# RINGS OF FUNCTIONS WITH INTEGER DERIVATIVES AT $\mathrm{X}=1$ 

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#### Abstract

Starting with results for $x$ to the power $x$, Dowe and Landy (1990) and Dowe (1995) have derived several properties of classes of functions for which the derivative of order $k$, evaluated at $x=1$, is an integer, or an integer divisible by $k(k \geq 1)$.

For all natural numbers $n$, we examine here some generalisations of these properties to the sets $E_{n}$ of functions $f$ (all defined in a neighbourhood of $x=1$ ), such that for $k \leq n$, the derivative of order $k$ of $f$ is divisible by $k!$ and for $k>n$, the derivative of order $k$ of $f$ at $x=1$ is divisible by:


$$
k(k-1) \ldots(k-n+1)=\frac{k!}{(k-n)!}=n!\binom{k}{n}=n!C_{k}^{n}
$$

## 1. Introduction

Let $\Omega$ be the set of real functions defined on a neighbourhood of $x=1$ and infinitely differentiable in their domain (without necessarily being analytic about $x=1$ ). For each integer $n$, define the following subsets of $\Omega$ : $E_{0}=\left\{f \in \Omega ; \forall k \in \mathbb{N}, f^{(k)}(1) \in \mathbb{Z}\right\}$ and

[^0]for all $n \geq 1, E_{n}$ is the set of functions $f \in \Omega$ having the following properties:
(i) $\forall \ell \in \mathbb{N}, f^{(\ell)}(1) \in \mathbb{Z}$;
(ii) $k!\mid f^{(k)}(1)$ for $k=0, \ldots, n$;
(iii) $\left.\frac{k!}{(k-n)!} \right\rvert\, f^{(k)}(1)$ for $k \geq n$.
where $a \mid b$ stands for " $a$ divides $b "$ and where $f^{(k)}(1)$ is the derivative of order $k$ of $f$ at $x=1$. Examples of functions belonging to (all) $E_{n}$ include $x \mapsto x^{m}, m \in \mathbb{Z}$, since:
\[

$$
\begin{gather*}
\forall m \geq 0,\left(\forall k \leq m,\left.\left(x^{m}\right)^{(k)}\right|_{x=1}=\frac{m!}{(m-k)!},\right.  \tag{1a}\\
\left.\forall k>m,\left.\left(x^{m}\right)^{(k)}\right|_{x=1}=0\right), \\
\forall m<0, \forall k,\left.\left(x^{m}\right)^{(k)}\right|_{x=1}=(-1)^{k} \frac{(-m-1+k)!}{(-m-1)!} . \tag{1b}
\end{gather*}
$$
\]

Dowe and Landy ([4]) and Dowe ([2],[3]) gave detailed attention to other examples and classes of examples and obtained several results, including: $x \mapsto x^{x} \in E_{1}$; for $f \in E_{0}$ (resp. $E_{1}$ ) then $x \mapsto$ $x^{f(x)} \in E_{0}\left(\right.$ resp. $\left.E_{1}\right) ; P \circ f \in E_{0}$ (resp. $E_{1}$ ) where $f \in E_{0}$ (resp. $E_{1}$ ) and $P$ is a polynomial with integer coefficients; $x \mapsto \frac{1}{f(x)} \in E_{0}$ for $f \in E_{0}$ and $f(1)=1$, etc.. Dowe ([2], p.2, p.5) also proves closure of $E_{0}$ and $E_{1}$ by addition/subtraction and multiplication.

The paper examines generalisations of these results in the following manner: in section 2, a mapping of $E_{n}$ onto $E_{n-1}$ is introduced, for use in the rest of the paper; in section 3, we show that the sets $E_{n}$ are embedded rings; in section 4, we examine closure properties of the sets $E_{n}$ for composition and in section 5 , we take the same approach for exponentiation, yielding more limited results. Section 6 concludes with a discussion on further generalisations of algebraic and analytic properties of the sets $E_{n}$.

## 2. A Mapping of $E_{n}$ onto $E_{n-1}$

Let $O$ be the set of functions defined in a neighbourhood of $x=1$. Several of the results obtained in the above mentioned earlier papers use the following mapping of $\Omega$ into $O$ :

$$
\begin{aligned}
& \Delta: \Omega \rightarrow O \\
& \Delta f(x)=\frac{f(x)-f(1)}{x-1} \text { if } x \neq 1 \\
& \Delta f(1)=f^{\prime}(1)
\end{aligned}
$$

Since this mapping will be used several times in the remainder of the paper, we begin with the following result:

## Theorem 2.1.

(a) $\Delta$ is a linear mapping. Its kernel is the set of constant functions in a neighbourhood of $x=1$;
(b) (i) The kernel of the restriction of $\Delta$ to $E_{n}$ is the set of constant integer functions in a neighbourhood of $x=1$;
(ii) $\Delta$ maps $E_{n}$ onto $E_{n-1}$ for all $n \geq 1$;
(c) $\forall n \geq 1,\left(f \in E_{n}\right) \Leftrightarrow\left(\Delta f \in E_{n-1}\right.$ and $f(1)$ is an integer $)$.

Proof. (a) and (b)(i) are straightforward by definition of $\Delta$. To begin proving (b)(ii), we next show that for $f \in \Omega, \Delta f$ is infinitely differentiable at $x=1$ with:

$$
\begin{equation*}
(\Delta f)^{(k)}(1)=\frac{f^{(k+1)}(1)}{k+1} \text { for all } k \geq 0 \tag{2a}
\end{equation*}
$$

For $f$ analytic about 1 , the proof is straightforward by manipulation of the Taylor series. Otherwise, suppose first that $f(1)=f^{\prime}(1)=$ $\ldots=f^{(k)}(1)=0$. In some neighbourhood of $x=1$,

$$
\left(\frac{f(x)-f(1)}{x-1}\right)^{(k)}=\sum_{p=0}^{k} C_{k}^{p} \frac{(-1)^{p} p!}{(x-1)^{p+1}}(f(x)-f(1))^{(k-p)} .
$$

By (repeated use of) de l'Hospital's ${ }^{1}$ rule,

$$
\lim _{x \rightarrow 1} \frac{(f(x)-f(1))^{(k-p)}}{(x-1)^{p+1}}=\frac{f^{(k+1)}(1)}{(p+1)!},
$$

thus:

$$
\lim _{x \rightarrow 1}\left(\frac{f(x)-f(1)}{x-1}\right)^{(k)}=f^{(k+1)}(1) \sum_{p=0}^{k} C_{k}^{p} \frac{(-1)^{p}}{p+1}=\frac{f^{(k+1)}(1)}{k+1}
$$

since:

$$
\begin{gathered}
\int_{0}^{x}(1+t)^{k} d t=\int_{0}^{x} \sum_{p=0}^{k} C_{k}^{p} t^{p} d t=\sum_{p=0}^{k} C_{k}^{p} \int_{0}^{x} t^{p} d t \\
=\sum_{p=0}^{k} C_{k}^{p} \frac{x^{p+1}}{p+1}=\frac{(1+x)^{k+1}-1}{k+1}
\end{gathered}
$$

in particular for $x=-1$.
In the general case where $f^{(j)}(1) \neq 0$ for some $j \leq k$, the result follows using:

$$
g(x)=f(x)-\sum_{p=0}^{k} \frac{f^{(p)}(1)}{p!}(x-1)^{p},
$$

since $g(1)=g^{\prime}(1)=\ldots=g^{(k)}(1)=0$ and since in some neighbourhood of $x=1$ one has $f^{(\ell)}(x)=g^{(\ell)}(x)$ for $\ell>k$ and $\left(\frac{f(x)-f(1)}{x-1}\right)^{(k)}=$ $\left(\frac{g(x)-g(1)}{x-1}\right)^{(k)}$. Thus $(\Delta f)^{(k)}(1)=(\Delta g)^{(k)}(1)=\frac{g^{(k+1)}(1)}{k+1}=\frac{f^{(k+1)}(1)}{k+1}$ for $k \geq 0$, establishing (2a).

Now if $f \in E_{n},(\Delta f)^{(k)}(1)=\frac{f^{(k+1)}(1)}{k+1}$ is an integer for $k \geq 0$; and $\Delta f \in E_{n-1}$ since:

$$
\begin{equation*}
k!\left\lvert\, \frac{f^{(k+1)}(1)}{k+1}\right. \text { for all } k \leq n-1 ; \tag{2b}
\end{equation*}
$$

as $(k+1)!\mid f^{(k+1)}(1)$ and

[^1]\[

$$
\begin{equation*}
\frac{k!}{(k-(n-1))!} \left\lvert\, \frac{f^{(k+1)}(1)}{k+1}\right. \text { for all } k>n-1 . \tag{2c}
\end{equation*}
$$

\]

as $\left.\frac{(k+1)!}{(k+1-n)!} \right\rvert\, f^{(k+1)}(1)$.

Finally let $f \in E_{n-1} . \Delta((x-1) f)=f$ and $(x-1) f \in E_{n}$, since:

$$
\begin{equation*}
((x-1) f)^{(k)}(1)=k f^{(k-1)}(1) \text { for } k \geq 1 . \tag{2d}
\end{equation*}
$$

Thus by (b)(i), the reciprocal image of $f \in E_{n-1}$ is the set $\{(x-$ 1) $f+t ; t \in \mathbb{Z}\} \subset E_{n}$. This completes the proof of (b)(ii); (c) is then straightforward, using (2a), (2b) and (2c) above.

Corollary 2.2. ${ }^{2}$ Let $s \geq 1$ and $f \in \Omega$. Then $f \in E_{s}$ if and only if $f=(x-1)^{s} g+P$, where $g \in E_{0}$ and $P$ is the null polynomial, or a polynomial of degree less than $s-1$ with integer coefficients.

Proof. By induction on $s \geq 1$. For $f \in E_{1}, \Delta f \in E_{0}$ and $f(1)=$ $P$ is an integer by Theorem 2.1 (c). Conversely, using Theorem 2.1 (a) as well as (2d) above, if $f=(x-1) g+P$ where $\Delta f=$ $g \in E_{0}$ and $P$ is a constant (integer) polynomial, then $f \in E_{1}$. Assume that the result holds for some $s \geq 1$ and take $f \in E_{s+1}$. Since $\Delta f \in E_{s}$ by Theorem 1 (c), $f=(x-1)^{s+1} g+(x-1) P$, where $g \in E_{0}$ and $P$ is the null polynomial, or a polynomial of degree less than $s-1$ with integer coefficients. $(x-1) P$ is thus the null polynomial, or a polynomial of degree less than $s$ with integer coefficients. Conversely, if $f=(x-1)^{s+1} g+P$, where $g \in E_{0}$ and $P$ is the null polynomial, or a polynomial of degree less than $s$ with integer coefficients, then $\Delta f=(x-1)^{s} g+Q$, where $g \in E_{0}$ and $Q$ is the null polynomial, or a polynomial of degree less than $s-1$ with integer coefficients; hence $\Delta f \in E_{s}$; and since $f(1)$ is an integer, $f \in E_{s+1}$ by Theorem 2.1 (c).

[^2]
## 3. The Rings $E_{n}$

We now begin to use Theorem 2.1 in order to derive:

## Theorem 3.1.

(a) (i) $E_{n+1} \subset E_{n}$;
(ii) $\forall n \geq 0, E_{n}$ is a ring;
(b) Define $E_{\infty}$ as the set of functions $f \in \Omega$ such that:
(i) $\forall k, f^{(k)}(1) \in \mathbb{Z}$;
(ii) $\forall k, k!\mid f^{(k)}(1)$.

Then $E_{\infty}$ is a ring;
(c) $F=\left\{f \in \Omega ; \forall k, f^{(k)}(1)=0\right\}$ is an ideal of $\Omega$.

Proof. (a)(i) Let $f \in E_{n+1}$.
If $i=n$ then $n!|(n+1)!| f^{(i)}(1)$; if $i>n$ then $\frac{i!}{(i-n)!}\left|\frac{i!}{(i-(n+1))!}\right| f^{(i)}(1)$. Thus $f \in E_{n}$.
(a) (ii) Let $f, g \in E_{n}$, where $f$ is defined on a neighbourhood $V$ and $g$ on a neighbourhood $W$ of 1 . Then $f \pm g$ and $f g$ are defined on $V \cap W$ and:

- It is straightforward that $f \pm g \in E_{n}$ as derivatives add up;
- We then notice that for $n=0, f, g \in E_{0} \Rightarrow f g \in E_{0}$, by Leibniz's identity (see e.g. [1], Théorème $A$, p.141), which generalises the derivation of the product rule. In order to prove that $E_{n}$ is closed under multiplication, we proceed by induction, assuming that the result holds for some $n-1 \geq 0$. By theorem 2.1(c) above, it is necessary and sufficient to prove that $\Delta(f g) \in E_{n-1}$, since $f g(1)$ is an integer.

Suppose first that $g(1)=0$. Since $f \in E_{n-1}$ ((a)(i) above), and $\Delta g \in E_{n-1}$ (theorem 1(c) above), it follows that $f \Delta g \in E_{n-1}$. Now:

$$
\forall x \neq 1, \Delta(f g)(x)=\frac{f g(x)}{x-1}=f(x) \Delta g(x)
$$

$$
\Delta(f g)(1)=(f g)^{\prime}(1)=f(1) g^{\prime}(1)=f(1) \Delta g(1)
$$

Thus, $f \Delta g=\Delta(f g)$ and so $\Delta(f g) \in E_{n-1}$, hence $f g \in E_{n}$ when $g(1)=0$.

In the general case where $g(1) \neq 0$, it suffices to use $h=g-g(1)$ to conclude, since $f g=f h+f g(1) \in E_{n}$.
(b) $E_{\infty}=\bigcap_{n=0}^{\infty} E_{n}$ is a ring as an intersection of rings;
(c) is clear from Leibniz's identity mentioned earlier, since for $f \in \Omega$ and $g \in F$,

$$
(f g)^{(k)}(1)=\sum_{p=0}^{k} C_{k}^{p} f^{(p)}(1) g^{(k-p)}(1)=0 \text { for all } k \geq 0
$$

The following is immediate:
Corollary 3.2. If $n$ is an integer ${ }^{3}$, $P$ is a polynomial with integer coefficients and $f \in E_{n}$, then $P \circ f \in E_{n}$.

## 4. Closure Results under Composition

Theorem 4.1. If $f, g \in E_{s}$ and $g(1)=1$, then $f \circ g \in E_{s}$ for $s \geq 0$.

Proof. By induction on $s$. Take $s=0$. For $n \geq 1$ and $f, g \in E_{0}$ :

$$
\begin{aligned}
& (f \circ g)^{(n)}(1) \\
= & \sum_{k_{1}+2 k_{2}+\ldots+n k_{n}=n} \frac{n!}{k_{1}!\ldots k_{n}!}\left(\frac{g^{(1)}(1)}{1!}\right)^{k_{1}} \ldots\left(\frac{g^{(n)}(1)}{n!}\right)^{k_{n}} f^{\left(k_{1}+\ldots+k_{n}\right)}(g(1)),
\end{aligned}
$$

(Faa di Bruno's formula, see e.g. [1], Théorème $A$, p.148).

[^3]The differenciation rules for the composition and product of functions imply that the coefficients $\left(\frac{n!}{k_{1}!\ldots k_{n}!}\right)\left(\frac{1}{1!}\right)^{k_{1}} \ldots\left(\frac{1}{n!}\right)^{k_{n}}$ in the above formula are integers ( see also [2], Corollary 3 p.2, or [1], Théorème $A$, p.145); and $(f \circ g)^{(n)}(1)$ is obviously an integer.

Suppose the result holds for some $s$. In order to prove that it also holds for $s+1$, we use theorem 2.1 above. Take $f, g \in E_{s+1}$ with $g(1)=1$. First, suppose that there is some neighbourhood of 1 where $g(x) \neq 1$ except for $x=1$. In this neighbourhood put:

$$
\begin{gathered}
\varphi(x)=\frac{(f \circ g)(x)-(f \circ g)(1)}{g(x)-g(1)}=\frac{(f \circ g)(x)-f(1)}{g(x)-1}, x \neq 1 \\
\text { and } \varphi(1)=f^{\prime}(1) .
\end{gathered}
$$

Then $\varphi(x)=\Delta f(g(x))$. Since $\Delta f \in E_{s}$ (theorem 2.1(b)) and $g \in E_{s+1} \subset E_{s}$ (theorem 3.1 (a)(i)), it follows that $\varphi \in E_{s}$. Now, $\Delta(f \circ g)=\varphi \Delta g$ and $\Delta g \in E_{s}$, thus $\Delta(f \circ g) \in E_{s}$ by theorem 3.1 (a)(ii). Finally, $f \circ g \in E_{s+1}$ by theorem 2.1(c), since $f \circ g(1)$ is an integer.

If there exists $k \geq 1$ such that $g^{(k)}(1) \neq 0$, there is a neighbourhood of 1 where $g(x) \neq 1$ except for $x=1$. Otherwise, if $\forall k \geq 1, g^{(k)}(1)=0$, then by Faa di Bruno's formula above, $(f \circ g)^{(k)}(1)=0, k \geq 1$ and $f \circ g \in E_{s+1}$.

Corollary 4.2. Suppose $f \in E_{0}$; then:
(a) If $f(1)=1$ then $\log f$ and $\frac{1}{f} \in E_{0}$;
(b) If $e^{f(1)}$ is an integer then $\exp f \in E_{0}$.

Proof. Either: [2], theorem 5 (iv), (vi) and (vii) pp.2-3, or:
(a) is straightforward using theorem 4.1.

For $(\mathrm{b})$, put $\delta(x)=f(x)-f(1)+1 . \delta \in E_{0}$ and $\delta(1)=1$.

By theorem 4.1, $g: x \mapsto \exp (\delta(x)-1)=\exp ((f(x)-f(1))) \in$ $E_{0}$, since $x \mapsto e^{x-1} \in E_{0}$. Thus,

$$
x \mapsto \exp f(x)=\exp (f(x)-f(1)) \exp f(1) \in E_{0} .
$$

Corollary 4.3. The group of multiplication units of $E_{s}$ is $U_{s}=$ $\left\{f \in E_{s} ; f(1)= \pm 1\right\}$.

Proof. Note that since $x \mapsto \frac{1}{x} \in E_{\infty}$ (see (1(b) above), $\frac{1}{f} \in E_{s}$ when $f \in E_{s}$ and $f(1)=1$ or $f(1)=-1$, by theorem 4.1.

Corollary 4.2(b) above leads us to looking at further extensions of closure results for exponentiation.

## 5. Closure Results under Exponentiation

The following seems to tell us that we cannot go very far in that direction.

Theorem 5.1. If $f, g \in E_{s}$ and $f(1)=1$ then $f^{g} \in E_{s}$ for $s=0,1$. For $s>1, f^{g} \notin E_{s}$ in general.

Note at once that the result is straightforward for $s=0$, using corollary $4.2(\mathrm{~b})$ of theorem 4.1 above, since $\log f \in E_{0}$ and $e^{g \log f(1)}=1$ is an integer.

For $s=1$, put $f=1+u,(u(1)=0), H_{0}=1$ and for $\ell \geq 1$, let:

$$
H_{\ell}(x)=\left(\prod_{m=0}^{\ell-1}(g-m)\right)(x)=\prod_{m=0}^{\ell-1}(g(x)-m)
$$

We now use two lemmas.
Lemma 5.2. For $s \geq 0$ and $f, g \in E_{s}$, in some neighbourhood of $x=1$,

$$
\left(f^{g}\right)^{(p)}(x)=\sum_{\ell=0}^{\infty}\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(x) \text { for } p \geq 0 .
$$

Proof. For $\alpha$ real, recall the Taylor expansion of $y \mapsto(1+y)^{\alpha}$ about zero:

$$
\begin{align*}
(1+y)^{\alpha}=1 & +\sum_{k=1}^{n} \frac{\alpha(\alpha-1) \ldots(\alpha-k+1)}{k!} y^{k} \\
& +\frac{\alpha(\alpha-1) \ldots(\alpha-n)}{(n+1)!}(1+c)^{\alpha-n-1} y^{n+1} \tag{5.1}
\end{align*}
$$

for $n \geq 1$, where $|c| \leq|y|$.
Now put $\alpha=g(x), y=u(x)$ and write (for $p=0$ ):

$$
f^{g}(x)=\sum_{\ell=0}^{N} H_{\ell}(x) \frac{u^{\ell}(x)}{\ell!}+\frac{H_{N+1}(x)}{(N+1)!}(u(x))^{N+1}(1+c(x))^{g(x)-N-1},
$$

for $N \geq 0$, where $|c(x)| \leq|u(x)|$.
Choosing a neighbourhood of $x=1$ where e.g. $|u(x)|<1 / 2$, $|g|$ being bounded in that neighbourhood, it is easily checked that the above series is uniformly convergent as $N$ goes to infinity. This completes the proof of lemma 5.2 for $p=0$.

Recall also that given a set of functions $f_{1}, \ldots f_{m}$, (where $m \geq 1$ ), all having a derivative of order $k \leq p$ for some $x$, Leibniz's identity can be further generalised (see e.g. [1], Théorème C, p.143) to:

$$
\begin{align*}
\left(f_{1} \ldots f_{m}\right)^{(p)}(x) & =\sum_{k_{1}+\ldots+k_{m}=p} \frac{p!}{k_{1}!\ldots k_{m}!}\left(f_{1}\right)^{\left(k_{1}\right)}(x) \ldots\left(f_{m}\right)^{\left(k_{m}\right)}(x) \\
& =\sum_{k_{1}+\ldots+k_{m}=p} \frac{p!}{k_{1}!\ldots k_{m}!} \prod_{n=1}^{m}\left(f_{n}\right)^{\left(k_{n}\right)}(x) \tag{5.2}
\end{align*}
$$

Let $\ell \geq 1$ and let $K$ be the set of $\left(k_{1}, \ldots, k_{\ell}\right)=k$ such that $k_{1}+\ldots+k_{\ell}=p$. For $p \geq 1$, according to (5.2), write:

$$
\begin{equation*}
\left(\frac{H_{\ell}}{\ell!} u^{\ell}\right)^{(p)}(x)=\frac{1}{\ell!} \sum_{k \in K} \frac{p!}{k_{1}!\ldots k_{\ell}!} \prod_{n=0}^{\ell-1}(u(g-n))^{\left(k_{n}\right)}(x) \tag{5.3}
\end{equation*}
$$

For $s \geq 0$, choose an upper bound $M_{s}$ of $\left|(u g)^{(i)}\right|$ and an upper bound $u_{s}$ of $\left|u^{(i)}\right|$, for $i=0, \ldots s$, in a neighbourhood of $x=1$.

For $\ell \geq p$, each summand in (5.3) contains at least $\ell-p k_{i}$ 's which are null.

The summand is therefore bounded above by:

$$
\left(M_{0}+(\ell-1) u_{0}\right)^{\ell-p}\left(M_{p}+(\ell-1) u_{p}\right)^{p} \frac{p!}{k_{1}!\ldots k_{\ell}!} .
$$

Now, choose a neighbourhood of $x=1$ where $u_{0}$ is smaller than $\frac{1}{e}(u(1)=0)$.

In this neighbourhood, the absolute value of the sum in (5.3) is bounded above by:

$$
\begin{aligned}
\left(M_{0}+\right. & \left.(\ell-1) u_{0}\right)^{\ell-p}\left(M_{p}+(\ell-1) u_{p}\right)^{p} \sum_{k_{1}+\ldots k_{\ell}=p} \frac{p!}{k_{1}!\ldots k_{\ell}!} \\
& =\left(M_{0}+(\ell-1) u_{0}\right)^{\ell-p}\left(M_{p}+(\ell-1) u_{p}\right)^{p} \ell^{p}
\end{aligned}
$$

Now the series with general term:

$$
a_{\ell}=\frac{\left(M_{0}+(\ell-1) u_{0}\right)^{\ell-p}\left(M_{p}+(\ell-1) u_{p}\right)^{p} \ell^{p}}{\ell!}
$$

is convergent as $\frac{a_{\ell+1}}{a_{\ell}} \underset{l \rightarrow \infty}{\approx} \frac{\left(M_{0}+\ell u_{0}\right) e}{\ell+1} \rightarrow u_{0} e<1$.
Thus the series with general term $f_{\ell}(x)=\left(\frac{H_{\ell}}{\ell!} u^{\ell}\right)^{(p)}(x)$ is uniformly convergent in a neighbourhood of 1 and the result follows by induction on $p$. This completes the proof of Lemma 5.2 for all $p$.

## Lemma 5.3.

(a) $\frac{u^{\ell}}{\ell!} \in E_{0}$;
(b) $\left(\frac{u^{\ell} \ell}{\ell!}\right)^{(k)}(1)=0$ for $\ell>k \geq 0$.

Proof. (a) By theorem 4.1, since $x \mapsto \frac{(x-1)^{\ell}}{\ell!} \in E_{0}$ and $f=u+1 \in$ $E_{s} \subset E_{0}$.
(b) is clear using e.g. Faa di Bruno's formula, and the fact that $u(1)=0$.

## Proof of Theorem 5.1:

We only need to turn to $s=1$. Using lemma 5.2 we get:

$$
\left(f^{g}\right)^{(p)}(1)=\sum_{\ell=0}^{\infty}\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1) \text { for } p \geq 0
$$

and using lemma 5.3(b) above:

$$
\left(f^{g}\right)^{(p)}(1)=\sum_{\ell=0}^{p}\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1) \text { for } p \geq 0
$$

It now suffices to prove that $\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1)$ is an integer divisible by $p$ for $\ell \leq p$.

$$
\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1)=\sum_{k=\ell}^{p} C_{p}^{k} H_{\ell}^{(p-k)}(1)\left(\frac{u^{\ell}}{\ell!}\right)^{(k)}(1) .
$$

(Leibniz's identity, and lemma 5.3(b), which implies that the above summation starts from $k=\ell$ ).

Since $E_{1}$ is a ring, $H_{\ell} \in E_{1}$. For $j \geq 1$, put $H_{\ell}^{(j)}(1)=j H_{j}$. Then:

$$
\begin{gathered}
\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1)=\frac{H_{\ell}(1)}{\ell!}\left(u^{\ell}\right)^{(p)}(1)+\sum_{k=\ell}^{p-1} \frac{p!}{(p-k)!k!}(p-k) H_{p-k}\left(\frac{u^{\ell}}{\ell!}\right)^{(k)}(1), \\
\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1)=\frac{H_{\ell}(1)}{\ell!}\left(u^{\ell}\right)^{(p)}(1)+p\left(\sum_{k=\ell}^{p-1} C_{p-1}^{k} H_{p-k}\left(\frac{u^{\ell}}{\ell!}\right)^{(k)}(1)\right) .
\end{gathered}
$$

Now, it is easily checked that $\frac{H_{\ell}(1)}{\ell!}$ is an integer and since $u^{\ell} \in E_{1}$, $p \mid\left(u^{\ell}\right)^{(p)}(1)$.
¿From lemma 2(a), $\sum_{k=\ell}^{p-1} C_{p-1}^{k} H_{p-k}\left(\frac{u^{\ell} \ell!}{l!}\right)^{(k)}(1)$ is an integer.
And thus, $\left(H_{\ell} \frac{u^{\ell}}{\ell!}\right)^{(p)}(1)$ is an integer divisible by $p$ for $\ell \leq p$.
Finally, the result does not extend to $s>1$ : for example $x \mapsto x \in$ $E_{2}$ but $x \mapsto x^{x} \notin E_{2}$, as $\left(x^{x}\right)^{(3)}(1)=3$ is not divisible by $3 * 2=6$.

Dowe ([2], corollary 11, p.9) further shows that for $k \geq 2, k(k-1)$ divides $\left(x^{x}\right)^{(k)}(1)$ if and only if $(k-1)$ divides $(k-2)$ !.

Corollary 5.4. ${ }^{4}$ If $f \in E_{0}$ (resp. $E_{1}$ ), then $x \mapsto x^{f(x)} \in E_{0}$ (resp. $E_{1}$ ).

## Notes

- Beyond corollary 4.2 of theorem 4.1 and theorem 5.1 above, results concerning exponentiation include:
$\left(h \in E_{1}\right) \Rightarrow\left(x^{h} \in E_{2}\right) \Leftrightarrow\left(h^{\prime} \log x \in E_{1}\right)$ (a consequence of a private communication by H.S. Wilf, see [2] theorem 9, p.8);
- The converse of theorem 5.1 is not true, since e.g. $g: x \mapsto$ $\exp (x-1) \in E_{0} ; g(x)=x^{f(x)}$ with $f: x \mapsto \frac{x-1}{\log x}, f(1)=1$, but $f \notin E_{0}$ as $f^{\prime}(1)=\frac{1}{2}$. However Dowe ([2], theorem 5 (viii), p.3) derives partial converse results, which can be generalised as follows, using theorems 4.1 and 5.1 above:

Suppose $u=f^{g}, u(1)=1$ and $u \in E_{0}$ (resp. $E_{1}$ ). Then:
(i) $\log f \in E_{0}, \log f(1) \in\{-1,1\} \Rightarrow g \in E_{0}$
(ii) $g \in E_{0}$ (resp. $E_{1}$ ), $g(1) \in\{-1,1\} \Rightarrow f \in E_{0}$ (resp. $E_{1}$ ).

## 6. Discussion and Conclusions

The above closure results could lead to further generalisations, such as:

- For $n \geq 1$, generate $E_{n}$, building from "minimal" subsets, as well as from the ring and restricted closure properties examined above. From the corollary to theorem 2.1, one first remark is that it is enough to work in $E_{0}$. Further examination of the algebraic and analytic properties of $E_{0}$ could be performed.

[^4]- Given a subset $\Sigma$ of the set of complex numbers and $n \geq 0$, extend the definition of the sets $E_{n}$ to sets $E_{n}(\Sigma)$ of functions $f \in \Omega$ such that:
(i) $\frac{f^{(k)}(1)}{k!} \in \Sigma$ for $k=0, \ldots, n$;
(ii) $(k-n)!\frac{f^{(k)}(1)}{k!} \in \Sigma$ for $k \geq n$.

The following could be investigated: when $\Sigma$ is endowed with algebraic properties (group, ring), these are transported to $E_{n}(\Sigma)$ and all the above results (where $\Sigma=\mathbb{Z}$ ) hold. When $\Sigma$ is a unit ring with unit group U , the units of $E_{n}(\Sigma)$ are the functions $f$ for which $f(1) \in U$ (generalising corollary 4.3 of theorem 4.1 above).

Further properties could be examined, that could give better hindsight into the algebraic structure of $E_{n}(\Sigma)$.

- Following Dowe's suggestion ([2], p.7), formulate and prove extensions of the above to fractional derivatives - thus departing from induction methods largely used in the present paper.


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[^1]:    ${ }^{1}$ The use of the expression "l'Hospital's rule" is debatable.

[^2]:    ${ }^{2}$ See also [2], lemma 6, p. 4 and corollary 7, p.5.

[^3]:    ${ }^{3}$ See also [2], theorems 5 and 8 ((i) to (iii)) concerning results for $n=0,1$.

[^4]:    ${ }^{4}$ See also [2], theorem 5 (v) p.3, theorem 8 (iv) p. 5 .

