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ON LOCAL GAMMA FACTORS FOR ORTHOGONAL GROUPS AND UNITARY GROUPS

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Dedicated to Freydoon Shahidi

Abstract. In this paper, we find a relation between the proportionality factors which arise from the functional equations of two families of local Rankin-Selberg convolutions for irreducible admissible representations of orthogonal groups, or unitary groups. One family is that of local integrals of the doubling method, and the other family is that of local integrals expressed in terms of spherical Bessel models.
Keywords: Gamma factors, Rankin-Selberg convolutions, intertwining operators.
MSC(2010): Primary: 11F70; Secondary: 22E55.

1. Introduction

Let G' be an orthogonal group, or a unitary group, in $n+1$ variables over a local field F, and consider a subgroup $G \subset G'$, which is the stabilizer of an anisotropic vector, so that G is an orthogonal group, or a unitary group, in n variables. Let π be an irreducible, admissible representation of G'. In this paper, we compare two families of local Rankin-Selberg integrals, which represent, at the unramified level, the standard L-function of π, twisted by characters. The first family is the one arising from the doubling method, first introduced by Piatetski-Shapiro and Rallis in [7], and later deeply studied by Lapid and Rallis in [6]. The second family is of integrals expressed in terms of a "spherical Bessel model" of π, that is a nontrivial element c in

$$\mathrm{Hom}_G(\pi \otimes \sigma, 1),$$

where σ is an irreducible, admissible representation of G, with a diagonal action of G on $\pi \otimes \sigma$. These integrals are a special case of the family of integrals studied by Ginzburg, Piatetski-Shapiro and Rallis in [4], representing standard L-functions for pairs of representations of orthogonal groups and general linear

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groups. See [5] for an analogue for unitary groups. We compare the local functional equations satisfied by the above two families. Let us denote the proportionality factor, which appears in the functional equation of the second family, by $\Gamma^{sph}(\pi, \sigma, \chi, s)$. This is a meromorphic function in the complex variable s. Here χ is a character of F^* in case our group is orthogonal, or of E^* in case our group is unitary and corresponds to a quadratic extension E/F. Fix a nontrivial character ψ of F. We prove in a straightforward way, that up to an explicit function of s, which depends on χ, ψ and G',G (but not on π, or σ), $\Gamma^{sph}(\pi, \sigma, \chi, s)$ is

$$\frac{\gamma(\pi \times \chi, s, \psi)}{\gamma(\hat{\sigma} \times \chi, s - \frac{1}{2}, \psi)},$$

where the functions in the numerator and denominator are the local gamma factors of irreducible, admissible representations of G', G respectively (twisted by characters). These are the local gamma factors, which fit in the theory of standard L-functions for these groups. They were defined by Lapid and Rallis in [6, Section 9].

The starting point of the proof appears already in [4, Section 1], where it is shown that integrals of the second family, corresponding to a certain choice of sections in the parabolic induction from $\chi|\cdot|^s \otimes \sigma$, are equal to typical integrals of the first family. The main work of this paper is to push this further and see how the last relation of integrals from both families is affected when we apply local intertwining operators. We had a sketch of the result presented here quite sometime ago. The proof turned to be quite delicate, as the order of applying the local functional equations is important. Sadly, Steve Rallis passed away in 2012, but, nevertheless, he should have his signature on this paper.

It is a pleasure to dedicate this paper to Freydoon Shahidi, who made such fundamental contributions to automorphic forms and representation theory of reductive groups, in particular, through his theory of local coefficients, which also lead to the definition of local gamma factors of generic representations, thus relating to the main topic of this contribution.

David Soudry

2. **Notations and preliminaries**

Let F be a local field of characteristic zero. If F is a p-adic field, we denote by q the number of elements in its residue field. Let E be either F or a quadratic extension of F. For $x \in E$, denote, in the first case, $\bar{x} = x$, and in the second case, denote by $\bar{x}$ the Galois conjugate of x over F. When $[E : F] = 2$, we take the absolute value $|a|_E = |a\bar{a}|_F$. In both cases, we will simply denote the absolute value of $a \in E$ by $|a|$.

Let $V' = V \oplus Eu_0$ be a finite dimensional space over E, of dimension $n+1$, equipped with a non-degenerate E-bilinear form b', which is symmetric in case $E = F$ and Hermitiean in case $[E : F] = 2$. Assume that u_0 is orthogonal to V.

Denote by b and b_0 the restrictions of b' to $V\times V$ and to $Eu_0\times Eu_0$ respectively. Let $\mathrm{U}(V')$ (resp. $\mathrm{U}(V)$) denote the isometry group of (V',b') (resp. (V,b)). Thus, when $E=F$, $\mathrm{U}(V')=\mathrm{O}(V')=\mathrm{O}_{n+1}$ is an orthogonal group in $n+1$ variables, and when E/F is quadratic, $\mathrm{U}(V')=\mathrm{U}_{n+1}$ is a unitary group in $n+1$ variables. Similarly, for $\mathrm{U}(V)$. We regard $\mathrm{U}(V)$ as the subgroup of elements of $\mathrm{U}(V')$, which fix u_0.
In the sequel, we will need to fix a norm, $h\mapsto\|h\|$, on the group $\mathrm{U}(V')$. When we realize $\mathrm{U}(V')$ as a matrix group, we will simply take the maximum absolute value of the matrix coordinates of h.

Consider the doubled space

$$(W',\langle,\rangle)=(V'\oplus V',b'\oplus -b'),$$

the orthogonal sum of (V',b') and $(V',-b')$. We identify $\mathrm{U}(V')\times\mathrm{U}(V')$ as a subgroup of $\mathrm{U}(W')$. For $g_1,g_2\in\mathrm{U}(V')$, we think of (g_1,g_2) as the element of $\mathrm{U}(W')$, which acts on W' by

$$(g_1,g_2)\cdot(u_1,u_2)=(g_1u_1,g_2u_2),\quad u_1,u_2\in V'.$$

Let

$$V'_\Delta=\{(u,u)|u\in V'\},\ V'_{-\Delta}=\{(u,-u)|u\in V'\}.$$

These are two dual maximal isotropic subspaces of W', and we have the polarization

$$W'=V'_\Delta+V'_{-\Delta}.$$

Similarly, we have the doubled space with polarization $W=V_\Delta+V_{-\Delta}$, when we replace V', b' by V and b, and we identify, as above, $\mathrm{U}(V)\times\mathrm{U}(V)$ as a subgroup of $\mathrm{U}(W)$. Denote $P'=P(V'_\Delta)$ (resp. $P=P(V_\Delta)$) the maximal parabolic subgroup of $\mathrm{U}(W')$ (resp. $\mathrm{U}(W)$) which preserves V'_Δ (resp. V_Δ). For g in the Levi part of P', $N_{E/F}(\det(g))=1$ (if $E=F$, this means that $\det(g)=1$). Let $g\in\mathrm{U}(V')$. Then

$$(g,g)\in P'. \tag{2.1}$$

Indeed, for $u\in V'$, $(g,g)\cdot(u,u)=(gu,gu)\in V'_\Delta$. Similar statements hold for P. Let

$$H^{1,1}=(Eu_0\oplus Eu_0,b_0\oplus(-b_0)).$$

Let $V_1=V\oplus 0\subset W$ and $V_2=0\oplus V\subset W$. Similarly, let $V'_1=V'\oplus 0\subset W'$, $V'_2=0\oplus V'\subset W'$. Then we have the following chains of inclusions, with identifications as above,

$$\begin{gathered}\mathrm{U}(V'_1)\times\mathrm{U}(V_2)\subset\mathrm{U}(V_1\oplus H^{1,1})\times\mathrm{U}(V_2)\subset\mathrm{U}(W'),\\ \mathrm{U}(V'_1)\times\mathrm{U}(V_2)\subset\mathrm{U}(V'_1)\times\mathrm{U}(V'_2)\subset\mathrm{U}(W'),\\ \mathrm{U}(V_1)\times\mathrm{U}(V_2)\subset\mathrm{U}(W)\subset\mathrm{U}(W'),\\ \mathrm{U}(V_1)\times\mathrm{U}(V_2)\subset\mathrm{U}(V'_1)\times\mathrm{U}(V_2).\end{gathered} \tag{2.2}$$

The spaces V_1 and V_2 are naturally identified with V, and the groups $\mathrm{U}(V_1)$, $\mathrm{U}(V_2)$ are naturally identified with $\mathrm{U}(V)$. We will make such identifications when convenient. We introduced these slight distinctions due to $H^{1,1}$, which "occupies two coordinates". Thus in the inclusion $\mathrm{U}(V_1 \oplus H^{1,1}) \times \mathrm{U}(V_2) \subset \mathrm{U}(W')$, we point to the fact $H^{1,1}$ is taken together with the first copy of V. In practice, we will write the elements of $\mathrm{U}(V_1 \oplus H^{1,1}) \times \mathrm{U}(V_2) \subset \mathrm{U}(W')$ as (g_1, g_2), where $g_1 \in \mathrm{U}(V_1 \oplus H^{1,1})$ and $g_2 \in \mathrm{U}(V)$. This is the element that acts as follows: for all $v_1, v_2 \in V, a_1, a_2 \in E$,

$$(g_1, g_2) \cdot (v_1 + a_1u_0, v_2 + a_2u_0) = g_1 \cdot (v_1 + a_1u_0, a_2u_0) + (0, g_2v_2). \tag{2.3}$$

In order to be fully consistent with our notation, we should have denoted the last element as $((g_1, 1_V), (1_{V_1 \oplus H^{1,1}}, g_2))$, but this is too cumbersome, and we will avoid this. We use similar interpretations and identifications in all other inclusions in (2.2).

Let $Q \subset \mathrm{U}(V_1 \oplus H^{1,1})$ be the parabolic subgroup which preserves the isotropic line $E(u_0, u_0)$. Note the decomposition

$$V_1 \oplus H^{1,1} = E(u_0, u_0) + V_1 + E(u_0, -u_0).$$

Of course, V_1 is orthogonal to $H^{1,1} = E(u_0, u_0) + E(u_0, -u_0)$, $(u_0, \pm u_0)$ are isotropic and $\langle (u_0, u_0), (u_0, -u_0) \rangle = 2b'(u_0, u_0)$. Let $r \in Q$. Then there is $t \in E^*$, such that

$$r \cdot (u_0, u_0) = t(u_0, u_0), \tag{2.4}$$

and there are $m \in \mathrm{U}(V)$, $x \in V^*$, such that for all $v \in V$ (i.e. for all $(v, 0) \in V_1$),

$$r \cdot (v, 0) = (mv, 0) + x(v)(u_0, u_0). \tag{2.5}$$

Lemma 2.1. *Let $r \in Q$. Then, with notation as in (2.4), (2.5), $(r, m) \in P'$, and*

$$\det{}_{V'_\Delta}(r, m) = t \det(m).$$

Proof. As explained in (2.3), the element (r, m) lies in $\mathrm{U}(V_1 \oplus H^{1,1}) \times \mathrm{U}(V_2)$, which, as in (2.2), we view as a subgroup of $\mathrm{U}(W')$. Take an element $(v + au_0, v + au_0) \in V'_\Delta$, where $v \in V$, $a \in E$. Then

$$(r, m) \cdot (v + au_0, v + au_0) = (r, m) \cdot (v, 0) + (r, m) \cdot (0, v) + a(r, m) \cdot (u_0, u_0) =$$

$$(mv, 0) + x(v)(u_0, u_0) + (0, mv) + at(u_0, u_0).$$

This is equal to $(mv + (x(v) + at)u_0, mv + (x(v) + at)u_0) \in V'_\Delta$. This shows that $(r, m) \in P'$, and its determinant on V'_Δ is $t \det(m)$. ∎

3. A certain family of sections in $\mathrm{Ind}_Q^{\mathrm{U}(V_1\oplus H^{1,1})}(\chi|\cdot|^s\otimes\sigma)$

Let χ be a character of E^* and s a complex number. Consider the parabolic inductions (normalized)

$$\rho'_{\chi,s} = \mathrm{Ind}_{P'}^{\mathrm{U}(W')}\chi(\det_{V'_\Delta}\cdot)|\det_{V'_\Delta}\cdot|^s, \tag{3.1}$$

$$\rho_{\chi,s} = \mathrm{Ind}_{P}^{\mathrm{U}(W)}\chi(\det_{V_\Delta}\cdot)|\det_{V_\Delta}\cdot|^s. \tag{3.2}$$

Let $f_{\chi,s}$ be a smooth holomorphic section in $\rho'_{\chi,s}$. When F is p-adic, we assume that it is a polynomial (in $q^{\pm s}$) section. We have the following very easy lemma.

Lemma 3.1. *The restriction of $f_{\chi,s}$ to $\mathrm{U}(W)$ lies in $\rho_{\chi,s+\frac{1}{2}}$.*

Let σ be an irreducible admissible representation of $\mathrm{U}(V)$. Let v_σ be a vector in V_σ, a (smooth) realization space of σ that we fix. We will restrict $f_{\chi,s}$ to $\mathrm{U}(V_1\oplus H^{1,1})\times\mathrm{U}(V_2)$, view σ as a representation of $\mathrm{U}(V_2)$ and integrate $\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma$ against the kernel function $f_{\chi,s}(g_1,g_2)$ ($g_1\in\mathrm{U}(V_1\oplus H^{1,1})$, $g_2\in\mathrm{U}(V)$) to obtain a function on $\mathrm{U}(V_1\oplus H^{1,1})$, with values in V_σ. See (2.3). Thus, define, for $g_1\in\mathrm{U}(V_1\oplus H^{1,1})$,

$$\Lambda_{v_\sigma,f_{\chi,s}}(g_1) = \int_{\mathrm{U}(V)} f_{\chi,s}(g_1,g_2)\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2. \tag{3.3}$$

Before we make sense out of this integral, let us show formally

Lemma 3.2. *The function $\Lambda_{v_\sigma,f_{\chi,s}}$ lies in the space of $\mathrm{Ind}_Q^{\mathrm{U}(V_1\oplus H^{1,1})}(\chi|\cdot|^s\otimes\sigma)$.*

Proof. Let $r\in Q$ and use the notation as in Lemma 2.1. Then

$$\Lambda_{v_\sigma,f_{\chi,s}}(rg_1) = \int_{\mathrm{U}(V)} f_{\chi,s}(rg_1,g_2)\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2.$$

Change variable $g_2\mapsto mg_2$. We get

$$\chi^{-1}(\det(m))\sigma(m)(\int_{\mathrm{U}(V)} f_{\chi,s}((r,m)(g_1,g_2))\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2). \tag{3.4}$$

By Lemma 2.1,

$$\begin{aligned} f_{\chi,s}((r,m)(g_1,g_2)) &= \chi(t\det(m))|t|^{s+\frac{n+\dim E-1}{2}}f_{\chi,s}(g_1,g_2) \\ &= \chi(t\det(m))|t|^s\delta_Q^{\frac{1}{2}}(r)f_{\chi,s}(g_1,g_2). \end{aligned} \tag{3.5}$$

The last equality is easy to check. Using (3.4), (3.5), we get

$$\Lambda_{v_\sigma,f_{\chi,s}}(rg_1) = \chi(t)|t|^s\delta_Q^{\frac{1}{2}}(r)\sigma(m)(\Lambda_{v_\sigma,f_{\chi,s}}(g_1)).$$

□

Let us interpret (3.3) precisely. We first view it in the weak sense, that is as a linear functional on $V_{\hat{\sigma}}$, the space of smooth linear functionals on V_σ. Thus, let $\xi_{\hat{\sigma}} \in V_{\hat{\sigma}}$, and consider the integral

$$\ell_{g_1,f_{\chi,s},v_\sigma}(\xi_{\hat{\sigma}}) = \int_{\mathrm{U}(V)} f_{\chi,s}(g_1,g_2)\chi^{-1}(\det(g_2))\langle\sigma(g_2)v_\sigma,\xi_{\hat{\sigma}}\rangle dg_2. \tag{3.6}$$

Since $f_{\chi,s}(g_1,g_2) = \rho'_{\chi,s}(g_1,I_V)f_{\chi,s}(I_{V_1\oplus H^{1,1}},g_2)$, we may assume that g_1 is the identity element. The point now is that $(I_{V_1\oplus H^{1,1}},g_2)$, as an element of $\mathrm{U}(V_1\oplus H^{1,1})\times\mathrm{U}(V_2)\subset\mathrm{U}(W')$, is the same element as $(I_V,g_2)\in\mathrm{U}(V)\times\mathrm{U}(V)\subset\mathrm{U}(W)$. Thus, by Lemma 3.1, it is enough to consider a smooth holomorphic section $\phi_{\chi,s+\frac{1}{2}}$ in $\rho_{\chi,s+\frac{1}{2}}$ and consider the integral

$$l_{\phi_{\chi,s+\frac{1}{2}},v_\sigma}(\xi_{\hat{\sigma}}) = \int_{\mathrm{U}(V)} \phi_{\chi,s+\frac{1}{2}}(I_V,g_2)\chi^{-1}(\det(g_2))\langle\sigma(g_2)v_\sigma,\xi_{\hat{\sigma}}\rangle dg_2. \tag{3.7}$$

This integral is equal to

$$\int_{\mathrm{U}(V)} \phi_{\chi,s+\frac{1}{2}}(g,I_V)\langle v_\sigma,\hat{\sigma}(g)\xi_{\hat{\sigma}}\rangle dg, \tag{3.8}$$

which is a local integral that arises in the doubling method for $\hat{\sigma}$. We know that it converges absolutely in a certain right half plane, which depends only on σ, and it continues to a meromorphic function in the complex plane. In case F is p-adic, this function is a rational function of q^{-s}. We know that $\phi_{\chi,s+\frac{1}{2}}$ can be written as a finite sum of sections of the form

$$\varphi\star\phi'_{\chi,s+\frac{1}{2}} = \int_{\mathrm{U}(V)} \varphi(x)\rho_{\chi,s+\frac{1}{2}}(x,I_V)\phi'_{\chi,s+\frac{1}{2}}dx,$$

where $\varphi\in C_c^\infty(\mathrm{U}(V))$ and $\phi'_{\chi,s+\frac{1}{2}}$ is a smooth holomorphic section in $\rho_{\chi,s+\frac{1}{2}}$. In the p-adic case, we may take $\phi'_{\chi,s+\frac{1}{2}} = \phi_{\chi,s+\frac{1}{2}}$ and φ the characteristic function of a small compact open subgroup $K_0\subset\mathrm{U}(V)$, divided by its measure, and K_0 is such that $K_0\times I_V$ fixes $\phi_{\chi,s+\frac{1}{2}}$. In the Archimedean case, this is a consequence of the theorem of Dixmier-Malliavin [3]. Thus, let us assume that $\phi_{\chi,s+\frac{1}{2}}$ is of the form $\varphi\star\phi'_{\chi,s+\frac{1}{2}}$. Note that

$$\phi'_{\chi,s+\frac{1}{2}}((I_V,g_2)(x,I_V)) = \chi(\det(x))\phi'_{\chi,s+\frac{1}{2}}(I_V,x^{-1}g_2).$$

Substituting in (3.7), we switch the order of integrations (for $\mathrm{Re}(s)$ large), change variable $g_2\mapsto xg_2$ and switch the order of integrations again,

$$\begin{aligned} &l_{\varphi\star\phi'_{\chi,s+\frac{1}{2}},v_\sigma}(\xi_{\hat{\sigma}}) = \\ &\int_{\mathrm{U}(V)}\int_{\mathrm{U}(V)} \phi'_{\chi,s+\frac{1}{2}}(I_V,g_2)\chi^{-1}(\det(g_2))\varphi(x)\langle\sigma(g_2)v_\sigma,\hat{\sigma}(x^{-1})\xi_{\hat{\sigma}}\rangle dx dg_2 \\ &\qquad = l_{\phi'_{\chi,s+\frac{1}{2}},v_\sigma}(\hat{\sigma}(\check{\varphi})\xi_{\hat{\sigma}}). \end{aligned} \tag{3.9}$$

Here, $\check{\varphi}(x)=\varphi(x^{-1})$. This shows that the functional $l_{\varphi\star\phi'_{\chi,s+\frac{1}{2}},v_\sigma}$ is a smooth functional on $V_{\hat{\sigma}}$. Going back to (3.3), we conclude that there is a unique vector in V_σ, which we denote by $\Lambda_{v_\sigma,f_{\chi,s}}(g_1)$, such that for all $\xi_{\hat{\sigma}}\in V_{\hat{\sigma}}$,

$$\langle\Lambda_{v_\sigma,f_{\chi,s}}(g_1),\xi_{\hat{\sigma}}\rangle=\int_{\mathrm{U}(V)} f_{\chi,s}(g_1,g_2)\chi^{-1}(\det(g_2))\langle\sigma(g_2)v_\sigma,\xi_{\hat{\sigma}}\rangle dg_2, \tag{3.10}$$

first for $\mathrm{Re}(s)$ sufficiently large and then the left hand side of (3.10) is equal to the meromorphic continuation of the right hand side (rational in q^{-s}, in the p-adic case). Finally, $\Lambda_{v_\sigma,f_{\chi,s}}$ is smooth. The argument is similar. Again, we may assume that, for all $g\in\mathrm{U}(W')$,

$$f_{\chi,s}(g)=\varphi_1\star f'_{\chi,s}(g)=\int_{\mathrm{U}(V')}\varphi_1(y)f'_{\chi,s}(g(y,I_V))dy,$$

where $\varphi_1\in C_c^\infty(\mathrm{U}(V'))$ and $f'_{\chi,s}$ is a smooth holomorphic section. (We keep using the symbol $\star$ to denote convolutions.) Then, for all $g_1\in\mathrm{U}(V')$,

$$\begin{aligned}\Lambda_{v_\sigma,\varphi_1\star f'_{\chi,s}}(g_1) &= (\varphi_1\star\Lambda_{v_\sigma,f'_{\chi,s}})(g_1)\\ &= \int_{\mathrm{U}(V')}\varphi_1(y)\Lambda_{v_\sigma,f'_{\chi,s}}(g_1y)dy.\end{aligned}$$

Thus, $\Lambda_{v_\sigma,f_{\chi,s}}$ is a smooth meromorphic (rational in q^{-s}, in the p-adic case) section in $\mathrm{Ind}_Q^{\mathrm{U}(V_1\oplus H^{1,1})}(\chi|\cdot|^s\otimes\sigma)$.

4. Local Rankin-Selberg convolutions for $\mathrm{U}(V')$ (spherical models)

Let π be an irreducible, admissible representation of $\mathrm{U}(V')$, which admits a spherical model with respect to σ. By definition, this means that there is a nontrivial (continuous) bilinear form $c: V_\pi\times V_\sigma\mapsto\mathbb{C}$, such that for all $g\in\mathrm{U}(V)$, $v_\pi\in V_\pi$, $v_\sigma\in V_\sigma$,

$$c(\pi(g)v_\pi,\sigma(g)v_\sigma)=c(v_\pi,v_\sigma). \tag{4.1}$$

Recall that we view $\mathrm{U}(V)$ as a subgroup of $\mathrm{U}(V')$. By [1], the space of forms satisfying (4.1) is at most one dimensional.

The local Rankin-Selberg integrals from [4] ([5] for unitary groups) corresponding to L-functions for $\mathrm{U}(V')\times\mathrm{Res}_{E/F}\mathrm{GL}_1$, for π with the given form c, have the form

$$\mathcal{L}(v_\pi,\eta_{\sigma,\chi,s})=\int_{\mathrm{U}(V)\backslash\mathrm{U}(V')} c(\pi(h)v_\pi,\eta_{\sigma,\chi,s}(h))dh, \tag{4.2}$$

where $\eta_{\sigma,\chi,s}$ is a smooth holomorphic (polynomial in $q^{\pm s}$ for p-adic F) section in $\mathrm{Ind}_Q^{\mathrm{U}(V_1\oplus H^{1,1})}(\chi|\cdot|^s\otimes\sigma)$, and we view $V\subset V'$ as the subspaces $V_1\subset V_1'\subset V_1\oplus H^{1,1}$. We know that the integral (4.2) converges absolutely for $\mathrm{Re}(s)$ sufficiently large, depending on π, σ only, and that it has a meromorphic continuation to the complex plane, being rational in q^{-s} in the p-adic case.

(We may also take $\eta_{\sigma,\chi,s}$ to be a smooth meromorphic section, rational in q^{-s} in the p-adic case.) Recall that if $E = F$ is p-adic, π, σ and χ are unramified, $v_\pi = v_\pi^0$ is an unramified vector, $\eta_{\sigma,\chi,s} = \eta^0_{\sigma,\chi,s}$ is an unramified section, such that its value at the identity element is v_σ^0, an unramified vector in V_σ, and finally, the form c is such that $c(v_\pi^0, v_\sigma^0) = 1$, then when n is even

$$\mathcal{L}(v_\pi^0, \eta^0_{\sigma,\chi,s}) = \frac{L(\pi \times \chi, s + \frac{1}{2})}{L(\sigma \times \chi, s + 1)},$$

and when n is odd,

$$\mathcal{L}(v_\pi^0, \eta^0_{\sigma,\chi,s}) = \frac{L(\pi \times \chi, s + \frac{1}{2})}{L(\sigma \times \chi, s + 1)L(\chi^2, 2s + 1)}.$$

If E/F is a quadratic unramified extension of the p-adic field F, then with similar assumptions and notations, we have

$$\mathcal{L}(v_\pi^0, \eta^0_{\sigma,\chi,s}) = \frac{L(\pi \times \chi, s + \frac{1}{2})}{L(\sigma \times \chi, s + 1)L(\chi, Asai, 2s + 1)}.$$

Let us substitute in (4.2) the meromorphic section $\Lambda_{v_\sigma, f_{\chi,s}}$ instead of $\eta_{\sigma,\chi,s}$.

Lemma 4.1. *With notation as before, we have, for* $\mathrm{Re}(s)$ *sufficiently large,*

$$\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \int_{\mathrm{U}(V')} c(\pi(h)v_\pi, v_\sigma) f_{\chi,s}(h, I_{V'})dh. \tag{4.3}$$

Proof. Given $v'_\pi \in V_\pi$, the functional on V_σ, $v_\sigma \mapsto c(v'_\pi, v_\sigma)$ is $\mathrm{U}(V)$-smooth. This follows from (4.1). By (3.10), we conclude that for $h \in \mathrm{U}(V')$ (and $\mathrm{Re}(s)$ large),

$$c(\pi(h)v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}(h)) = \int_{\mathrm{U}(V)} f_{\chi,s}(h, g_2)\chi^{-1}(\det(g_2))c(\pi(h)v_\pi, \sigma(g_2)v_\sigma)dg_2.$$

Since $(g_2, g_2) \in P'$, $f_{\chi,s}(h, g_2) = \chi(\det(g_2))f_{\chi,s}(g_2^{-1}h, I_V)$. Using this and (4.1), we get

$$\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \int_{\mathrm{U}(V)\backslash \mathrm{U}(V')} \int_{\mathrm{U}(V)} f_{\chi,s}(g_2^{-1}h, I_V)c(\pi(g_2^{-1}h)v_\pi, v_\sigma)dg_2 dh.$$

Collapsing the integrations, we get

$$\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \int_{\mathrm{U}(V')} c(\pi(h)v_\pi, v_\sigma) f_{\chi,s}(h, I_V)dh. \tag{4.4}$$

This step is justified by the fact that the last integral converges absolutely, for $\mathrm{Re}(s)$ sufficiently large (depending on π, σ). Indeed, by the work of Sakellaridis and Venkatesh [9], the function $c(\pi(h)v_\pi, v_\sigma)$ is of moderate growth in h, which means that there are positive numbers $\alpha = \alpha(v_\pi, \sigma)$ and $d = d(\pi, \sigma)$, such that for all $h \in \mathrm{U}(V')$,

$$|c(\pi(h)v_\pi, v_\sigma)| \leq \alpha \|h\|^d. \tag{4.5}$$

Note that the element (h, I_V) in (4.4) viewed as an element of $\mathrm{U}(V_1') \times \mathrm{U}(V_2)$, which is a subgroup of $\mathrm{U}(V_1 \oplus H^{1,1}) \times \mathrm{U}(V_2) \subset \mathrm{U}(W')$, is the same element as $(h, I_{V'})$, viewed as an element of $\mathrm{U}(V_1') \times \mathrm{U}(V_2') \subset \mathrm{U}(W')$. Thus, the integrand in (4.4) is majorized by $\alpha |f_{\chi,s}(h, I_{V'})| \|h\|^d$. For $\mathrm{Re}(s)$ sufficiently large, depending on d only, the last function is integrable. This is what one proves for the integrals of the doubling method to converge absolutely in a right half plane. □

Remark 4.2. Technically, the last proof is valid in case the group $\mathrm{U}(V)$ is split over F. The reason is that this is the assumption made in [9]. Yiannis Sakellaridis told the second named author that the only obstacle to extending their results to the non-split case is that the theory of compactification of spherical varieties has not been developed yet in this case. However, in our case (and for many other specific spherical varieties) he can easily describe a compactification using the Galois action on the (known) compactification over the algebraic closure, and then the rest will follow verbatim.

The integral (4.3) looks like a typical local integral constructed in the doubling method for the representation π, only that the function $h \mapsto c(\pi(h)v_\pi, v_\sigma)$ is not a matrix coefficient of π. However, we may replace it in the integrand by a matrix coefficient, as follows.

Lemma 4.3. *There is a matrix coefficient of π, $h \mapsto \langle \pi(h)v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot})\rangle$, where $\check{u}(v_\sigma, f_{\chi,\cdot})$ is in the smooth dual of V_π and depends on v_σ and on the section $s \mapsto f_{\chi,s}$, such that, for* $\mathrm{Re}(s)$ *sufficiently large,*

$$\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \int_{\mathrm{U}(V')} \langle \pi(h)v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot})\rangle f_{\chi,s}(h, I_{V'})dh. \tag{4.6}$$

Proof. As before, we may assume that $f_{\chi,s}$ has the following form

$$f_{\chi,s} = \varphi \star f'_{\chi,s} = \int_{\mathrm{U}(V')} \varphi(r)\rho'_{\chi,s}(I_{V'}, r)f'_{\chi,s}dr, \tag{4.7}$$

where $\varphi \in C_c^\infty(\mathrm{U}(V'))$ and $f'_{\chi,s}$ is a smooth holomorphic section in $\rho'_{\chi,s}$. Since, for $r, h \in \mathrm{U}(V')$,

$$f'_{\chi,s}((h, I_{V'})(I_{V'}, r)) = \chi(\det(r))f'_{\chi,s}(r^{-1}h, I_{V'}),$$

we get, for $\mathrm{Re}(s)$ large,

(4.8)

$$\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \int_{\mathrm{U}(V')}\int_{\mathrm{U}(V')} \chi(\det(r))\varphi(r)c(\pi(rh)v_\pi, v_\sigma)f'_{\chi,s}(h, I_{V'})drdh.$$

The functional

$$v'_\pi \mapsto \int_{\mathrm{U}(V')} \chi(\det(r))\varphi(r)c(\pi(r)v'_\pi, v_\sigma)dr = c(\pi(\varphi\cdot\chi(det))v'_\pi, v_\sigma)$$

is a smooth functional on V_π, and hence there is a unique vector $\check{u}(v_\sigma, f_{\chi,\cdot})$ in the smooth dual of V_π, such that, for all $v'_\pi \in V_\pi$,

$$c(\pi(\varphi \cdot \chi(det))v'_\pi, v_\sigma) = \langle v'_\pi, \check{u}(v_\sigma, f_{\chi,\cdot})\rangle. \tag{4.9}$$

Thus, the function $h \mapsto \int_{\mathrm{U}(V')} \chi(\det(r))\varphi(r)c(\pi(rh)v_\pi, v_\sigma)dr$ is the matrix coefficient of π, $h \mapsto \langle \pi(h)v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot})\rangle$. □

The integral on the right hand side of (4.6) is now an integral which arises in the doubling method for the representation π on $\mathrm{U}(V')$. We know that, as such, it has a meromorphic continuation to the whole plane, where in the p-adic case it is a rational function of q^{-s}. Of course, we have already known this, since $\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}})$ is such a function. We will also denote this meromorphic function by $L(v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot}), f_{\chi,s})$.

5. Applying intertwining operators

In Lemma 4.3, we obtained the equality of two local integrals, one arising from Rankin-Selberg integrals for π on $\mathrm{U}(V')$, expressed in terms of spherical models of π, and the other local integral is arising from the doubling method for π on $\mathrm{U}(V')$. For both families of local integrals, we have local functional equations obtained by applying intertwining operators to the sections which appear in the integrals. We have two parabolic inductions to consider, one for each local integral, and hence we need to consider two local intertwining operators. Our main goal is to find the relation between the two proportionality factors in the two local functional equations.

Let us write the two local functional equations for π without normalization of intertwining operators.

$$\Gamma^{sph}(\pi, \sigma, \chi, s + \frac{1}{2})\mathcal{L}(v_\pi, \Lambda_{v_\sigma, f_{\chi,s}}) = \mathcal{L}(v_\pi, M_{\chi,\sigma,s}(\Lambda_{v_\sigma, f_{\chi,s}})), \tag{5.1}$$

where $M_{\chi,\sigma,s}$ is the intertwining operator on $\mathrm{Ind}_Q^{\mathrm{U}(V_1 \oplus H^{1,1})}(\chi|\cdot|^s \otimes \sigma)$, corresponding to the Weyl element w in $\mathrm{U}(V_1 \oplus H^{1,1})$, which flips (u_0, u_0), $(u_0, -u_0)$ and acts as the identity on V_1. $\Gamma^{sph}(\pi, \sigma, \chi, s + \frac{1}{2})$ depends on the choice of a Haar measure on N_Q, the unipotent radical of Q.

$$\Gamma^{dbl}(\pi, \chi, s + \frac{1}{2})L(v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot}), f_{\chi,s}) = L(v_\pi, \check{u}(v_\sigma, f_{\chi,\cdot}), M_{\chi,s}(f_{\chi,s})), \tag{5.2}$$

where $M_{\chi,s}$ is the intertwining operator on $\rho'_{\chi,s}$ corresponding to the Weyl element $(I_{V'}, -I_{V'})$ in $\mathrm{U}(W')$. Note that this element flips V'_Δ and $V'_{-\Delta}$. (With respect to the basis (5.3) below, its matrix, modulo the diagonal subgroup, is the standard long Weyl element.) Again, $\Gamma^{dbl}(\pi, \chi, s + \frac{1}{2})$ depends on the choice of Haar measure on $N_{P'}$, the unipotent radical of P'. The functions $\Gamma^{sph}(\pi, \sigma, \chi, s + \frac{1}{2})$, $\Gamma^{dbl}(\pi, \chi, s + \frac{1}{2})$ are meromorphic. When F is p-adic, they are rational in q^{-s}. The main theorem of this paper is

Theorem 5.1. *There are compatible choices of Haar measures on the unipotent radicals N_Q, N_P, $N_{P'}$, such that*

$$\Gamma^{sph}(\pi,\sigma,\chi,s)=\frac{\Gamma^{dbl}(\pi,\chi,s)}{\Gamma^{dbl}(\hat{\sigma},\chi,s-\frac{1}{2})}.$$

The rest of this section is devoted to the proof of the theorem.
We will pass to coordinates using a basis of W' as follows. Start with an orthogonal basis $\{v_1,...,v_n\}$ of V and then take
(5.3)

$$\{(u_0,u_0),(v_1,v_1),...,(v_n,v_n),\frac{1}{2\alpha_n}(v_n,-v_n),...,\frac{1}{2\alpha_1}(v_1,-v_1),\frac{1}{2\alpha_0}(u_0,-u_0)\},$$

where $\alpha_i=b'(v_i,v_i)$, $0\le i\le n$. The Gram matrix of this basis is ϵ_{2n+2}, where ϵ_m is the $m\times m$ permutation matrix, which has 1 along the main skew diagonal.
We start with the functional equation (5.1). We consider the intertwining operator, given for $\mathrm{Re}(s)$ large by

$$M_{\chi,\sigma,s}(\Lambda_{v_\sigma,f_{\chi,s}})(h)=\int_{N_Q}\Lambda_{v_\sigma,f_{\chi,s}}(wuh)du.$$

We will choose a Haar measure on N_Q after passing to coordinates. The matrix of (w,I_V) with respect to the basis (5.3) is

$$J_w=\begin{pmatrix} & & \frac{1}{2\alpha_0}\\ & I_{2n} & \\ 2\alpha_0 & & \end{pmatrix}.$$

As in (2.5), for $u\in N_Q$, the matrix of (u,I_V) has the following form

$$N_u=\begin{pmatrix}1 & x & x(2D)^{-1}\epsilon_n & y\\ & I_n & 0 & -(2D)^{-1}\bar{x}^t\\ & & I_n & -\epsilon_n\bar{x}^t\\ & & & 1\end{pmatrix},$$

where $D=diag(\alpha_1,...,\alpha_n)$, $x=(x_1,...,x_n)\in E^n$, and $y+\bar{y}=-\sum_{i=1}^n\frac{1}{\alpha_i}x_i\bar{x}_i$. Thus, $y=-\sum_{i=1}^n\frac{1}{2\alpha_i}x_i\bar{x}_i+\mathrm{Im}(y)$, where $\overline{\mathrm{Im}(y)}=-\mathrm{Im}(y)$. In case $E=F$, this means that $\mathrm{Im}(y)=0$. We have
(5.4)

$$J_wN_u=\begin{pmatrix}1 & & & \\ -\alpha_0D^{-1}\bar{x}^t & I_n & & \\ & & I_n & \\ & & \alpha_0xD^{-1}\epsilon_n & 1\end{pmatrix}J_w\begin{pmatrix}1 & x & 0 & z\\ & I_n & 0 & 0\\ & & I_n & -\epsilon_n\bar{x}^t\\ & & & 1\end{pmatrix},$$

where $z=y+\sum_{i=1}^n\frac{1}{2\alpha_i}x_i\bar{x}_i$ (so that $z+\bar{z}=0$). Denote the third matrix in the r.h.s. of (5.4) by $u_1(x,z)$. Since $\det\begin{pmatrix}1 & \\ -\alpha_0D^{-1}\bar{x}^t & I_n\end{pmatrix}=1$, we get, for

$g_2 \in \mathrm{U}(V)$, $h \in \mathrm{U}(V')$,

$$f_{\chi,s}(wuh, g_2) = f_{\chi,s}(J_w u_1(x,z)(h,g_2)).$$

Thus, for $\mathrm{Re}(s)$ large,

(5.5) $M_{\chi,\sigma,s}(\Lambda_{v_\sigma, f_{\chi,s}})(h) =$
$$\int_{E^n \times F^{dim E-1}} \int_{\mathrm{U}(V)} f_{\chi,s}(J_w u_1(x,z)(h,g_2))\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2 d(x,z).$$

Here, when $E = F$, $z = 0$ and there is no dz-integration, and when $[E : F] = 2$, the dz-integration is over $z \in E$ with $z + \bar{z} = 0$, a subspace which we identify with F. We fix a choice of a Haar measure dt on E, and when $[E : F] = 2$, we choose a Haar measure dz on F. We let $d(x,z) = dx_1...dx_n dz$- the product measure. Again, when $E = F$, there is no dz integration. It is easy to see that the integral (5.5) converges absolutely for $\mathrm{Re}(s)$ large in the sense explained in the end of Section 3. We may then switch the order of integration in (5.5) and get, for $\mathrm{Re}(s)$ large,

$$M_{\chi,\sigma,s}(\Lambda_{v_\sigma, f_{\chi,s}})(h) = \int_{\mathrm{U}(V)} f'_{\chi,s}(h,g_2)\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2, \tag{5.6}$$

where, for $\mathrm{Re}(s)$ large, $g \in \mathrm{U}(W')$,

$$f'_{\chi,s}(g) = \int_{E^n \times F^{dim E-1}} f_{\chi,s}(J_w u_1(x,z)g)d(x,z). \tag{5.7}$$

This is an intertwining integral on $\rho'_{\chi,s}$, with respect to the Weyl element J_w. Note that the integration can be realized along the unipotent radical of the parabolic subgroup of $\mathrm{U}(W')$, preserving the line through (u_0, u_0), modulo the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & x & 0 \\ & I_n & 0 & -\epsilon_n \bar{x}^t \\ & & I_n & 0 \\ & & & 1 \end{pmatrix}, \ x \in E^n.$$

Thus, $f'_{\chi,s}(g)$ has meromorphic continuation to the whole plane, and in the p-adic case, it is rational in q^{-s}. Keep denoting this meromorphic function by $f'_{\chi,s}$. Denote by $\tilde{f}_{\chi,s}$ the restriction of $f'_{\chi,s}$ to $\mathrm{U}(W)$. Then it is easy to check that $\tilde{f}_{\chi,s}$ is an element of $\rho_{\chi,s-\frac{1}{2}}$. Denote $h \cdot f_{\chi,s} = \rho'_{\chi,s}(h, I_V)f_{\chi,s}$. Then, we may write the r.h.s. of (5.6) as

$$\int_{\mathrm{U}(V)} \widetilde{h \cdot f_{\chi,s}}(I_V, g_2)\chi^{-1}(\det(g_2))\sigma(g_2)v_\sigma dg_2,$$

and this integral, which converges absolutely for $\mathrm{Re}(s)$ large, has a meromorphic continuation to the complex plane, rational in q^{-s} in the p-adic case. We denote it (as a meromorphic function) by $\tilde{\Lambda}_{v_\sigma, f_{\chi,s}}(h)$. Thus (5.6) reads as

$$M_{\chi,\sigma,s}(\Lambda_{v_\sigma, f_{\chi,s}})(h) = \tilde{\Lambda}_{v_\sigma, f_{\chi,s}}(h). \tag{5.8}$$

Let us denote by $\tilde{M}_{\chi, s-\frac{1}{2}}$ the intertwining operator on $\rho_{\chi, s-\frac{1}{2}}$ analogous to $M_{\chi,s}$. Each one comes with a choice of a Haar measure on the relevant unipotent radical. Denote

$$\tilde{M}_{\chi,s-\frac{1}{2}}(\tilde{f}_{\chi,s})(h, g_2) = \tilde{M}_{\chi,s-\frac{1}{2}}(\widetilde{h \cdot f_{\chi,s}})(I_V, g_2).$$

Note, again, that as elements in $\mathrm{U}(W')$, (I_V, g_2) and $(I_{V'}, g_2)$ are the same. We may now consider, first for $\mathrm{Re}(s)$ small, and then as meromorphic functions,

$$\tilde{\Lambda}_{v_\sigma, \tilde{M}_{\chi,s-\frac{1}{2}}\tilde{f}_{\chi,s}}(h) = \int_{\mathrm{U}(V)} \tilde{M}_{\chi,s-\frac{1}{2}}(\tilde{f}_{\chi,s})(h, g_2)\chi(\det(g_2))\sigma(g_2)v_\sigma dg_2.$$

Lemma 5.2. *For all $h \in \mathrm{U}(V')$, v_σ, $f_{\chi,s}$,*

$$\Gamma^{dbl}(\hat{\sigma}, \chi, s)\tilde{\Lambda}_{v_\sigma, f_{\chi,s}}(h) = \tilde{\Lambda}_{v_\sigma, \tilde{M}_{\chi,s-\frac{1}{2}}\tilde{f}_{\chi,s}}(h). \tag{5.9}$$

Proof. Let $\xi_{\hat{\sigma}}$ be an element in $V_{\hat{\sigma}}$. For $\mathrm{Re}(s)$ sufficiently large,

$$\begin{aligned} \langle \tilde{\Lambda}_{v_\sigma, f_{\chi,s}}(h), \xi_{\hat{\sigma}} \rangle &= \int_{\mathrm{U}(V)} \langle \sigma(g_2)v_\sigma, \xi_{\hat{\sigma}} \rangle \chi^{-1}(\det(g_2))\widetilde{h \cdot f_{\chi,s}}(I_V, g_2)dg_2 \\ &= \int_{\mathrm{U}(V)} \langle v_\sigma, \hat{\sigma}(g_2)\xi_{\hat{\sigma}} \rangle \widetilde{h \cdot f_{\chi,s}}(g_2, I_V)dg_2. \end{aligned}$$

The last integral is a local integral arising in the doubling method for $\hat{\sigma}$. Now apply the local functional equation, as in (5.2), with $\hat{\sigma}$ replacing π, and $s - \frac{1}{2}$ instead of s. We get

$$\Gamma^{dbl}(\hat{\sigma}, \chi, s)\langle \tilde{\Lambda}_{v_\sigma, f_{\chi,s}}(h), \xi_{\hat{\sigma}} \rangle = \langle \tilde{\Lambda}_{v_\sigma, \tilde{M}_{\chi,s-\frac{1}{2}}\tilde{f}_{\chi,s}}(h), \xi_{\hat{\sigma}} \rangle.$$

Since this is true for all $\xi_{\hat{\sigma}}$, we get (5.9). □

Recall that $\Gamma^{dbl}(\hat{\sigma}, \chi, s)$ depends on the choice of a Haar measure on N_P. From (5.8) and Lemma 5.2, we get

$$\Gamma^{dbl}(\hat{\sigma}, \chi, s)M_{\chi,\sigma,s}(\Lambda_{v_\sigma, f_{\chi,s}})(h) = \tilde{\Lambda}_{v_\sigma, \tilde{M}_{\chi,s-\frac{1}{2}}\tilde{f}_{\chi,s}}(h). \tag{5.10}$$

Lemma 5.3. *There is a compatible choice of Haar measures on N_P, $N_{P'}$, such that*

$$\tilde{M}_{\chi,s-\frac{1}{2}}(\tilde{f}_{\chi,s})(h, g_2) = M_{\chi,s}(f_{\chi,s})(h, g_2). \tag{5.11}$$

Proof. It is enough to prove (5.11) for $g_2 = I_V$, $h = I_{V'}$ and for $\mathrm{Re}(s)$ sufficiently large (so that the following integral converges absolutely). In this case, the l.h.s. of (5.11) is

$$(5.12) \qquad \int_{Y_n} \tilde{f}_{\chi,s}(\begin{pmatrix} 1 & & \\ & J_{2n,D} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & I_n & T & \\ & & I_n & \\ & & & 1 \end{pmatrix})dT.$$

Here Y_n is the space of matrices $T \in M_n(E)$, such that $(\overline{\epsilon_n T})^t = \epsilon_n T$. Y_n is isomorphic to N_P. The matrix $J_{2n,D}$ is the matrix of $(I_V, -I_V)$ as an element of $\mathrm{U}(W)$.

$$J_{2n,D} = \begin{pmatrix} & \epsilon_n (2D)^{-1} \\ 2D\epsilon_n & \end{pmatrix}.$$

We fix a choice of a Haar measure dT, for example a product of Haar measures on E, F, according to the matrix coordinates of T. Denote

$$J^1_{2n,D} = \begin{pmatrix} 1 & & \\ & J_{2n,D} & \\ & & 1 \end{pmatrix}.$$

Denote, also, for $T \in Y_n$,

$$v_n(T) = \begin{pmatrix} 1 & & & \\ & I_n & T & \\ & & I_n & \\ & & & 1 \end{pmatrix}.$$

By definition, $\tilde{f}_{\chi,s}$ is the restriction of (5.7) to $\mathrm{U}(W)$. Substitute this in (5.12), and we get, for $\mathrm{Re}(s)$ large,

$$(5.13) \qquad \int_{Y_n} \int_{E^n \times F^{dimE-1}} f_{\chi,s}(J_w u_1(x,z) J^1_{2n,D} v_n(T)) d(x,z) dT.$$

We have

$$u_1(x,z) J^1_{2n,D} = J^1_{2n,D} \begin{pmatrix} 1 & 0 & x\epsilon_n (2D)^{-1} & z \\ & I_n & 0 & -\epsilon_n (2D)^{-1} \epsilon_n \bar{x}^t \\ & & I_n & 0 \\ & & & 1 \end{pmatrix},$$

and

$$J_w J^1_{2n,D} = J_{2n+2,D'} = \begin{pmatrix} & \epsilon_{n+1} (2D')^{-1} \\ 2D' \epsilon_{n+1} & \end{pmatrix},$$

where $D' = diag(\alpha_0, D)$. Note that $J_{2n+2,D'}$ is the matrix of $(I_{V'}, -I_{V'})$ according to the basis (5.3). Hence, in the integrand of (5.13),

$$J_w u_1(x,z) J^1_{2n,D} v_n(T) = J_{2n+2,D'} \begin{pmatrix} 1 & 0 & x\epsilon_n (2D)^{-1} & z \\ & I_n & T & -\epsilon_n (2D)^{-1} \epsilon_n \bar{x}^t \\ & & I_n & 0 \\ & & & 1 \end{pmatrix}.$$

Change variable $x \mapsto x(2D)\epsilon_n$. The measure $d(x,z)$ changes by $|2|^n|\det(D)|$, and (5.13) becomes

$$(5.14)\qquad |2|^n|\det(D)|\int_{Y_{n+1}} f_{\chi,s}(\epsilon_{2n+2}\begin{pmatrix} I_{n+1} & Y \\ & I_{n+1}\end{pmatrix})dY,$$

where dY is the product measure $d\begin{pmatrix} x & z \\ T & -\epsilon_n\bar{x}^t\end{pmatrix} = d(x,z)dT$. This defines a Haar measure on $N_{P'}$, such that (5.14) is equal to

$$\int_{N_{P'}} f_{\chi,s}((I_{V'},-I_{V'})u)du = M_{\chi,s}(f_{\chi,s})(I).$$

□

From Lemma 5.3, (5.10) and the definition of $\tilde{\Lambda}_{v_\sigma,\tilde{M}_{\chi,s-\frac{1}{2}}\tilde{f}_{\chi,s}}(h)$, we conclude that

$$(5.15)\qquad \Gamma^{dbl}(\hat{\sigma},\chi,s)M_{\chi,\sigma,s}(\Lambda_{v_\sigma,f_{\chi,s}})(h) = \Lambda_{v_\sigma,M_{\chi,s}(f_{\chi,s})}(h).$$

Now, multiply (5.1) by $\Gamma^{dbl}(\hat{\sigma},\chi,s)$. By (5.15), we get

$$(5.16)\quad \Gamma^{dbl}(\hat{\sigma},\chi,s)\Gamma^{sph}(\pi,\sigma,\chi,s+\frac{1}{2})\mathcal{L}(v_\pi,\Lambda_{v_\sigma,f_{\chi,s}}) = \mathcal{L}(v_\pi,\Lambda_{v_\sigma,M_{\chi,s}(f_{\chi,s})}).$$

By Lemma 4.3, we can rewrite (5.16) as

$$(5.17)\qquad \begin{aligned}&\Gamma^{dbl}(\hat{\sigma},\chi,s)\Gamma^{sph}(\pi,\sigma,\chi,s+\frac{1}{2})L(v_\pi,\check{u}(v_\sigma,f_{\chi,\cdot}),f_{\chi,s})\\ &= L(v_\pi,\check{u}(v_\sigma,M_{\chi,\cdot}(f_{\chi,\cdot})),M_{\chi,s}(f_{\chi,s})).\end{aligned}$$

In the proof of Lemma 4.3, we see that the vector $\check{u}(v_\sigma,f_{\chi,\cdot})$ can be taken to be the same vector as $\check{u}(v_\sigma,M_{\chi,\cdot}(f_{\chi,\cdot}))$. Indeed, according to the proof, we need to write $f_{\chi,s}$ as a finite sum of convolutions $\varphi_i\star f^i_{\chi,s}$, $1\le i\le N$, as in the proof of the lemma. The point is that

$$M_{\chi,s}(\varphi_i\star f^i_{\chi,s}) = \varphi_i\star M_{\chi,s}(f^i_{\chi,s}),$$

for all i. Now, look at (4.8), for $M_{\chi,s}(f^i_{\chi,s})$, and $\mathrm{Re}(s)$ small, and then look at (4.9) to see that each vector $\check{u}(v_\sigma,M_{\chi,\cdot}(f^i_{\chi,\cdot}))$ is equal to $\check{u}(v_\sigma,f^i_{\chi,\cdot})$, and hence

$$\check{u}(v_\sigma,M_{\chi,\cdot}(f_{\chi,\cdot})) = \sum_{i=1}^N \check{u}(v_\sigma,M_{\chi,\cdot}(f^i_{\chi,\cdot})) = \sum_{i=1}^n \check{u}(v_\sigma,f^i_{\chi,\cdot}) = \check{u}(v_\sigma,f_{\chi,\cdot}).$$

Now, (5.17) becomes

$$(5.18)\qquad \begin{aligned}&\Gamma^{dbl}(\hat{\sigma},\chi,s)\Gamma^{sph}(\pi,\sigma,\chi,s+\frac{1}{2})L(v_\pi,\check{u}(v_\sigma,f_{\chi,\cdot}),f_{\chi,s})\\ &= L(v_\pi,\check{u}(v_\sigma,f_{\chi,\cdot}),M_{\chi,s}(f_{\chi,s})).\end{aligned}$$

From (5.2), we conclude that

$$\Gamma^{dbl}(\pi,\chi,s+\frac{1}{2})=\Gamma^{dbl}(\hat{\sigma},\chi,s)\Gamma^{sph}(\pi,\sigma,\chi,s+\frac{1}{2}).$$

This concludes the proof of Theorem 5.1.

Let ψ be a nontrivial character of F. In [6, Sections 5,6,9], the local gamma factors $\gamma(\pi\times\chi,s,\psi)$ and $\gamma(\hat{\sigma}\times\chi,s,\psi)$ are obtained by dividing the doubling Γ factors above by certain meromorphic functions (rational functions of q^{-s} in the p-adic case), which do not depend on the representations:

$$\gamma(\pi\times\chi,s,\psi)=\frac{\Gamma^{dbl}(\pi,\chi,s)}{\beta_{\mathrm{U}(V')}(\chi,s,\psi)};$$

$$\gamma(\hat{\sigma}\times\chi,s,\psi)=\frac{\Gamma^{dbl}(\hat{\sigma},\chi,s)}{\beta_{\mathrm{U}(V)}(\chi,s,\psi)}.$$

The functions β in the denominators depend on the choice of Haar measures defining the intertwining operators above, so that the local gamma factors are independent of these choices. Thus, from Theorem 5.1 follows

Theorem 5.4.

$$\Gamma^{sph}(\pi,\sigma,\chi,s)=\frac{\beta_{\mathrm{U}(V')}(\chi,s,\psi)}{\beta_{\mathrm{U}(V)}(\chi,s-\frac{1}{2},\psi)}\cdot\frac{\gamma(\pi\times\chi,s,\psi)}{\gamma(\hat{\sigma}\times\chi,s-\frac{1}{2},\psi)}. \tag{5.19}$$

We remark that the denominator in (5.19) is, in case $\mathrm{U}(V)$ is an even orthogonal group, the gamma factor related to the intertwining operator $M_{\chi,\sigma,s}$. In the other cases, there is a missing gamma factor related to χ, for example, when $\mathrm{U}(V)$ is an odd orthogonal group, $\gamma(\chi^2,2s-1,\psi)$. To remedy this, one needs to write a better expression, suited to our setup, of the ratio of the functions β in (5.19), and, also, find a "natural" normalization of $M_{\chi,\sigma,s}$.

Finally, let ω_π, ω_σ denote the central characters of π, σ. We know from [8, 2], that when F is p-adic, $\omega_\pi(-1)\Gamma^{dbl}(\pi,\chi,s)$ is stable, in the sense that if τ is another irreducible admissible representation of $\mathrm{U}(V')$, then for χ sufficiently ramified,

$$\omega_\pi(-1)\Gamma^{dbl}(\pi,\chi,s)=\omega_\tau(-1)\Gamma^{dbl}(\tau,\chi,s).$$

Similarly, of course, for $\omega_\sigma(-1)\Gamma^{dbl}(\hat{\sigma},\chi,s)$. Thus, we conclude from Theorem 5.1:

Theorem 5.5. *Assume that F is p-adic. Then $\omega_\pi\omega_\sigma(-1)\Gamma^{sph}(\pi,\sigma,\chi,s)$ is stable.*

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