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Author(s):

M. Ashenab and H. Momenaee Kermani

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CHARACTERIZATION OF 2×2 FULL DIVERSITY SPACE-TIME CODES AND INEQUIVALENT FULL RANK SPACES

M. ASHENAB AND H. MOMENAEE KERMANI*

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Abstract. In wireless communication systems, space-time codes are applied to encode data when multiple antennas are used in the receiver and transmitter. The concept of diversity is very crucial in designing space-time codes. In this paper, using the equivalent definition of full diversity space-time codes, we first characterize all real and complex 2×2 rate one linear dispersion space-time block codes that are full diversity. This characterization is used to construct full diversity codes which are not derived from Alamouti scheme. Then, we apply our results to characterize all real subspaces of $M_2(\mathbb{C})$ and $M_2(\mathbb{R})$ whose nonzero elements are invertible. Finally, for any natural number $n > 1$, we construct infinitely many inequivalent subspaces of $M_n(\mathbb{C})$ whose nonzero elements are invertible.
Keywords: Space-time coding, linear dispersion, full diversity, full rank.
MSC(2010): Primary: 15A03; Secondary: 94A05, 15A15.

1. Introduction

The use of multiple antennas in wireless communication systems has been shown to be an effective approach in order to improve the performance of the system. To encode data in a Multiple Input Multiple Output (MIMO) communication system, space-time codes have been introduced. Since the introduction of these codes a lot of effort has been devoted to apply different areas of mathematics such as linear algebra, Galois theory, noncommutative algebra, number theory etc. to construct codes satisfying certain conditions [4, 6, 9–11, 15, 16]. Among these areas, linear algebra plays the key role since these codes are in fact matrices whose entries are linear combinations of variables. That is:

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*Corresponding author.

Definition 1.1. Let t, n and k be natural numbers. A linear dispersion (LD) space-time block code is a $t \times n$ matrix $C(s)$ whose entries are linear combinations of complex variables $s_1, \dots, s_k$ and their conjugates $\bar{s_1}, \dots, \bar{s_k}$ where $s = (s_1, \dots, s_k)$. A real LD code is a $t \times n$ matrix $C(x)$ whose entries are linear combinations of real variables $x_1, \dots, x_k$ with real coefficients where $x = (x_1, \dots, x_k)$.

In this definition, n represents the number of antennas in the transmitter and t is the number of time slots used to transmit k symbols. The rate of C is defined as $R = \frac{k}{t}$. One of the main advantages of MIMO systems is that they are capable of providing higher *diversity* at the receiver [7]. *Rank criterion* [18] is the main motivation for the mathematical definition of a *full diversity* space-time code.

Definition 1.2. An LD code is called full diversity on the set $S \subseteq \mathbb{C}$ (constellation set) if for any two distinct vectors $z, w \in S^k$ the determinant of the $n \times n$ matrix $(C(z)-C(w))^*(C(z)-C(w))$ is nonzero. We call an LD code full diversity if it is full diversity on $\mathbb{C}$; Similarly a real LD code is called full diversity if $\det(C(z) - C(w))^T(C(z) - C(w)) \neq 0$ for all distinct vectors $z, w \in \mathbb{R}^k$.

Although most of the well-known LD codes such as the golden code and LD codes based on division algebras [4,16] are full diversity on certain subsets of $\mathbb{C}$, those LD codes which are full diversity on the whole complex numbers are of special importance since they can be applied in different communication systems using different constellation sets. According to Definition 1.2, a square LD code $C(s)$ is full diversity on $\mathbb{C}$ if and only if $\det(C(z)) \neq 0$ for every vector $0 \neq z \in \mathbb{C}^k$. Apparently any full diversity LD code defines a real vector space whose nonzero matrices are full rank. We call such spaces *full rank* spaces. Conversly, choosing a basis for a full rank space one can define a full diversity LD code. Therefore, the maximum rates of full diversity real and complex $n \times n$ LD codes are equal to the maximum dimensions of real and complex full rank spaces divided by n, which are equal to $\frac{\rho(n)}{n}$ and $\frac{b+1}{n}$ respectively where $n = (2a+1)2^b$, $b = c + 4d$, $\rho(n) = 2^c + 8d$ and a, b, c, d are nonnegative integers with $0 \leq c < 4$ [1]. So, the maximum rate of 2×2 real and complex full diversity LD codes is 1. Full diversity LD codes with maximum rate are called *full diversity full rate*. Multiplying a full diversity full rate LD code by fixed invertible matrices P and Q from left and right respectively, one can obtain a new full diversity full rate LD code. The class of orthogonal space-time codes is an important class of full diversity LD codes [11,15]. Alamouti code $C = \begin{pmatrix} s_1 & s_2 \\ -\bar{s_2} & \bar{s_1} \end{pmatrix}$ is one of these LD codes [2]. In fact for any two complex numbers $u, v \neq 0$ the LD code

$$C = \begin{pmatrix} s_1 & s_2 \\ -v\bar{s_2} & u\bar{s_1} \end{pmatrix}$$

is a full diversity full rate linear dispersion (FDFRLD) code if u is not a negative multiple of v. One may conjecture that all 2×2 FDFRLD codes have the form

$$C = P\begin{pmatrix} f & g \\ -v\bar{g} & u\bar{f} \end{pmatrix}Q, \tag{1.1}$$

where f and g are linear combinations of $s_1, s_2, \bar{s_1}$ and $\bar{s_2}$. In this paper, we first characterize all real full diversity LD codes and then by characterizing all 2×2 FDFRLD codes we show that there exist 2×2 FDFRLD codes which are not of form (1.1). It is worth to mention that full diversity LD codes are usually of high performance in practice. Therefore, once full diversity LD codes are characterized, this characterization can be used to construct high performance LD codes with additional properties. For instance, the well known Sezginer's code [17], which has a low decoding complexity, is the summation of two full diversity LD codes characterized in Theorem 3.5.

In the last section we study full rank spaces. Spaces of matrices whose nonzero elements have constant rank k have been studied by many researchers under the name of $k-$*spaces* [3, 5, 12, 13, 20, 21]. Two $k-$spaces V and W in $M_n(\mathbb{C})$ are called equivalent if $V = PWQ$ for some invertible matrices $P, Q \in M_n(\mathbb{C})$. The construction of inequivalent $k-$spaces has been also considered [3]. We characterize 2×2 full rank spaces and construct infinitely many $n\times n$ inequivalent full rank spaces of $M_n(\mathbb{C})$.

Notations: Superscripts $(\cdot)^T$, $(\cdot)^*$, and $\bar{(\cdot)}$ indicate transpose, Hermitian transpose and complex conjugations, respectively. I_a denotes the identity matrix of size $a\times a$ and $M_a(\mathbb{R})$ denotes the set of all $a\times a$ matrices with real entries. $\mathbb{C}$ stands for the complex field and $\mathbb{R}$ stands for the field of real numbers. The notation $\|\cdot\|$ indicates ordinary vector norm. The notations $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote determinant and trace of a matrix respectively. For a set $X \subseteq M_n(\mathbb{C})$, the term $\mathrm{span}_{\mathbb{R}}(X)$ denotes the real span of X.

2. Characterization of 2×2 real full diversity full rate linear dispersion codes

In this section we characterize all 2×2 real FDFRLD codes. We start with a simple lemma.

Lemma 2.1. *If $C(x)$ is a full diversity full rate 2×2 real LD code then $det(C(x)) = f^2 + g^2$ or $det(C(x)) = -f^2 - g^2$ where f and g are linear combinations of x_1 and x_2.*

Proof. Since $\det(C(x))$ is a quadratic form, by [14, Theorem 3.1.5], $\det(C(x))$ has one of the forms $f^2 + g^2$, $-f^2 - g^2$ or $f^2 - g^2$ but since this code is full diversity the last one is not possible and so the lemma has been proved. □

Theorem 2.2. *Every rate one full diversity 2×2 real LD code $C(x)$ has the form*

$$P\begin{pmatrix} f & g \\ -g & f \end{pmatrix}Q, \tag{2.1}$$

where $[f, g]^T = A[x_1, x_2]^T$ and A, P and Q are real 2×2 invertible matrices.

Proof. Without loss of generality by Lemma 2.1 assume that $\det(C) = f^2 + g^2$. Let $[f, g]^T = A[x_1, x_2]^T$. Since the code is full diversity, A is invertible and so from $[x_1, x_2]^T = A^{-1}[f, g]^T$, we can write the entries of the code as linear combinations of f and g. Hence the code can be written as $C = fA_1 - gA_2$ where $A_1, A_2 \in M_2(\mathbb{R})$. Note that since A is invertible, f and g can take all real values independently. If we let $f = 1$ and $g = 0$, we conclude that $\det(A_1) = 1$. Multiplying the code by A_1^{-1} dose not change the determinant and so we can assume that $A_1 = I_2$. Let $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so the code takes the form

$$\begin{pmatrix} f - ag & -bg \\ -cg & f - dg \end{pmatrix}$$

and so $f^2 + g^2 = \det(C) = f^2 - (a + d)fg + (ad - bc)g^2$. Therefore $\mathrm{tr}(A_2) = a + d = 0$ and $\det(A_2) = ad - bc = 1$. So the characteristic polynomial of A_2 is $x^2 + 1$. Using the rational form of A_2, there exists a nonsingular matrix $S \in M_2(\mathbb{R})$ such that $SA_2S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Multiplying the code by S from the left and by S^{-1} from the right leaves $A_1 = I_2$ and so the code takes the form (2.1). □

3. Characterization of 2×2 complex full diversity full rate codes

In this section we characterize all 2×2 FDFRLD codes. We start with the following definition.

Definition 3.1. Given a square LD code $C(s)$, the determinant area of $C(s)$ is defined as the set $\det(C(s)) = \{\det(C(z)) : z \in \mathbb{C}^k\}$.

If we consider the determinant of an LD code as a continuous function from $\mathbb{C}^k$ to $\mathbb{C}$, then the image of this function is the determinant area of the code. The determinant area of Alamouti code is the set of nonnegative numbers. But, the determinant area of the code $C = \begin{pmatrix} s_1 & s_2 \\ -v\bar{s_2} & u\bar{s_1} \end{pmatrix}$ when u and v are two fixed complex numbers, is the set of all nonnegative combinations of u and v as illustrated in Figure 1.

Definition 3.2. A subset D of a real vector space is called a convex cone if for any two vectors x and $y \in D$ and any pair of positive scalars α and β we

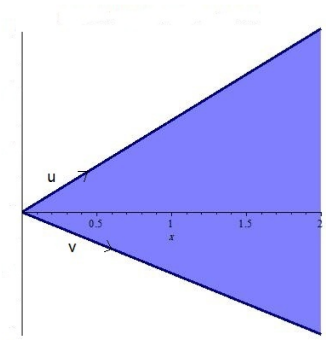

FIGURE 1. proper closed convex cone

have $\alpha x + \beta y \in D$. We call a convex cone a proper closed convex cone if it is closed and $-z \notin D$ for any $0 \neq z \in D$.

Let P and Q be two invertible $n \times n$ matrices and $C(s)$ be a full diversity LD code. Then, $PC(s)Q$ is also full diversity and its determinant area is just a rotation of the determinant area of $C(s)$. In the following theorem the determinant area of a 2×2 full diversity LD code is described.

Theorem 3.3. *The determinant area of a 2×2 FDFRLD code $C(s)$ is a proper closed convex cone.*

Proof. To prove the theorem we first prove that the determinant area is closed and connected. To this end, note that the function $f(s) = \det(C(s))$ is a continuous function from $\mathbb{C}^2$ to $\mathbb{C}$, $f(0) = 0$, if $s \neq 0$ then $f(s) \neq 0$, and for every $t \in \mathbb{R}$, $f(ts) = t^2 f(s)$. Let $A = \{s \in \mathbb{C}^2 : \| s \| \leq 1\}$. A is a compact set so $f(A)$ is a compact set as well. For every $s \in \mathbb{C}^2$ there exists $\varepsilon > 0$ such that $\varepsilon s \in A$ and so $\varepsilon^2 f(s) \in f(A)$. So, $\det(C(s)) = \{My : M \geq 0\ ,\ y \in f(A)\}$. Now $\det(C(s))$ is a closed set because if $M_k y_k \to z$ then since $f(A)$ is compact, there exists a subsequence y_{k_n} such that $y_{k_n} \to y \in f(A)$. We can assume that $y_k \neq 0$. So there exists $M \in \mathbb{R}$ such that $M_{k_n} \to M$ and $M_{k_n} y_{k_n} \to My = z$. So $z \in \det(C(s))$. For the connectedness, note that the set $\mathbb{C}^2 \setminus \{0\}$ is a connected set and hence the set $f(\mathbb{C}^2 \setminus \{0\}) = \det(C) \setminus \{0\}$ is also connected.
Now we prove that if $0 \neq z \in \det(C(s))$ then $-z \notin \det(C(s))$. We consider the real vector space corresponding to $C(s)$. By contradiction suppose that there exist two nonzero matrices A_1 and A_2 in this space such that $\det(A_1) = -\det(A_2)$. Multiplying the code by A_1^{-1}, we have a new code whose corresponding vector space, called V, contains I_2 and $A = A_1^{-1} A_2$. I_2 and A are linearly independent since if $A = xI_2$ then $\det(A) = x^2 > 0$ but $\det(A) = -1$. So for every $x \in \mathbb{R}$, $A + xI_2 \neq 0$ and since the new code is full diversity, $\det(A + xI_2) \neq 0$. So A dose not have any real eigenvalue and

since $\det(A) = -1$, its eigenvalues lie on one side of the x-axis. Without loss of generality we assume that the eigenvalues are above the x-axis. Recalling that $\dim(V) = 4$, choose $B \in V$ such that the set $\{I_2, A, B\}$ is linearly independent. We can assume that B has an eigenvalue with negative imaginary part (otherwise consider $-B$). Now by continuity of eigenvalues, for some $t \in \mathbb{R}$, $tA + (1-t)B$ has a real eigenvalue. This is a contradiction since $tA + (1-t)B$ is not a scalar multiple of I_2 and again dose not have any real eigenvalue. So the determinant area of such an LD code is a closed subset of $\mathbb{C}$ such that $\det(C(s)) \setminus \{0\}$ is connected; for any $z \in \det(C(s))$ and any positive number a, $az \in \det(C(s))$ and $-z \notin \det(C(s))$ which means that $\det(C(s))$ is a proper closed convex cone. $\square$

If $s_j = x_j + iy_j$, then any linear combination of s_j and $\bar{s_j}$ is a linear combination of x_j and y_j and conversely any linear combination of x_j and y_j is a linear combination of s_j and $\bar{s_j}$. So we can consider entries of an LD code as complex linear combinations of $x_1, y_1, \ldots, x_k, y_k$ where $x_1, y_1, \ldots, x_k, y_k$ are real variables. If A is a real invertible $2k \times 2k$ matrix and $[f_1, \ldots, f_{2k}]^T = A[x_1, y_1, \ldots, x_k, y_k]^T$ then for every full diversity LD code $C(x_1, y_1, \ldots, x_k, y_k)$ the new code $C_{new} = C(f_1, \ldots, f_{2k})$ is also full diversity. If $F(x_1, y_1, \ldots, x_k, y_k) = \det(C(s))$ then $\det(C_{new}(s)) = F(f_1, \ldots, f_{2k})$. Obviously $F(x_1, y_1, \ldots, x_k, y_k)$ is a form of degree n; that is a polynomial whose monomials are of degree n. Specially if $n = 2$ then this form is a quadratic form. Any quadratic form can be written as xMx^T where $x = (x_1, y_1, \ldots, x_k, y_k)$ and M is a symmetric $2k \times 2k$ matrix. Notice that every $m \times m$ real symmetric matrix can be written as $Q^T \mathrm{diag}(\lambda_1, \ldots, \lambda_m)Q$ where Q is a real orthogonal matrix and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Moreover, every real positive definite matrix is in form $P^T P$ for some $P \in M_m(\mathbb{R})$ [8].

Theorem 3.4. *Let C be a 2×2 FDFRLD code. Then*

$$\det(C(s)) = a_1 f_1^2 + a_2 f_2^2 + a_3 g_1^2 + a_4 g_2^2 \tag{3.1}$$

for some complex numbers $a_1, \ldots, a_4$ and $(f_1, f_2, g_1, g_2)^T = A(x_1, y_1, x_2, y_2)^T$ for some real invertible 4×4 matrix A.

Proof. Using Theorem 3.3 and multiplying the code by a constant matrix and rotating the determinant area, we can assume that the determinant area lies on the right side of the y-axis and so the determinant of any nonzero matrix in the corresponding space has a positive real part. Assume that $\det(C(s)) = xMx^T$ and $M = M_1 + iM_2$ where $M_1, M_2 \in M_4(\mathbb{R})$ are symmetric. Since for all $x \in \mathbb{R}^4$, $xM_1x^T > 0$, M_1 is positive definite. Therefore there exists a nonsingular real 4×4 matrix P such that $M_1 = P^T P$. Now $(P^{-1})^T M_2 P^{-1}$ is a real symmetric matrix so there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ and a real orthogonal 4×4 matrix Q such that $(P^{-1})^T M_2 P^{-1} = Q^T \Lambda Q$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Then $M = P^T Q^T (I + i\Lambda) QP$. Let $A = QP$ and $a_i = 1 + i\lambda_i$ and $(f_1, f_2, g_1, g_2)^T = Ax^T$.

Then

$$\begin{aligned}\det(C(s)) &= xMx^T = xA^T diag(a_1,a_2,a_3,a_4)Ax^T\\ &= (f_1,f_2,g_1,g_2)diag(a_1,a_2,a_3,a_4)(f_1,f_2,g_1,g_2)^T\\ &= a_1f_1^2+a_2f_2^2+a_3g_1^2+a_4g_2^2.\end{aligned}$$

□

Since the LD code in the previous theorem is full diversity, any nonzero linear combination of a_1,a_2,a_3 and a_4 with nonnegative coefficients is nonzero or equivalently, a_1,a_2,a_3 and a_4 lie on one side of a line passing through the origin.

Before proving the result of this section we note that the code

$$C = \begin{pmatrix} b_1f_1+ib_2f_2 & b_3g_1+ib_4g_2 \\ -(b_3g_1-ib_4g_2) & b_1f_1-ib_2f_2 \end{pmatrix}$$

has determinant equal to $a_1f_1^2+a_2f_2^2+a_3g_1^2+a_4g_2^2$, providing $b_i^2 = a_i$ for $i=1,2,3,4$.

Theorem 3.5. *Every FDFRLD 2×2 code has the form*

$$P\begin{pmatrix} b_1f_1+ib_2f_2 & b_3g_1+ib_4g_2 \\ -u(b_3g_1-ib_4g_2) & b_1f_1-ib_2f_2 \end{pmatrix}Q, \tag{3.2}$$

where b_1,b_2,b_3,b_4 and u are complex numbers such that any nonzero linear combination of b_1^2,b_2^2,ub_3^2 and ub_4^2 with nonnegative coefficients is nonzero, P and Q are 2×2 invertible matrices and $(f_1,f_2,g_1,g_2)^T = A(x_1,y_1,x_2,y_2)^T$ for some invertible real 4×4 matrix A.

Proof. Let C be a 2×2 FDFRLD code. By Theorem 3.4 , the determinant of C is in form (3.1) where all a_i's lie on one side of a line passing through the origin and $(f_1,f_2,g_1,g_2)^T = A(x_1,y_1,x_2,y_2)^T$ for some real invertible 4×4 matrix A. Using the same argument as in the proof of Theorem 2.2, the code can be written as $f_1A_1+f_2A_2+g_1A_3+g_2A_4$ where A_i's are 2×2 complex matrices. Note that since A is invertible, f_1,f_2,g_1 and g_2 can take all real values independently. Letting $f_i=1$ for some $i\in\{1,2\}$ and $g_1=g_2=f_j=0$ for $j\neq i$, since the code is full diversity, we conclude that the matrix A_i is invertible. Similarly, A_3 and A_4 are invertible. Multiplying the code by A_1^{-1}, we can assume that $A_1=I_2$. Upper triangularizing A_2 leaves $A_1=I_2$. So, letting $A_2 = \begin{pmatrix} a & b \\ 0 & c\end{pmatrix}$ and $g_1=g_2=0$ the code takes the form

$$\begin{pmatrix} f_1+af_2 & bf_2 \\ 0 & f_1+cf_2 \end{pmatrix}$$

whose determinant is $f_1^2+acf_2^2+(a+c)f_1f_2$ with $a,c\neq 0$. Since the coefficient of f_1f_2 in (3.1) is zero, we have $a=-c$, hence the eigenvalues of A_2 are distinct and so A_2 is diagonalizable. Therefore we can assume that $b = 0$ and so letting $A_3 = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ and $A_4 = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$, the code takes the form

$$\begin{pmatrix} f_1 + af_2 + d_1g_1 + e_1g_2 & d_2g_1 + e_2g_2 \\ d_3g_1 + e_3g_2 & f_1 - af_2 + d_4g_1 + e_4g_2 \end{pmatrix}.$$

Again calculating the determinant of the above code and comparing with (3.1) we have $d_1 = d_4 = e_1 = e_4 = 0$ and $d_2e_3 + e_2d_3 = 0$. Letting $-\frac{d_3}{d_2} = \frac{e_3}{e_2} = u$ and $b_1 = 1, b_2 = -ia, b_3 = d_2, b_4 = -ie_2$ we observe that the code takes the desired form. Note that since A_3, A_4 are nonsingular, then $d_2, d_3, e_2, e_3 \neq 0$. To complete the proof, we note that codes of form (3.2) are full diversity. $\square$

We call the code (3.2) an LD code with parameters $(A, P, Q, b_1, b_2, b_3, b_4, u)$. We use the characterization in Theorem 3.5 to construct LD codes which are not of form (1.1).

Corollary 3.6. *Let all complex numbers b_1^2, b_2^2, b_3^2 and b_4^2 lie on one side of a line passing through the origin and have distinct arguments. Then, the LD code with parameters $(A, P, Q, b_1, b_2, b_3, b_4, 1)$ is not of form (1.1).*

Proof. Without loss of generality assume that b_2 and b_3 are positive combinations of b_1 and b_4 and $A = I_4$. Then the code has determinant equal to $b_1^2x_1^2 + b_2^2y_1^2 + b_3^2x_2^2 + b_4^2y_2^2$. It suffices to show that it can not be written as $b_1^2|f|^2 + b_4^2|g|^2$. Let $b_2^2 = \alpha b_1^2 + \beta b_4^2$ and $b_3^2 = \gamma b_1^2 + \theta b_4^2$ where α, β, γ and θ are positive real numbers. If $b_1^2x_1^2 + b_2^2y_1^2 + b_3^2x_2^2 + b_4^2y_2^2 = b_1^2|f|^2 + b_4^2|g|^2$ then $b_1^2(x_1^2 + \alpha y_1^2 + \gamma x_2^2) + b_4^2(y_2^2 + \beta y_1^2 + \theta x_2^2) = b_1^2|f|^2 + b_4^2|g|^2$ and so $|f|^2 = f_1^2 + f_2^2 = x_1^2 + \alpha y_1^2 + \gamma x_2^2$, where f_1 and f_2 represent real and imaginary part of f respectively. Let $f_1 = a_1x_1 + a_2y_1 + a_3x_2$ and $f_2 = a_1'x_1 + a_2'y_1 + a_3'x_2$. Since $f_1^2 + f_2^2 = x_1^2 + \alpha y_1^2 + \gamma x_2^2$, we have $a_1a_2 = -a_1'a_2'$, $a_1a_3 = -a_1'a_3'$, $a_2a_3 = -a_2'a_3'$, $a_1^2 + a_1'^2 = 1$, $a_2^2 + a_2'^2 = \alpha > 0$, and $a_3^2 + a_3'^2 = \beta > 0$. Thus $(a_1a_2a_3)^2 = (a_1a_2)(a_1a_3)(a_2a_3) = -(a_1'a_2'a_3')^2$ and so $a_1a_2a_3 = 0$. Without loss of generality assume that $a_1 = 0$ so $a_1' \neq 0$ and equalities $a_1a_2 = -a_1'a_2'$ and $a_1a_3 = -a_1'a_3'$ imply that $a_2' = a_3' = 0$. So $a_2, a_3 \neq 0$ but this contradicts $a_2a_3 = -a_2'a_3'$. $\square$

As mentioned before the determinant area of Alamouti code is a half line. Next, we characterize all full diversity LD codes whose determinant areas are half lines.

Corollary 3.7. *If the determinant area of an LD code is a half line then the code has the form*

$$P\begin{pmatrix} f & g \\ -a\bar{g} & \bar{f} \end{pmatrix}Q, \tag{3.3}$$

where a is a positive real number and f and g are two linear combinations of $s_1, \bar{s_1}, s_2$ and $\bar{s_2}$.

Proof. Let $(A, P, Q, b_1, ib_2, b_3, ib_4, u)$ be parameters of an LD code whose determinant area is a half line. Then, since the numbers b_1^2, b_2^2, ub_3^2 and ub_4^2 have the same argument, there exists a real number θ such that $e^{2i\theta}b_1^2, e^{2i\theta}b_2^2, e^{2i\theta}ub_3^2$, and $e^{2i\theta}ub_4^2$ are positive numbers and so $e^{i\theta}b_1$ and $e^{i\theta}b_2$ are both real. Multiplying the code by $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, the code will take the form

$$\begin{pmatrix} b_1'f_1 + ib_2'f_2 & b_3'g_1 + ib_4'g_2 \\ -u(b_3'g_1 - ib_4'g_2) & b_1'f_1 - ib_2'f_2 \end{pmatrix},$$

where $b_j' = e^{i\theta}b_j$. Since $ub_3'^2, ub_4'^2$ are positive numbers so $\frac{ub_3'}{\bar{b_3'}} = \frac{ub_3'^2}{|b_3'|^2}$ and $\frac{ub_4'}{\bar{b_4'}} = \frac{ub_4'^2}{|b_4'|^2}$ are positive. Letting $a = \frac{ub_3'}{\bar{b_3'}} = |\frac{ub_3'}{\bar{b_3'}}| = |u| = |\frac{ub_4'}{\bar{b_4'}}| = \frac{ub_4'}{\bar{b_4'}}$, $f = b_1'f_1 + ib_2'f_2$ and $g = b_3'g_1 + ib_4'g_2$ the code takes the form (3.3). □

4. Full rank subspaces of $M_n(\mathbb{C})$

Although the study of full rank spaces began long time ago, recently it has been shown that these spaces are very applicable specially in the theory of space-time codes [11, 19]. In this section, as mentioned before, we introduce infinitely many inequivalent full rank real subspaces of $M_n(\mathbb{C})$. If an LD code is full diversity, its corresponding subspace is full rank. In the following corollary, using Theorems 2.2 and 3.5 we characterize all real full rank subspaces of $M_2(\mathbb{R})$ and $M_2(\mathbb{C})$.

Corollary 4.1. *Any full rank subspace of $M_2(\mathbb{R})$ of dimension 2 is equivalent to* $\mathrm{Span}_{\mathbb{R}}\{I_2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ *and any full rank subspace of dimension 4 in $M_2(\mathbb{C})$ is equivalent to a space of the form*

$$\mathrm{Span}_{\mathbb{R}}\{I_2, \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix}, \begin{pmatrix} 0 & b_3 \\ b_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_4 \\ -b_4 & 0 \end{pmatrix}\}$$

where $b_2, b_3, b_4 \in \mathbb{C}$ and any linear combination of $1, b_2^2, b_3^2$ and b_4^2 with nonnegative coefficients is nonzero.

Similar to space-time codes, we define the determinant area of a full rank space V as $\det V = \{\det(A) : A \in V\}$. So, the determinant areas of two equivalent full rank subspaces differ by a rotation. To prove our main result, we use the theory of orthogonal design. A $[p, n, k]$ orthogonal design is a $p \times n$ matrix C whose nonzero entries are $z_1, \ldots, z_k, -z_1, \ldots, -z_k$ and their conjugates $\bar{z_1}, \ldots, \bar{z_k}, -\bar{z_1}, \ldots, -\bar{z_k}$ such that $C^*C = (|z_1|^2 + \cdots + |z_k|^2)I_n$ [11]. Clearly, an $[n, n, k]$ orthogonal design is a full diversity LD code whose corresponding real space is of dimension $2k$. So $k \leq b + 1$, where $n = (2a + 1)2^b$. When n

is odd, $k = 1$. In [11] the author introduced an $n \times n$ orthogonal design with $k = b + 1$ for any even number n having the form

$$C = \begin{pmatrix} N & M \\ -M^* & N^* \end{pmatrix}, \tag{4.1}$$

where $N = z_1 I_{\frac{n}{2}}$ and M is an $\frac{n}{2} \times \frac{n}{2}$ matrix whose entries are $0, \pm z_2, \ldots, \pm z_k$ and $\pm \bar{z_1}, \ldots, \pm \bar{z_k}$. Now, using this construction we prove the following theorem.

Theorem 4.2. *If n is even and $n = 2^a(2b + 1)$, then for any $l \in \mathbb{N}$ with $2 \leq l \leq 2b + 2$, there exist infinitely many inequivalent full rank spaces of dimension l.*

Proof. Letting $z_1 = 0$ In (4.1), we have $MM^* = M^*M = (|z_2|^2 + \cdots + |z_k|^2) I_{\frac{n}{2}}$. Let $0 \leq \theta < \pi$, $u = e^{\frac{i\theta}{n}}$, $L_1 = (ux_1 + iy_1) I_{\frac{n}{2}}$ and $L_2 = (ux_1 - iy_1) I_{\frac{n}{2}}$ where x_1 and y_1 are real and imaginary parts of z_1 respectively. Let

$$C_\theta = \begin{pmatrix} L_1 & M \\ -M^* & L_2 \end{pmatrix}. \tag{4.2}$$

Then, $\det(C_\theta)$ $=\det(L_1 L_2 + MM^*)$ $=\det((u^2 x_1^2 + y_1^2) I_{\frac{n}{2}} + (|z_2|^2 + \cdots + |z_k|^2) I_{\frac{n}{2}}) = (u^2 x_1^2 + y_1^2 + |z_2|^2 + \cdots + |z_k|^2)^{\frac{n}{2}}$. So, C_θ is full diversity and the determinant area of C_θ, being the set of all linear combinations of u^2 and 1 with positive coefficients, is a proper closed convex cone with angel θ. When $\theta_1 \neq \theta_2$, the spaces corresponding to C_{θ_1} and C_{θ_2} are inequivalent since the determinant area of equivalent spaces differ by a rotation. So, the theorem is established. ☐

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(Mohammad Ashenab) Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
and
Young Researchers Society, Shahid Bahonar University of Kerman, Kerman, Iran.
E-mail address: `mohammadashenab@math.uk.ac.ir`

(Hossein Momenaee Kermani) Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.
E-mail address: `momenaee@uk.ac.ir`