a_1-1}{2}} \frac{1 - \frac{1}{p^{(a_1+1)/2}}}{1 - \frac{1}{p}} \left(1 + \left(\frac{-\epsilon_2}{p}\right)\right).$$

- If  $a_1 = 0 \pmod 2, a_2 = 1 \pmod 2$ , then

$$(2.5) \quad \alpha_p(n, t) = p^{(a_1-2)/2} \frac{1 - \frac{1}{p^{a_1/2}}}{1 - \frac{1}{p}} \left(1 + \left(\frac{-\epsilon_1}{p}\right)\right) + p^{a_1/2} \sum_{k=0}^{a_2-a_1} \left(\frac{-\epsilon_1}{p}\right)^k.$$

- If  $a_1, a_2 = 0 \pmod 2$ , then

$$(2.6) \quad \alpha_p(n, t) = 2p^{(a_1-2)/2} \frac{1 - \frac{1}{p^{a_1/2}}}{1 - \frac{1}{p}} + p^{a_1/2} \sum_{k=0}^{a_2-a_1} \left(\frac{-\epsilon_1}{p}\right)^k.$$

In particular, if  $p \nmid 2n$  then  $a_1 = 0$  and  $\epsilon_1 = n$ , so that  $\left(\frac{-\epsilon_1 \epsilon_2}{p}\right) = \chi_{-n}(p)$  and

$$\alpha_p(n, t) = \sum_{j=0}^{\text{ord}_p(n^2-t^2)} \chi_{-n}(p^j).$$

Moreover, if  $n$  is square-free and  $p \mid n$  is odd then the above formulas show that if  $p \nmid t$  (which is equivalent to  $p \nmid n^2 - t^2$  in that case) then  $\alpha_p(n, t) = 1$ , while if  $p \mid \gcd(n, t)$ , so that  $a_1 = 1$  and  $p^2 \mid n^2 - t^2$ , then  $\alpha_p(n, t) \leq 2$ .

We use (2.3), (2.4), (2.5), (2.6) to bound  $A(n, t)$  by the value of a multiplicative function at  $n^2 - t^2$ : First assume that  $n$  is squarefree. Let  $f_n$  be the multiplicative function whose values on prime powers are:  $f_n(2^k) = 1$ , while for  $p \neq 2$ ,

$$(2.7) \quad f_n(p^k) = \begin{cases} \sum_{j=0}^k \chi_{-n}(p^j), & p \nmid n \\ 1, & p \mid n \text{ and } k = 1 \\ 2, & p \mid n \text{ and } k \geq 2. \end{cases}$$

Then the above computations yield that if  $n$  is square-free and  $p$  is odd, then  $\alpha_p(n, t) \leq f_n(p^k), k = \text{ord}_p(n^2 - t^2)$ , hence:

**Lemma 2.1.** *If  $n$  is square-free, and  $|t| < n$ , then*

$$A(n, t) \leq 24f_n(n^2 - t^2).$$

More generally, for  $n$  which are not square-free, we have

**Lemma 2.2.** *Let*

$$m = \prod_{\text{ord}_p(\text{gcd}(n,t)) \geq 2} p^{\text{ord}_p(\text{gcd}(n,t))}$$

and write  $n = mn_1$ ,  $t = mt_1$ . Let  $f_{m,n}$  be the multiplicative function defined by  $f_{n,m}(2^k) = 1$ , and for  $p$  odd

$$(2.8) \quad f_{m,n}(p^k) = \begin{cases} k + 1, & p \mid m \\ \sum_{j=0}^k \left(\frac{-n}{p}\right)^j, & p \nmid n \\ 1, & p \nmid m, p \mid n, k = 1 \\ 2, & p \nmid m, p \nmid n, k \geq 2. \end{cases}$$

Then for all  $\epsilon > 0$ ,

$$A(n, t) \ll m^{\frac{1}{2}} \tau(m) f_{n,m}(n_1^2 - m_1^2),$$

where  $\tau(m)$  is the divisor function.

**2.3. Linnik’s fundamental lemma.** We will need an upper bound for  $A(n, t)$  valid for general  $n$ :

**Proposition 2.3.** *If  $|t| < n$ , then*

$$A(n, t) \ll \text{gcd}(n, t)^{1/2} n^\epsilon, \quad \forall \epsilon > 0.$$

This kind of bound, a consequence of Lemma 2.2, was stated and used by Linnik [17], who omitted the factor of  $\text{gcd}(n, t)^{1/2}$ . A correct version was given by Pall [23, Section 7], [24, Theorem 4], see also [8, Section 4] for a discussion of the case when  $n$  is square-free.

Proposition 2.3 allows us to deduce a mean equidistribution statement for regions which on average contain *one* lattice point. To do so, divide the sphere  $\sqrt{n}S^2$  into boxes  $\{A_j\}$  of size  $\approx n^{1/4}$  (so there are about  $n^{1/2}$  such boxes); so one expects that there should be at most  $n^\epsilon$  lattice points in each such box. We show that this expectation is met in the mean square, that is

**Theorem 2.4.**

$$\sum_j \left( \#A_j \cap \mathcal{E}(n) \right)^2 \ll n^{1/2+\epsilon}, \quad \forall \epsilon > 0.$$

*Proof.* Theorem 2.4 is an immediate consequence of Proposition 2.3, since

$$\begin{aligned} \sum_j \left( \#A_j \cap \mathcal{E}(n) \right)^2 &\ll \#\{x, y \in \mathcal{E}(n) : |x - y| \ll n^{1/4}\} \\ &\ll \sum_{n-n^{1/2} \leq t \leq n} A(n, t). \end{aligned}$$

Applying Proposition 2.3 now gives

$$\sum_j \left( \#A_j \cap \mathcal{E}(n) \right)^2 \ll n^\epsilon \sum_{n-n^{1/2} \leq t \leq n} \gcd(n, t)^{1/2}.$$

Thus it suffices to show that the mean value of  $\gcd(n, t)$  over the interval  $I = [n - \sqrt{n}, n]$  is at most  $n^\epsilon$ . Writing

$$\gcd(n, t) = \sum_{d|n, d|t} 1,$$

and switching order of summation gives

$$\begin{aligned} \sum_{t \in I} \gcd(n, t) &= \sum_{d|n} \#\{t \in I : d | t\} \leq \sum_{d|n} \frac{|I|}{d} + O(1) \\ &\ll |I| \log n + O(n^\epsilon) \ll \sqrt{n} \log n, \end{aligned}$$

proving Theorem 2.4.  $\square$

### 3. Electrostatic energy

In this section, we show that  $\widehat{\mathcal{E}}(n)$  give points with asymptotically optimal  $s$ -energy:

$$E_s(P_1, \dots, P_N) := \sum_{i \neq j} \frac{1}{|P_i - P_j|^s}.$$

In what follows we take  $0 < s < 2$ .

**Theorem 3.1.** *Fix  $0 < s < 2$ . Suppose  $n \rightarrow \infty$  such that  $n \not\equiv 0, 4, 7 \pmod{8}$ . Then there is some  $\delta > 0$  so that*

$$E_s(\widehat{\mathcal{E}}(n)) = I(s)N_n^2 + O(N_n^{2-\delta}),$$

where

$$I(s) = \iint_{S^2 \times S^2} \frac{1}{\|x - y\|^s} d\sigma(x) d\sigma(y) = \frac{2^{1-s}}{2-s}.$$

3.1. **A division into close and distant pairs.** We denote by

$$\mathbf{x} \mapsto \widehat{x} = \frac{\mathbf{x}}{\sqrt{n}}$$

the projection from the sphere  $|\mathbf{x}|^2 = n$  to the unit sphere  $S^2$ . We fix a small  $\rho > 0$  and divide the pairs of points in  $\mathcal{E}(n) \times \mathcal{E}(n)$  into close pairs and distant pairs, depending on whether  $\|\widehat{x} - \widehat{y}\| < n^{-\rho}$  or not. The projected points  $\widehat{\mathcal{E}}(n)$  are well separated:  $\|\widehat{x} - \widehat{y}\| \geq n^{-1/2}$ , hence we may take  $\rho \leq 1/2$ . We treat the contribution of close pairs by using the upper bound of Proposition 2.3 for the number  $A(n, t)$  of pairs  $\mathbf{x}, \mathbf{y} \in \mathcal{E}(n)$  with inner product  $\langle \mathbf{x}, \mathbf{y} \rangle = t$ , and that of the distant pairs by using a quantitative form of the equidistribution of the sets  $\widehat{\mathcal{E}}(n)$  on the sphere.

3.1.1. *The contribution of nearby points.*

**Lemma 3.2.** *The contribution of nearby pairs is bounded by*

$$\sum_{\substack{\mathbf{x} \neq \mathbf{y} \in \mathcal{E}(n) \\ \|\widehat{x} - \widehat{y}\| < n^{-\rho}}} \frac{1}{\|\widehat{x} - \widehat{y}\|^s} \ll n^{1-\rho(2-s)+\epsilon}.$$

*Proof.* The squares of the distances between points in  $\mathcal{E}(n)$  are of the form  $\|\widehat{x} - \widehat{y}\|^2 = 2h/n$  for some integer  $h$ , since

$$\|\widehat{x} - \widehat{y}\|^2 = \frac{\|\mathbf{x} - \mathbf{y}\|^2}{n} = \frac{2n - 2\langle \mathbf{x}, \mathbf{y} \rangle}{n},$$

and  $\|\mathbf{x} - \mathbf{y}\|^2 = 2h$  is equivalent to  $\langle \mathbf{x}, \mathbf{y} \rangle = n - h$ . Hence the number of pairs of points  $\mathbf{x}, \mathbf{y} \in \mathcal{E}(n)$  at distance  $\|\mathbf{x} - \mathbf{y}\|^2 = 2h$  is  $A(n, n - h)$ , that is

$$\#\{\mathbf{x}, \mathbf{y} \in \mathcal{E}(n) : \|\widehat{x} - \widehat{y}\| = \sqrt{\frac{2h}{n}}\} = A(n, n - h).$$

Therefore the contribution of close pairs to the sum  $E_s$ , that is pairs of points with  $\|\widehat{x} - \widehat{y}\| < n^{-\rho}$ , is:

$$\sum_{\substack{\mathbf{x} \neq \mathbf{y} \in \mathcal{E}(n) \\ \|\widehat{x} - \widehat{y}\| < n^{-\rho}}} \frac{1}{\|\widehat{x} - \widehat{y}\|^s} = n^{s/2} \sum_{1 \leq h \leq \frac{1}{2}n^{1-2\rho}} \frac{A(n, n - h)}{(2h)^{s/2}}.$$

According to Proposition 2.3,

$$A(n, n - h) \ll n^\epsilon \gcd(n, n - h)^{1/2} = n^\epsilon \gcd(n, h)^{1/2}.$$

Hence the contribution of close pairs is bounded by

$$\sum_{\substack{\mathbf{x} \neq \mathbf{y} \in \mathcal{E}(n) \\ \|\widehat{x} - \widehat{y}\| < n^{-\rho}}} \frac{1}{\|\widehat{x} - \widehat{y}\|^s} \ll n^{s/2+\epsilon} \sum_{1 \leq h \leq n^{1-2\rho}} \frac{\gcd(n, h)}{(2h)^{s/2}} \ll n^{1-\rho(2-s)+\epsilon},$$

as claimed. □

As a consequence, we may replace the potential  $\|\widehat{x} - \widehat{y}\|^{-s}$  by its truncated form

$$F_n(\widehat{x}, \widehat{y}) = \min\left(\frac{1}{\|\widehat{x} - \widehat{y}\|^s}, \frac{1}{n^{s\rho}}\right)$$

to get

$$(3.1) \quad E_s(\mathcal{E}(n)) = \sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{E}(n)} F_n(\widehat{x}, \widehat{y}) + O(n^{1-\rho(2-s)+\epsilon}),$$

where the remainder term is negligible relative to the main term  $N_n^2 I(s)$  since  $N_n^2 \gg n^{1-\epsilon}$  by Siegel's theorem.

3.1.2. *Distant pairs.* For a fixed  $\mathbf{x}_0 \in \mathcal{E}(n)$ , consider the  $s$ -energy sum

$$S(\mathbf{x}_0) := \frac{1}{N_n} \sum_{\substack{\mathbf{x} \in \mathcal{E}(n) \\ \mathbf{x} \neq \mathbf{x}_0}} F_n(\widehat{x}, \widehat{x}_0) = \frac{1}{N_n} \sum_{\substack{\mathbf{x} \in \mathcal{E}(n) \\ \mathbf{x} \neq \mathbf{x}_0}} \min(n^{s\rho}, \frac{1}{\|\widehat{x} - \widehat{x}_0\|^s}),$$

where  $N_n = \#\mathcal{E}(n)$ .

**Proposition 3.3.** *For  $0 < s < 2$ , there is some  $\eta > 0$  so that as  $n \rightarrow \infty$ ,  $n \neq 0, 4, 7 \pmod{8}$ ,*

$$S(\mathbf{x}_0) = I(s) + O(n^{-\eta+s\rho} + n^{-\rho(2-s)}),$$

where

$$I(s) = \int_{S^2} \frac{1}{\|x - \widehat{x}_0\|^s} d\sigma(x) = \frac{2^{1-s}}{2-s}.$$

As an immediate consequence of Proposition 3.3 we see, on using  $N_n \gg n^{1/2-\epsilon}$ , that

$$\sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{E}(n)} F_n(\widehat{x}, \widehat{y}) = N_n^2 I(s) + O\left(n^\epsilon (n^{-\eta+s\rho} + n^{-\rho(2-s)})\right).$$

Taking into account (3.1) we get

$$E_s(\mathcal{E}(n)) = I(s)N_n^2 + O\left(n^\epsilon (n^{-\eta+s\rho} + n^{-\rho(2-s)})\right).$$

Taking  $\rho = \eta/2$  we find

$$E_s(\mathcal{E}(n)) = I(s)N_n^2(1 + O(n^{-\eta(1-\frac{s}{2})+\epsilon})),$$

which proves Theorem 3.1. It remains to prove Proposition 3.3.

### 3.2. Using equidistribution.

3.2.1. *Discrepancy on  $\mathbb{R}/\mathbb{Z}$ .* We begin with a short review of discrepancy on the circle, see [16]: For a sequence on the circle  $X \subset \mathbb{R}/\mathbb{Z}$ , we define Weyl sums by

$$W(k, N) := \frac{1}{N} \sum_{n \leq N} e(kx_n).$$

Uniform distribution of  $X$  is equivalent to  $W(k, N) \rightarrow 0$  for all  $k \neq 0$ .

The discrepancy of the sequence is defined as

$$D_N(X) := \sup_I \left| \frac{1}{N} \#\{n \leq N : x_n \in I\} - \text{length}(I) \right|,$$

where the supremum is over all intervals  $I \subset \mathbb{R}/\mathbb{Z}$ . Uniform distribution is equivalent to  $D_N \rightarrow 0$ . A quantitative measure, which also allows to treat shrinking intervals, is given by the Erdős-Turán inequality, one variant being: For all  $M \geq 1$ ,

$$D_N(X) \ll \frac{1}{M+1} + \sum_{k=1}^M \frac{1}{k} |W(k, N)|.$$

We also recall Koksma's inequality on  $\mathbb{R}/\mathbb{Z}$ , which bounds the sampling error in terms of the discrepancy: Let  $X \subset [0, 1]$  be a sequence of points, with discrepancy  $D_N(X)$ . If  $f$  is continuous on  $[0, 1]$  and of bounded variation, with total variation  $V(f)$ , then

$$(3.2) \quad \left| \frac{1}{N} \sum_{n \leq N} f(x_n) - \int_0^1 f(x) dx \right| \ll D_N(X) \cdot V(f).$$

3.2.2. *Spherical coordinates.* Fix a point  $x_0$  on the unit sphere  $S^2 \subset \mathbb{R}^3$ , and define spherical coordinates with  $x_0$  as the North Pole as follows: For a point  $x \in S^2$ , denote by  $\theta \in [0, \pi]$  the angle of inclination, that is the angle between the zenith direction (the ray between the origin and  $x_0$ ) and the ray from the origin to  $x$ , and by  $\phi \in [0, 2\pi)$  the azimuthal angle, which is the angle between a fixed direction in the plane through the origin orthogonal to the zenith direction, and the ray from the origin to the projection of  $x$  on that plane. Thus we have

$$|x - x_0|^2 = 2(1 - \cos \theta).$$

In these coordinates, the normalized area measure on  $S^2$  is  $d\sigma = \frac{1}{4\pi} \sin \theta d\theta d\phi$ .

We say that a function on  $S^2$  is *zonal* if it is invariant under rotation around the line between  $x_0$  and the origin, that is depends only on the angle of inclination  $\theta$ . For any even  $2\pi$ -periodic function  $g(\theta)$  we may define a zonal function on

the sphere  $S^2$  by setting  $G(x) = G(\phi, \theta) = g(\theta)$ . The average of  $G$  over the sphere is related to the average of  $g$  over the interval  $[0, \pi]$  via

$$\int_{S^2} G(x) d\sigma(x) = \frac{1}{2} \int_0^\pi g(\theta) \sin \theta d\theta.$$

3.2.3. *Uniform distribution and discrepancy on the sphere.* Let  $\mathcal{H}_\nu$  be the space of spherical harmonics of degree  $\nu$ . These are eigenfunctions of the Laplace-Beltrami operator on  $S^2$ , with eigenvalue  $\nu(\nu + 1)$ . The dimension of the space is  $\dim \mathcal{H}_\nu = 2\nu + 1$ . The span of all the spherical harmonics is dense in  $L^2(S^2)$ . Hence to prove equidistribution of the sets  $\mathcal{E}(n)$  on the sphere it suffices to show that for all spherical harmonics  $H \in \mathcal{H}_\nu$  of positive degree, the corresponding Weyl sums

$$W(H, n) := \frac{1}{\#\mathcal{E}(n)} \sum_{\mathbf{x} \in \mathcal{E}(n)} H_\nu\left(\frac{\mathbf{x}}{\sqrt{n}}\right)$$

tend to zero.

For a sequence of points  $X \subset S^2$ , the spherical cap discrepancy is defined as

$$D_N(X) := \sup_C \left| \frac{1}{N} \#\{n \leq N : x_n \in C\} - \sigma(C) \right|,$$

where the supremum is over all spherical caps, and  $\sigma$  is the normalized area measure.

A bound for the discrepancy on the sphere, analogous for the Erdős-Turán bound, is given by [11]: For all  $M \geq 1$ ,

$$(3.3) \quad D_N(X) \ll \frac{1}{M+1} + \sum_{\nu=1}^M \frac{1}{\nu} \sum_{j=1}^{\dim \mathcal{H}_\nu} |W(H_{\nu,j}, N)|,$$

where  $H_{\nu,j}$  denotes an orthonormal basis of  $\mathcal{H}_\nu$ .

3.2.4. *Weyl sums on the sphere.* A fundamental bound for Fourier coefficients of half-integer weight forms, due to Iwaniec [13], allows one to prove uniform distribution of the points  $\mathcal{E}(n)$  on the sphere [6, 10]. We will need a quantitative version of that bound given in [10], see also [7]: There are constants  $\gamma > 0$  (small) and  $A > 0$  so that if  $H_\nu \in \mathcal{H}_\nu$  is a spherical harmonic of degree  $\nu > 0$ , then

$$W(H_\nu, n) \ll \frac{n^{1/2}}{N_n} n^{-\gamma} \nu^A \|H_\nu\|_\infty$$

(recall  $N_n := \#\mathcal{E}(n)$ ).

We take  $n$ 's for which there is a primitive point in  $\mathcal{E}(n)$ , equivalently  $n \not\equiv 0, 4, 7 \pmod{8}$ , then  $\sqrt{n}/N_n \ll n^\epsilon, \forall \epsilon > 0$ . Moreover, we replace the  $L^\infty$  norm by the  $L^2$  norm via the inequality

$$\|H\|_\infty \leq \sqrt{\dim(\mathcal{H}_\nu)} \cdot \|H_\nu\|_2, \quad \forall H_\nu \in \mathcal{H}_\nu$$

which gives:

**Lemma 3.4.** *There are  $\delta > 0$ ,  $B > 0$  so that*

$$W(H_\nu, n) \ll n^{-\delta} \nu^B \|H_\nu\|_2$$

for all  $H_\nu \in \mathcal{H}_\nu$ ,  $\nu > 0$ .

Applying the discrepancy bound (3.3) with  $M \simeq n^{\delta/(B+1)}$ , we get that the spherical cap discrepancy  $D(\widehat{\mathcal{E}}(n))$  satisfies

$$D(\widehat{\mathcal{E}}(n)) \ll n^{-\eta}, \quad \eta = \frac{\delta}{B+1}.$$

**3.3. Proof of Proposition 3.3.** Consider the sequence of points in the interval  $[0, 1]$  given by

$$z_j = \frac{\|\widehat{x}_j - \widehat{x}_0\|^2}{4} = \frac{1 - \cos \theta_j}{2} \in [0, 1].$$

The area (with respect to  $\sigma$ ) of the cap  $\|\mathbf{x} - \mathbf{x}_0\| < 2\sqrt{t}$  is  $t$ , which is the length of the interval for the corresponding points  $0 \leq z = \|\mathbf{x} - \mathbf{x}_0\|^2/4 \leq t$ . Hence the discrepancy of the sequence  $z_j$  on the interval  $[0, 1]$  is bounded by the spherical cap discrepancy of the sequence  $\widehat{x}_j$ , which is  $\ll n^{-\eta}$ . Hence by Koksma’s inequality (3.2), for any continuous function  $g$  of bounded variation on  $[0, 1]$  we have

$$\left| \frac{1}{N} \sum g(z_j) - \int_0^1 g(t) dt \right| \ll n^{-\eta} \cdot V(g).$$

Now take

$$g_n(z) = \min\left(\frac{1}{(2z^{1/2})^s}, n^{s\rho}\right),$$

and

$$G_n(\widehat{x}) = g_n\left(\frac{\|\widehat{x} - \widehat{x}_0\|^2}{4}\right) = \min\left(\frac{1}{\|\widehat{x} - \widehat{x}_0\|^s}, n^{s\rho}\right).$$

The total variation of  $g_n$  is

$$V(g_n) \ll \max g_n = n^{s\rho}.$$

Hence we find that

$$\frac{1}{N} \sum_{\substack{\mathbf{x} \in \mathcal{E}(n) \\ \mathbf{x} \neq \mathbf{x}_0}} G_n\left(\frac{\mathbf{x}}{\sqrt{n}}\right) = \int_{S^2} G_n(x) d\sigma(x) + O(n^{-\eta+s\rho}).$$

The mean of  $G_n$  is

$$\int_{S^2} G_n(x) d\sigma(x) = \int_{S^2} \frac{1}{\|x - \widehat{x}_0\|^s} d\sigma(x) + O(n^{-\rho(2-s)}),$$

since the difference between the two integrals is certainly bounded by

$$\int_{|x - \widehat{x}_0| < n^{-\rho}} \frac{1}{\|x - \widehat{x}_0\|^s} d\sigma(x) = \int_0^{n^{-2\rho/4}} \frac{1}{(2\sqrt{z})^s} dz \ll n^{-\rho(2-s)}$$



(recall we assume that  $0 < s < 2$ ).

In conclusion, we find that

$$S(\mathbf{x}_0) = I(s) + O(n^{-\eta+s\rho} + n^{-\rho(2-s)}),$$

proving Proposition 3.3. □

#### 4. Upper bounds on Ripley’s function

4.1. **Nair’s Theorem.** We will need to use a result of M. Nair [22] on mean values of multiplicative functions of polynomial arguments over short intervals. Nair’s theorem, following several prior developments in the subject surveyed in [22], deals with the following situation: Let  $\mathcal{M}$  be the class of multiplicative, non-negative functions  $f$  satisfying

- $f(p^k) \leq A_0^k$ ,
- $f(n) \leq A_1(\epsilon)n^\epsilon$  for all  $\epsilon > 0$ .

We are given an integer polynomial  $P(t) = \sum_{j=0}^g a_j t^j \in \mathbb{Z}[t]$  of degree  $g$ , assumed to have distinct roots, with discriminant  $D$ , and such that  $P(t)$  has no fixed prime divisor. We define the height of  $P$  by  $\|P\| := \max_j |a_j|$ . Let

$$\rho(m) = \#\{x \pmod m : P(x) \equiv 0 \pmod m\},$$

and let

$$\overline{D} = \prod_{\substack{p^a \parallel D \\ \rho(p) \neq 0}} p^a.$$

**Theorem 4.1** (Nair [22]). *Fix  $\alpha, \delta \in (0, 1)$ . Then for  $f \in \mathcal{M}$ ,  $x^\alpha < y < x$ ,  $x \gg \|P\|^\delta$ ,*

$$\sum_{x-y < m < x} f(|P(m)|) \ll_{\alpha, \delta, A_1} c(\overline{D})y \prod_{p \leq x} \left(1 - \frac{\rho(p)}{p}\right) \exp\left(\sum_{p \leq x} \frac{f(p)\rho(p)}{p}\right),$$

where the implied constants depend only on the constant  $A_1$  for the family  $\mathcal{M}$ , on  $\alpha, \delta$  and on the reduced discriminant  $\overline{D}$ .

We want to use the result for the multiplicative functions  $f_n$  of (2.7), the polynomial  $P(t) = n^2 - t^2$ , and  $x = n - 1$ . In this case, we have

$$\rho(m) = \#\{x \pmod m : x^2 \equiv n^2 \pmod m\},$$

hence

$$\rho(p) = \begin{cases} 2 & p \nmid 2n, \\ 1 & p \mid 2n, \end{cases}$$

and moreover,  $\rho(p^k) = 2$  for  $p \nmid 2n$ . In particular

$$\overline{D} = D = -4n^2.$$

Thus the unspecified dependence on  $\overline{D}$  in Nair’s theorem is an issue we need to address.

Examining the proof of Nair's theorem shows that there are only two places where the dependence on  $\bar{D}$  appears:

a) In [22, Lemma 2(iii)], in the estimate

$$\sum_{m \leq t} \frac{F(m)\rho(m)}{m} \ll_{\bar{D}} \exp\left(\sum_{p \leq t} \frac{F(p)\rho(p)}{p}\right),$$

where  $F \in \mathcal{M}$ . The dependence (at the bottom of page 262) is in bounding the sum over higher prime powers

$$\sum_{p \leq t} \sum_{\ell \geq 2} \frac{F(p^\ell)\rho(p^\ell)}{p^\ell} \ll 1.$$

In our case, since  $\rho(p^\ell) \leq 2$  this bound is clearly uniform in  $\bar{D} \approx n^2$ .

b) In the proof of his main theorem, in [22, Equation (6.3) on p. 265], he employs the estimate

$$y \sum_{\substack{z^{1/2} < a \leq z \\ P^+(a) < \log x \log \log x}} \frac{\rho(a)}{a} \leq c(\bar{D})y^{7/8},$$

where  $z = y^{1/2}$  and  $P^+(a)$  denotes the greatest prime factor of  $a$ . In our case, use  $\rho(a) \ll a^\epsilon$  (independent of  $n$ ) to bound the sum by

$$y \sum_{\substack{z^{1/2} < a \leq z \\ P^+(a) < \log x \log \log x}} \frac{\rho(a)}{a} \ll y \frac{z^\epsilon}{z^{1/2}} \Psi(z; \log x \log \log x),$$

where  $\Psi(x, z)$  is the number of  $a < x$  with  $P^+(a) < z$ , which is known to satisfy ([22, Lemma 3])

$$\Psi(x; \log x \log \log x) \ll \exp\left(\frac{3 \log x}{\sqrt{\log \log x}}\right) \ll x^\epsilon.$$

Hence in our case, we certainly have

$$y \sum_{\substack{z^{1/2} < a \leq z \\ P^+(a) < \log x \log \log x}} \frac{\rho(a)}{a} \ll y^{7/8}$$

uniformly in  $n$  (recall  $x^\alpha < y < x$ ).

#### 4.2. Reduction to bounding mean values of multiplicative functions.

For  $0 \leq a < b < n$  we set

$$(4.1) \quad M(n; a, b) = \#\{|\mathbf{x}|^2 = |\mathbf{y}|^2 = n, a < |\mathbf{x} - \mathbf{y}|^2 < b\}$$

so that

$$K_r(\widehat{\mathcal{E}}(n)) = M(n; 0, r^2 n).$$

Recall that we denote by  $\chi_{-n}$  the quadratic character associated to the field  $\mathbb{Q}(\sqrt{-n})$ . We claim:

**Proposition 4.2.** *Fix  $0 < \alpha < 1$ . Assume that  $n$  is square-free,  $n \not\equiv 7 \pmod{8}$ ,  $a < b < n$  and  $n^\alpha < b - a < n$ . Then*

$$(4.2) \quad M(n; a, b) \ll (b - a) \cdot \exp\left(2 \sum_{p < n} \frac{\chi_{-n}(p)}{p}\right).$$

*Proof.* The condition  $a < |\mathbf{x} - \mathbf{y}|^2 < b$  is equivalent to the inner product of  $\mathbf{x}, \mathbf{y}$  satisfying

$$n - \frac{b}{2} < \mathbf{x} \cdot \mathbf{y} < n - \frac{a}{2},$$

and hence

$$M(n; a, b) = \sum_{n - \frac{b}{2} < t < n - \frac{a}{2}} A(n, t),$$

where  $A(n, t)$  is the number of pairs of vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{E}(n)$  with inner product  $\mathbf{x} \cdot \mathbf{y} = t$ , equivalently  $|\mathbf{x} - \mathbf{y}|^2 = 2(n - t)$ .

According to Lemma 2.1, we may bound  $A(n, t)$  by the value of a multiplicative function at  $n^2 - t^2$ :

$$A(n, t) \leq 24f_n(n^2 - t^2),$$

where  $f_n$  is the multiplicative function given by (2.7). Therefore we find that we can bound

$$M(n; a, b) \ll \sum_{n - \frac{b}{2} < t < n - \frac{a}{2}} f_n(n^2 - t^2).$$

This is a sum of a multiplicative function at polynomial values, summed over an interval  $(n - b/2, n - a/2)$ , for which one can give an upper bound using Nair's theorem [22] described in Section 4.1. The conclusion is that

$$M(n; a, b) \ll (b - a) \prod_{p < n - \frac{a}{2}} \left(1 - \frac{2}{p}\right) \exp\left(\sum_{p < n - \frac{a}{2}} \frac{2f_n(p)}{p}\right).$$

Since  $f_n(p) = 1 + \chi_{-n}(p)$  (for all  $p$  with the convention  $\chi_{-n}(p) = 0$  if  $p \mid 2n$ ), we get

$$M(n; a, b) \ll (b - a) \exp\left(2 \sum_{p < n - \frac{a}{2}} \frac{\chi_{-n}(p)}{p}\right).$$

This is (4.2) except that the sum is over primes  $p < n - a/2$  instead of  $p < n$ . To recover (4.2), note that since  $0 \leq a < n$ , we have

$$\left| \sum_{n - \frac{a}{2} < p < n} \frac{\chi_{-n}(p)}{p} \right| \leq \sum_{n/2 < p < n} \frac{1}{p} \ll \frac{1}{\log n}$$

by Mertens' theorem, and hence

$$M(n; a, b) \ll (b - a) \exp\left(2 \sum_{p < n} \frac{\chi_{-n}(p)}{p}\right)$$

as claimed. □

**Corollary 4.3.** *Assume that  $n$  is square-free,  $n \not\equiv 7 \pmod{8}$ . Then assuming the Generalized Riemann Hypothesis (GRH),*

$$K_r(\widehat{\mathcal{E}}(n)) \ll N_n^2 r^2.$$

*Proof.* Taking  $a = 0$  and  $b = r^2 n$  in Proposition 4.2 gives, for  $n^{-1/2+\delta} < r < n$ , that

$$K_r(\widehat{\mathcal{E}}(n)) = M(n; 0, r^2 n) \ll r^2 n \exp\left(2 \sum_{p < n} \frac{\chi_{-n}(p)}{p}\right),$$

Using the Gauss-Dirichlet formula  $N_n = c_n \sqrt{n} L(1, \chi_{d_n})$  of (2.1) gives for  $n$  squarefree

$$K_r(\widehat{\mathcal{E}}(n)) \ll r^2 N_n^2 \left(\frac{\exp\left(\sum_{p < n} \frac{\chi_{-n}(p)}{p}\right)}{L(1, \chi_{-n})}\right)^2.$$

It is a consequence of GRH, that

$$\sum_{p < x} \chi_{-n}(p) \ll x^{1/2} (nx)^\epsilon, \quad \forall \epsilon > 0.$$

This implies

$$\frac{1}{L(1, \chi_{-n})} \exp\left(\sum_{p < n} \frac{\chi_{-n}(p)}{p}\right) = O(1),$$

which gives our claim (in fact what we require is the absence of ‘‘Siegel zeros’’). □

We record the corresponding result when  $n$  is not necessarily squarefree:

**Corollary 4.4.** *Assume that  $n \not\equiv 7 \pmod{8}$ , and  $n^{-1/2+\delta} < r < 1$ . Then assuming the Generalized Riemann Hypothesis,*

$$K_r(\widehat{\mathcal{E}}(n)) \ll \left(\sum_{m|n}^b m^{-\frac{1}{2}+\epsilon}\right) \cdot n L(1, \chi_{-n})^2 r^2$$

for all  $\epsilon > 0$ , the sum  $\sum_m^b$  running over all  $m | n$  which are squarefull, that is such that  $m = \prod_p p^{k_p}$  with all  $k_p \geq 2$ .

*Proof.* We first show that, as in Proposition 4.2, that for  $n^{2\delta} < b - a < n$

$$(4.3) \quad M(n; a, b) \ll (b - a) \left(\sum_{m|n}^b m^{-\frac{1}{2}+\epsilon}\right) L(1, \chi_{-n})^2,$$

which will prove the Lemma, since  $K_r(\widehat{\mathcal{E}}(n)) = M(n; 0, r^2n)$ .

According to Lemma 2.2

$$A(n, t) \ll m^{1/2} \tau(m) f_{m,n}(n_1^2 - t_1^2),$$

where

$$m = \prod_{\text{ord}_p(\text{gcd}(n,t)) \geq 2} p^{\text{ord}_p(\text{gcd}(n,t))}$$

with  $n = mn_1$ ,  $t = mt_1$ , and  $f_{m,n}$  is the multiplicative function (2.8). Therefore, with  $a = ma_1$ ,  $b = mb_1$ ,

$$(4.4) \quad M(n; a, b) = \sum_{n - \frac{b}{2} < t < n - \frac{a}{2}} A(n, t) \ll \sum_m^b m^{1/2} \tau(m) \sum_{n_1 - \frac{b_1}{2} < t_1 < n_1 - \frac{a_1}{2}} f_{m,n}(n_1^2 - t_1^2),$$

the sum  $\sum_m^b$  running over all  $m \mid n$  which are squarefull, that is such that  $m = \prod_p p^{k_p}$  with all  $k_p \geq 2$ .

For  $m, n$  fixed estimate the inner sum using Nair's theorem, noting that  $b_1 - a_1 = \frac{b-a}{m} \in (n^{2\delta}, n)$ , obtaining the bound

$$\begin{aligned} \sum_{n_1 - \frac{b_1}{2} < t_1 < n_1 - \frac{a_1}{2}} f_{m,n}(n_1^2 - t_1^2) &\ll (b_1 - a_1) \prod_{p < n_1 - \frac{a_1}{2}} \left(1 - \frac{2}{p}\right) \exp\left(\sum_{p < n_1 - \frac{a_1}{2}} \frac{2f_{m,n}(p)}{p}\right) \\ &\ll \frac{b-a}{m} \exp\left(2 \sum_{p \mid m} \frac{1}{p} + 2 \sum_{\substack{p < n - \frac{a_1}{2} \\ p \nmid m}} \frac{\chi_{-n}(p)}{p}\right) \\ &\ll \frac{b-a}{m} (\log \log m)^C \exp\left(2 \sum_{p < n - \frac{a_1}{2}} \frac{\chi_{-n}(p)}{p}\right) \\ &\ll \frac{b-a}{m^{1-\epsilon}} L(1, \chi_{-n})^2 \end{aligned}$$

(the last inequality assumes GRH, or the absence of Siegel zeros). Inserting into (4.4) proves (4.3).  $\square$

**4.3. Proof of Corollary 1.6.** We now show that weak limits of the nearest neighbour spacing measures

$$(4.5) \quad \mu(\widehat{\mathcal{E}}(n)) := \frac{1}{N} \sum_{j=1}^N \delta_{\frac{N}{4} d_j^2}$$

are absolutely continuous, in fact that there is some  $c_4 > 0$  so that any weak limit  $\nu$  of (4.5) for  $n \neq 7 \pmod 8$  squarefree, satisfies  $\nu \leq c_4 dx$ . For this we need to show that for any fixed  $0 \leq \alpha < \beta < \infty$ , the proportion of normalized

nearest neighbour spacings  $\frac{N}{4}d_j^2$  which lie in the interval  $[\alpha, \beta]$  satisfies

$$\frac{1}{N} \#\{j \leq N : \alpha \leq \frac{N}{4}d_j^2 < \beta\} \leq c_4(\beta - \alpha).$$

Since the number of normalized nearest neighbour spacings in an interval is bounded by the number of all normalized spacings in that interval, it suffices to show that

$$\frac{1}{N_n} \#\{|\mathbf{x}|^2 = |\mathbf{y}|^2 = n : \alpha < \frac{N_n}{4} \left| \frac{\mathbf{x}}{\sqrt{n}} - \frac{\mathbf{y}}{\sqrt{n}} \right| < \beta\} \leq c_4(\beta - \alpha).$$

The LHS is, in the notation of (4.1), equal to

$$\frac{1}{N_n} M(n; \alpha \frac{4n}{N_n}, \beta \frac{4n}{N_n}) \ll \frac{(\beta - \alpha)n}{N_n^2} \exp\left(2 \sum_{p < n} \frac{\chi_{-n}(p)}{p}\right)$$

by Proposition 4.2. Using (2.1) for  $n \neq 7 \pmod 8$  squarefree, we replace  $n/N_n^2$  by  $1/L(1, \chi_{-n})^2$ , and as in the proof of Corollary 4.3 we use GRH to deduce that  $\exp\left(2 \sum_{p < n} \frac{\chi_{-n}(p)}{p}\right)/L(1, \chi_{-n})^2 = O(1)$ .  $\square$

### 5. The number variance

5.1. **Proof of Theorem 1.8.** For  $0 \leq \rho_1 < \rho_2$  and  $z \in S^2$  let

$$A_{\rho_1, \rho_2}(z) = \{w \in S^2 : \rho_1 \leq \text{dist}(z, w) \leq \rho_2\}.$$

(so for  $\rho_1 = 0$  we get a spherical cap).

The variance over all annuli of radii  $\rho_1 < \rho_2$  is

$$V(n; \rho_1, \rho_2) := \int_{S^2} \left( Z(n; A_{\rho_1, \rho_2}(z)) - A \right)^2 d\sigma(z),$$

where  $A = \text{area}(A_{\rho_1, \rho_2}(z))$  is the common area of all these annuli.

We want to show that, assuming the Lindelöf Hypothesis for standard  $GL(2)/\mathbb{Q}$  L-functions, for any sequence  $n \rightarrow \infty$ , with  $n \neq 7 \pmod 8$  squarefree, we have

$$V(n; \rho_1, \rho_2) := \int_{S^2} \left( Z(n; A_{\rho_1, \rho_2}(z)) - A \right)^2 d\sigma(z) \ll_{\epsilon} n^{\epsilon} N_n \cdot A \quad \forall \epsilon > 0,$$

where  $A = \text{area}(A_{\rho_1, \rho_2}(z))$  is the common area of the annuli.

For  $m = 0, 1, \dots$ , and  $j = 1, 2, \dots, 2m + 1$ , let  $\phi_{j,m}$  be an orthonormal basis of eigenfunction of the Laplacian  $\Delta$  of the sphere, i.e. of the spherical harmonics of degree  $m$ . For such a  $\phi_{j,m}$  the Weyl sum is defined by

$$(5.1) \quad W_{\phi_{j,m}}(n) := \sum_{\mathbf{x} \in \mathcal{E}(n)} \phi_{j,m}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right).$$

Let  $k(z, \zeta)$  be a point pair invariant on  $S^2$  [26]. Then (in  $L^2$ )

$$(5.2) \quad k(z, \zeta) = \sum_{m=0}^{\infty} h_k(m) \sum_{j=1}^{2m+1} \phi_{j,m}(z) \phi_{j,m}(\zeta)$$

with

$$h_k(m) = \int_{S^2} k(z, \zeta) \omega_m(\zeta) d\zeta,$$

where  $\omega_m(\zeta)$  is the zonal spherical harmonic about  $z$ , normalized to take value 1 at  $\zeta = z$ , and  $d\zeta = 4\pi d\sigma(z)$  is the un-normalized area measure on  $S^2$ . Thus (see e.g. [20])

$$h_k(m) = 2\pi \int_0^1 k(t) P_m(t) dt,$$

where  $P_m(t)$  is the Legendre polynomial.

We have

$$Z(n; A_{\rho_1, \rho_2}(z)) = \sum_{\mathbf{x} \in \mathcal{E}(n)} k\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

for the point pair invariant

$$k(z, \zeta) = \mathbf{1}_{A_{\rho_1, \rho_2}(z)}(\zeta),$$

where  $\mathbf{1}_\Omega$  is the indicator function of the set  $\Omega$ , and therefore we get from (5.2) that

$$(5.3) \quad V(n; \rho_1, \rho_2) = \sum_{m=1}^{\infty} h_{\rho_1, \rho_2}(m)^2 \sum_{j=1}^{2m+1} |W_{\phi_{j,m}}(n)|^2.$$

The key arithmetic ingredient is the explicit formula for the Weyl sums in terms of special values of L-functions. The particular version that we use is due to [3] and [2] as explicated in [21] and coupled with [15]. We choose the  $\phi_{j,m}$  to be an orthonormal basis of Hecke eigenfunctions for the action of the Hamilton quaternions on  $S^2$  (see [21]). Each such  $\phi_{j,m}$  has a Jacquet-Langlands lift to a holomorphic Hecke cusp form  $f_{j,m}$  for  $\Gamma_0(8)$ , of weight  $2m + 2$ . Let  $L(s, f)$  and  $L(s, \text{Sym}^2 f)$  denote the finite parts of the corresponding L-functions. Then for  $n$  squarefree

$$(5.4) \quad |W_{\phi_{j,m}}(n)|^2 = c \frac{n^{1/2} L(\frac{1}{2}, f_{j,m}) L(\frac{1}{2}, f_{j,m} \times \chi_{-n})}{L(1, \text{Sym}^2 f_{j,m})}.$$

Here  $c > 0$  is an absolute constant (independent of  $\phi_{j,m}$ ,  $m$  and  $n$ ) and  $\chi_{-n}$  is the quadratic Dirichlet character corresponding to the extension  $\mathbb{Q}(\sqrt{-n})$ . For the indefinite ternary form  $y^2 - xz$ , instead of the definite form  $x^2 + y^2 + z^2$  at hand, the explicit formula (5.4) is given in [19, (5.1)] and it follows in a similar way from [14] and [1].

From (5.4) and<sup>1</sup> the Lindelöf Hypothesis applied to the L-functions  $L(s, f_{j,m})$  and  $L(s, f_{j,m} \times \chi_{-n})$ , (5.3) becomes

$$\begin{aligned}
 (5.5) \quad V(n; \rho_1, \rho_2) &\ll_\epsilon \sum_{m=1}^\infty h_{\rho_1, \rho_2}(m)^2 n^{1/2} \sum_{j=1}^{2m+1} m^\epsilon n^\epsilon \\
 &= n^{\frac{1}{2}+\epsilon} \sum_{m=1}^\infty h_{\rho_1, \rho_2}(m)^2 m^{1+\epsilon}, \quad \forall \epsilon > 0.
 \end{aligned}$$

The simple estimate  $h_{\rho_1, \rho_2}(m) \ll m^{-3/2}$  (see [20, p. 169]) yields (for any  $X \gg 1$ )

$$\begin{aligned}
 V(n; \rho_1, \rho_2) &\ll_\epsilon X^\epsilon n^{1/2+\epsilon} \sum_{m \leq X} m h_{\rho_1, \rho_2}(m)^2 + n^{1/2+\epsilon} \sum_{m > X} m^{-2+\epsilon} \\
 &\ll X^\epsilon n^{1/2+\epsilon} \int_{S^2} |\chi_{A_{\rho_1, \rho_2}}(\zeta)|^2 d\sigma(\zeta) + n^{1/2+\epsilon} X^{-1+\epsilon}.
 \end{aligned}$$

Choosing  $X = n$  gives

$$V(n; \rho_1, \rho_2) \ll_\epsilon n^{1/2+\epsilon'} A \ll AN_n^{1+\epsilon''},$$

as claimed.

**5.2. Proof of Corollary 1.10.** We show that Conjecture 1.7 implies Corollary 1.10.

*Proof.* Assume the covering radius of  $\widehat{\mathcal{E}}(n)$  is bigger than  $\rho$ , so that there is some point  $\xi_0 \in S^2$  so that the cap  $\text{Cap}(\xi_0, \rho) \subset S^2$  contains no projected lattice point  $\frac{1}{\sqrt{n}}\mathcal{E}(n)$ . Therefore, if  $0 < \delta \leq \rho/2$ , then for all  $\xi \in \text{Cap}(\xi_0, \rho/2)$ , the caps  $\text{Cap}(\xi, \delta)$  also do not contain any projected lattice points, that is

$$Z(n; \text{Cap}(\xi, \delta)) = 0, \quad \forall \xi \in \text{Cap}(\xi_0, \rho).$$

It follows that

$$(5.6) \quad \int_{S^2} \left| Z(n; \text{Cap}(\xi, \delta)) - N_n \text{area}(\text{Cap}(\xi, \delta)) \right|^2 d\sigma(\xi) \gg \rho^2 N_n^2 \delta^4.$$

Combining (5.6) and Conjecture 1.7 gives

$$\rho^2 \delta^2 \ll N_n^{-1}.$$

Taking  $\delta = \rho/2$  we obtain

$$\rho \ll N_n^{-1/4},$$

as claimed. □

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<sup>1</sup>We also need a good lower bound for  $L(1, \text{Sym}^2 f_{j,m})$ , which unconditionally is due to Hoffstein-Lockhart [12]



### 6. Gaps between sums of two squares

We denote by  $\mathcal{S}_2 = \{n_1 < n_2 < \dots\}$  the sequence of integers which are sums of two squares. An old conjecture asserts that the gaps between consecutive elements of  $\mathcal{S}_2$  satisfy  $n_{i+1} - n_i \ll n_i^\epsilon$ , for all  $\epsilon > 0$ . Note that primes  $p = 1 \pmod 4$  are also conjectured to have this property, and since such primes are in  $\mathcal{S}_2$  this a fortiori implies the above conjecture. However, all that is known is the elementary bound  $n_{i+1} - n_i \ll n_i^{1/4}$ . In this section we point out that the covering radius conjecture 1.9 implies the above conjecture on gaps between sums of two squares.

For  $Y \gg 1$ , let  $\mathcal{S}_2(Y) = \mathcal{S}_2 \cap [Y, 2Y)$ , and let

$$G(Y) = \max\{n_{i+1} - n_i : n_i \in \mathcal{S}_2(Y)\}$$

be the maximal gap between sums of square in the interval  $[Y, 2Y)$ ,

$$G(Y) = n'' - n'$$

with  $n' < n''$  consecutive elements of  $\mathcal{S}_2(Y)$ . We want to show that Conjecture 1.9 implies that  $G(Y) \ll Y^\epsilon$ , for all  $\epsilon > 0$ .

Assume then that  $G(Y) > Y^\epsilon$ . By Brun's sieve, every interval of length  $\geq G(Y)/8$  contains an integer  $m$  which is not divisible by any small prime  $p \leq G(Y)^\delta$ , for  $\delta > 0$  sufficiently small. Hence we may find an integer  $m$  for which

$$(6.1) \quad \left| m - \frac{n' + n''}{4} \right| < \frac{1}{8}G(Y),$$

and free of any prime factors less than  $G(Y)^\delta$ :

$$(6.2) \quad p \mid m \Rightarrow p > G(Y)^\delta.$$

Take  $n = m^2$  and the point  $\mathbf{m} := (0, 0, m) \in \mathcal{E}(n)$  (note  $n = 1, 5 \pmod 8$ ). Then by Conjecture 1.9 there is  $\mathbf{x} = (x_1, x_2, x_3) \in \mathcal{E}(n)$ ,  $\mathbf{x} \neq \mathbf{m}$  so that

$$|\mathbf{m} - \mathbf{x}|^2 = x_1^2 + x_2^2 + (m - x_3)^2 < G(Y)^\delta m,$$

Thus

$$x_1^2 + x_2^2 < G(Y)^\delta m,$$

and since  $x_1^2 + x_2^2 + x_3^2 = m^2$ , we have

$$x_1^2 + x_2^2 = (m - x_3)(m + x_3).$$

We claim that  $m + x_3 \in \mathcal{S}_2$ . To see this, note that if  $p = 3 \pmod 4$  divides the sum of two squares  $x_1^2 + x_2^2$ , then  $\text{ord}_p(x_1^2 + x_2^2)$  is even. It follows that if  $p = 3 \pmod 4$  is a prime such that  $p \mid m + x_3$  and  $\text{ord}_p(m + x_3)$  is odd, then  $p \mid m - x_3$  and hence  $p \mid m$ . Since moreover,

$$m - x_3 = \frac{x_1^2 + x_2^2}{m + x_3} < \frac{x_1^2 + x_2^2}{m} < G(Y)^\delta$$

we conclude that  $p \leq m - x_3 < G(Y)^\delta$ , which is excluded by (6.2). Hence  $\text{ord}_p(m + x_3)$  is even for any prime  $p = 3 \pmod{4}$ , that is  $m + x_3 \in \mathcal{S}_2$  is a sum of two squares.

Since  $2m = (m + x_3) + (m - x_3)$ , we obtain

$$\text{dist}(2m, \mathcal{S}_2) < G(Y)^\delta.$$

Hence

$$\frac{1}{2}G(Y) = \text{dist}\left(\frac{n' + n''}{2}, \mathcal{S}_2\right) \leq \left|\frac{n' + n''}{2} - 2m\right| + \text{dist}(2m, \mathcal{S}_2) < \frac{1}{4}G(Y) + G(Y)^\delta$$

by (6.1). This is a contradiction for  $Y \gg 1$ .

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### REFERENCES

- [1] E. Baruch and Z. Mao, Central value of automorphic  $L$ -functions, *Geom. Funct. Anal.* **17** (2007), no. 2, 333–384.
- [2] S. Böcherer, P. Sarnak and R. Schulze-Pillot, Arithmetic and equidistribution of measures on the sphere, *Comm. Math. Phys.* **242** (2003), no. 1-2, 67–80.
- [3] S. Böcherer and R. Schulze-Pillot, The Dirichlet series of Koecher and Maass and modular forms of weight  $3/2$ , *Math. Z.* **209** (1992), no. 2, 273–287.
- [4] J. Bourgain, Z. Rudnick and P. Sarnak, Local statistics of lattice points on the sphere, in: *Modern Trends in Constructive Function Theory*, pp. 269–282, *Contemp. Math.* 661, Amer. Math. Soc. Providence, RI, 2016.
- [5] B. Dahlberg, On the distribution of Fekete points, *Duke Math. J.* **45** (1978), no. 3, 537–542.
- [6] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, *Invent. Math.* **92** (1988), no. 1, 73–90.
- [7] W. Duke and R. Schulze-Pillot, Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids, *Invent. Math.* **99** (1990), no. 1, 49–57.
- [8] J. Ellenberg, P. Michel and A. Venkatesh, Linnik's ergodic method and the distribution of integer points on spheres, in: *Automorphic Representations and  $L$ -Functions*, pp. 119–185, *Tata Inst. Fundam. Res. Stud. Math.* 22, Tata Inst. Fund. Res. Mumbai, 2013.
- [9] J. Ellenberg, P. Michel and A. Venkatesh, Linnik's ergodic method and the distribution of integer points on spheres, Arxiv:1001.0897 [math.NT].
- [10] E.P. Golubeva, O.M. Fomenko, Asymptotic distribution of lattice points on the three-dimensional sphere, (Russian), translated from *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* **160** (1987), *Anal. Teor. Chisel i Teor. Funktsii* **8** 54–71, 297, *J. Soviet Math.* **52** (1990), no. 3, 3036–3048.

- [11] P.J. Grabner, Erdős-Turán type discrepancy bounds, *Monatsh. Math.* **111** (1991), no. 2, 127–135.
- [12] J. Hoffstein and P. Lockhart, Coefficients of Maass forms and the Siegel zero, with an appendix by D. Goldfeld, J. Hoffstein and D. Lieman, *Ann. of Math. (2)* **140** (1994), no. 1, 161–181.
- [13] H. Iwaniec, Fourier coefficients of modular forms of half-integral weight, *Invent. Math.* **87** (1987), no. 2, 385–401.
- [14] S. Katok and P. Sarnak, Heegner points, cycles and Maass forms, *Israel J. Math.* **84** (1993), no. 1-2, 193–227.
- [15] W. Kohnen and D. Zagier, Values of  $L$ -series of modular forms at the center of the critical strip, *Invent. Math.* **64** (1981), no. 2, 175–198.
- [16] L. Kuipers, H. Niederreiter, Uniform Distribution of Sequences, Pure Appl. Math. John Wiley & Sons, 1974.
- [17] U. Linnik, Über die Darstellung grosser Zahlen durch positive ternäre quadratische Formen, *Bull. Acad. Sci. URSS. Ser. Math. [Izvestia Akad. Nauk SSSR]* **4** (1940) 363–402.
- [18] Y.V. Linnik, Ergodic Properties of Algebraic Fields, translated from the Russian by M.S. Keane, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 45, Springer-Verlag, 1968.
- [19] S.-C. Liu, R. Masri and M.P. Young, Subconvexity and equidistribution of Heegner points in the level aspect, *Compos. Math.* **149** (2013), no. 7, 1150–1174.
- [20] A. Lubotzky, R. Phillips and P. Sarnak, Hecke operators and distributing points on the sphere. I, in: *Frontiers of the Mathematical Sciences: 1985* (New York, 1985), *Comm. Pure Appl. Math.* **39** (1986), no. S, suppl. S149–S186.
- [21] W. Luo, A note on the distribution of integer points on spheres, *Math. Z.* **267** (2011), no. 3–4, 965–970.
- [22] M. Nair, Multiplicative functions of polynomial values in short intervals, *Acta Arith.* **62** (1992), no. 3, 257–269.
- [23] G. Pall, Quaternions and sums of three squares, *Amer. J. Math.* **64** (1942) 503–513.
- [24] G. Pall, Representation by quadratic forms, *Canad. J. Math.* **1** (1949) 344–364.
- [25] Z. Rudnick, I. Wigman and N. Yesha, Nodal intersections for random waves on the 3-dimensional torus, *Ann. Inst. Fourier*, to appear.
- [26] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc. (N.S.)* **20** (1956) 47–87.
- [27] D. Stoyan, W.S. Kendall and J. Mecke, Stochastic Geometry and its Applications, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, Chichester, 1987.
- [28] B.A. Venkov (Wenkov), Über die Klassenzahl positiver binärer quadratischer Formen, *Math. Z.* **33** (1931) 350–374.
- [29] B.A. Venkov, Elementary Number Theory, translated from the Russian and edited by H. Alderson, Wolters-Noordhoff, 1970.
- [30] G. Wagner, On the means of distances on the surface of a sphere (lower bounds), *Pacific J. Math.* **144** (1990) 389–398.
- [31] G. Wagner, On means of distances on the surface of a sphere. II. Upper bounds, *Pacific J. Math.* **154** (1992), no. 2, 381–396.

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