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Author(s):

E. Ansari-piri, R.G. Sanati and M. Kardel

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A CHARACTERIZATION OF ORTHOGONALITY PRESERVING OPERATORS

E. ANSARI-PIRI, R.G. SANATI* AND M. KARDEL

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Abstract. In this paper, we characterize the class of orthogonality preserving operators on an infinite-dimensional Hilbert space H as scalar multiples of unitary operators between H and some closed subspaces of H. We show that any circle (centered at the origin) is the spectrum of an orthogonality preserving operator. Also, we prove that every compact normal operator is a strongly orthogonality preserving operator.
Keywords: Orthogonality preserving operators, adjoint of operators, isometry, unitary operators.
MSC(2010): Primary: 46L08; Secondary: 46L05.

1. Introduction

In a real normed linear space $(X, \|.\|)$, one may encounter in various definitions of orthogonality between two elements $x, y \in X$ such as

- Roberts orthogonality (1934): $\|x - \lambda y\| = \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$;
- Birkhoff orthogonality (1935): $\|x\| \leq \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$;
- Isosceles orthogonality (1945): $\|x - y\| = \|x + y\|$;
- Pythagorean orthogonality (1945): $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

Sikorska [15], Alonso and Benitez [1, 2] studied and compared these concepts of orthogonality and the relations between them.

The usual definition of orthogonality in a complex normed linear space $(X, \|.\|)$ is the Birkhoff-James orthogonality which says that x is orthogonal to y in X (and in this case we write $x \perp y$), if for each $\lambda \in \mathbb{C}$, $\|x\| \leq \|x + \lambda y\|$. Note that this definition is not symmetric in general, unless the norm comes from an inner product.

Let H be an infinite-dimensional Hilbert space and $B(H)$ be the C^*-algebra of all bounded linear operators acting on H. The spectrum of an element T

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*Corresponding author.

of $B(H)$ is defined by $\mathrm{sp}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible }\}$. Also, the spectral radius of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}$.

An operator $T : H \to H$ is said to be orthogonality preserving (OP in short), if $Tx \perp Ty$ whenever $x \perp y$, and T is called strongly orthogonality preserving (SOP in short), when $Tx \perp Ty \Leftrightarrow x \perp y$. In the last two decades, there has been considerable interest in the concepts of OP and SOP operators in normed spaces (for instance, see [7, 12] and [13]). Note that an OP operator need not to be linear or continuous. For example, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x, y) = \begin{cases} (x, y), & x \neq 0, y \neq 0; \\ (1, 1), & x \neq 0, y = 0; \\ (-1, 1), & y \neq 0, x = 0; \\ (0, 0), & x = y = 0. \end{cases}$$

Then T is neither linear nor continuous, but it is certainly OP. For more details and examples see [7, Examples 1.1 and 1.2]. Under the assumption of linearity, it is known that every linear OP operator between inner product spaces is a scalar multiple of an isometry (cf. [7]). In fact, if V, W are inner product spaces and $T : V \to W$ is an OP map, then there is a positive scalar γ such that $\|Tx\| = \gamma\|x\|$ for all $x \in V$. It should be noticed that, when $V = W = H$ is a Hilbert space, the equality $\|Tx\| = \gamma\|x\|$ holds for all $x \in H$, if and only if the equality $\langle Tx, Ty\rangle = \gamma^2\langle x, y\rangle$ holds for all $x, y \in H$. This shows that the set of all bounded linear OP operators coincides with the set of SOP operators in $B(H)$. In the setting of real normed spaces, Koldobski [13], and then for the general case with the Birkhoff-James orthogonality, Blanco and Turnšek [5], generalized Chmieliński's theorem ([7, Theorem 2.1]). Later, Ilišević and Turnšek [10] proved the same result for a Hilbert C^*-module E over a C^*-algebra A with the orthogonality defined by its A-valued inner product $\langle \cdot, \cdot\rangle_A$. Actually, using the fact that a C^*-algebra A is a closed $*$-subalgebra of $B(H)$ for some Hilbert space H, they proved that every A-linear orthogonality preserving operator on a Hilbert A-module is a scalar multiple of an isometry when A contains the C^*-algebra $K(H)$ of all compact operators on H. Note that one can take into account three notions of orthogonality in E:

(1) Usual orthogonality: $x \perp y \Leftrightarrow \langle x, y\rangle_A = 0$, for each $x, y \in E$;
(2) Birkhoff-James orthogonality in E as a normed space;
(3) Strongly Birkhoff-James orthogonality: $x \perp y \Leftrightarrow \|x\| \leq \|x + ay\|$, for each $x, y \in E$ and $a \in A$.

Arambašić and Rajić [3] compared these notions of orthogonality and obtained some relations between them. It is known that a C^*-algebra A is a Hilbert C^*-module over itself via the A-valued inner product $\langle a, b\rangle_A = ab^*$, where $a, b \in A$. Burgos, Fernández-Polo, Garcés and Peralta [6] studied orthogonality additive maps on this special class of Hilbert modules as elements of a more general class of Banach spaces known under the name of JB^*-triples. For more details about the orthogonality of additive maps and JB^*-triples the reader is referred to [6, 11].

Moreover, many kinds of approximately orthogonality preserving mappings and their stability have been studied, among others, by Chmieliński [7], Wójcik [16, 17], Moslehian and Zamani [18, 19]. Also recently, Chmieliński [8] introduced the reverse orthogonality preserving operators on normed spaces in the sense that the relation $x \perp y$ implies that $Ty \perp Tx$, where T is a nonzero linear operator on a normed space X and $x, y \in X$. Considering the standard definition of angle between two elements of a real inner product space, Moslehian, Zamani and Frank obtained some interesting results about angle preserving mappings (see [19]).

Also, Frank, Mishchenko and Pavlov [9], showed that for an orthogonality preserving operator $T \in B(H)$, $T^*T = \lambda.I_H$ where λ is a positive scalar and I_H is the identity operator on H. Using the polar decomposition of T as an element of $B(H)$, they proved that there is an isometry $V \in B(H)$ such that $T = \sqrt{\lambda}V$, and so $V^*V = I_H$.

In this paper, we consider an OP operator $T \in B(H)$ onto $M := \mathrm{ran}(T)$ (the range of T) and prove that T is OP if and only if it is a positive scalar multiple of a unitary from H onto the closed subspace M of H. Moreover, we characterize the class of normal OP operators as the class of surjective OP operators in $B(H)$ and, as we will see in Section 2, an OP operator $T \in B(H)$ is normal if and only if T^* is OP. Indeed, the following statements are equivalent in $OP(H)$, the set of all bounded linear OP operators on H:

(i) $T^* \in OP(H)$.
(ii) T is normal.
(iii) T is surjective.

Finally, we shall show that for any circle $\Gamma_r := \{\lambda \in \mathbb{C} : |\lambda| = r\}$ there exists an OP operator T such that $\mathrm{sp}(T) = \Gamma_r$.

2. Orthogonality preserving normal operators in $B(H)$

The main theorem of this section is Theorem 2.2. To prove it, we need the following Lemma.

Lemma 2.1. *Let H be a Hilbert space and $T \in B(H)$. If T is an orthogonality preserving operator, then so are T^*T and $|T|$.*

Proof. Let x, y be two orthogonal elements of H. Since $\langle x, y\rangle = 0$ and T is OP, we have $\langle Tx, Ty\rangle = 0$. Hence $\langle x, T^*Ty\rangle = 0$. Therefore $\langle Tx, TT^*Ty\rangle = 0$, whence $\langle T^*Tx, T^*Ty\rangle = 0$, which means that T^*T is OP.
Similarly, since $\langle x, T^*Ty\rangle = 0$ and $|T| = (T^*T)^{\frac{1}{2}}$, we have $\langle x, |T|^2y\rangle = 0$, so $\langle |T|x, |T|y\rangle = 0$, which ensures that $|T|$ is also OP. $\square$

It is well known that every unitary operator on a Hilbert space is an isometry and so it is an OP operator. Now, we characterize the OP operators, whose adjoin is also OP.

Theorem 2.2. *Let $T \in OP(H)$. Then $T^* \in OP(H)$ if and only if T is normal.*

Proof. Let T be a normal OP operator. Then $T^*T = TT^*$. Now if $\langle x, y\rangle = 0$, then $\langle T^*x, T^*y\rangle = \langle x, TT^*y\rangle = \langle x, T^*Ty\rangle = \langle Tx, Ty\rangle = 0$, which ensures that T^* is OP.
Conversely, let T and T^* be OP. Since H is an inner product space, T and T^* are scalar multiples of isometries. Thus, there exist $\gamma, \gamma' > 0$ such that $\|Tx\| = \gamma\|x\|$ and $\|T^*x\| = \gamma'\|x\|$ for all $x \in H$. Therefore, $\gamma' = \|T^*\| = \|T\| = \gamma$.
The operators $TT^*, T^*T, |T|$ and $|T^*|$ are OP by Lemma 2.1. Therefore, they are all scalar multiples of isometries. These scalars are evidently γ^2 for TT^*, T^*T and γ for $|T|$ and $|T^*|$. Now we have

$$\begin{aligned}
\langle T^*Tx - TT^*x, T^*Tx - TT^*x\rangle &= \langle T^*Tx, T^*Tx\rangle + \langle TT^*x, TT^*x\rangle \\
&\quad - \langle T^*Tx, TT^*x\rangle - \langle TT^*x, T^*Tx\rangle \\
&= \|T^*Tx\|^2 + \|TT^*x\|^2 \\
&\quad - \langle T^*Tx, TT^*x\rangle - \langle TT^*x, T^*Tx\rangle \\
&= 2\gamma^4\|x\|^2 - \langle T^*Tx, TT^*x\rangle - \langle TT^*x, T^*Tx\rangle.
\end{aligned}$$

It follows from $\langle T^*Tx, TT^*x\rangle = \langle |T|x, |T|TT^*x\rangle$ that

$$\begin{aligned}
\langle |T|x, |T|TT^*x\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \langle |T|x + i^k|T|TT^*x, |T|x + i^k|T|TT^*x\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k \langle |T|(x + i^kTT^*x), |T|(x + i^kTT^*x)\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k \||T|(x + i^kTT^*x)\|^2 \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k \gamma^2 \|(x + i^kTT^*x)\|^2 \\
&= \frac{1}{4}\gamma^2\sum_{k=0}^{3} i^k \langle x + i^kTT^*x, x + i^kTT^*x\rangle \\
&= \gamma^2\langle x, TT^*x\rangle \\
&= \gamma^2\langle T^*x, T^*x\rangle
\end{aligned}$$

$$\begin{aligned} &= \gamma^2\|T^*x\|^2 \\ &= \gamma^4\|x\|^2. \end{aligned}$$

Hence, $\langle T^*Tx, TT^*x\rangle = \gamma^4\|x\|^2$. Also, $\langle TT^*x, T^*Tx\rangle = \gamma^4\|x\|^2$ by changing the role of T by T^*. Thus, we have

$$\langle T^*Tx - TT^*x, T^*Tx - TT^*x\rangle = 2\gamma^4\|x\|^2 - \gamma^4\|x\|^2 - \gamma^4\|x\|^2 = 0 \ (x \in H).$$

Hence $\|T^*T - TT^*\| = 0$ and so $TT^* = T^*T$, i.e., T is normal. □

Corollary 2.3. *If $T, T^* \in OP(H)$, then*

$$\|Tx\| = r(T)\|x\| = \|T^*x\|,$$

for all $x \in H$.

Proof. Let T and T^* be OP. Then, there is $\gamma > 0$ such that $\|Tx\| = \gamma\|x\| = \|T^*x\|$, for all $x \in H$. By Theorem 2.2, T is normal. Since T is an element of the C^*-algebra $B(H)$, $\gamma = \|T\| = r(T) = \|T^*\|$. □

Corollary 2.4. *If $T \in OP(H)$, then $\|Tx\| = r(|T|)\|x\|$, for all $x \in H$.*

Proof. Let T be an OP operator. Then, by Lemma 2.1, $|T|$ is also OP. Therefore, there exists $\gamma > 0$ such that

$$\|Tx\| = \gamma\|x\| = \||Tx|\| \ (x \in H).$$

But $|T|$ is self-adjoint and so is normal. Therefore, by Corollary 2.3,

$$\gamma = \||T|\| = r(|T|).$$

□

3. Characterization of OP operators as multiples of unitaries

The main results of this section are Theorem 3.2, Theorem 3.3 and Corollary 3.5. Before proving these, we need a lemma.

Lemma 3.1. *Let A be a unital C^*-algebra and U be a unitary element in A such that $\mathrm{sp}(U) \neq \Gamma_1$. Then, there exists a self adjoint element $a \in A$ such that $U = \exp(ia)$.*

Proof. [14, Theorem 1.2.12]. □

Theorem 3.2. *If $T \in OP(H)$, then T is a scalar multiple of a unitary between Hilbert spaces H and $\mathrm{ran}(T)$.*

Proof. Let T be OP. Using Corollary 2.4, we have $\|Tx\| = r(|T|)\|x\|$, for all $x \in H$. Set $M = \mathrm{ran}(T)$. Since T is a multiple of an isometry, T is injective and M is a closed subspace of H. Hence, $T : H \to M$ is a bijective linear continuous OP map and so it is invertible. Let $S : M \to H$ be the inverse of T and $y \in M$. Then, there is $x_0 \in H$ such that $Tx_0 = y$. Therefore,

$$\|Sy\| = \|STx_0\| = \|x_0\| = \frac{1}{r(|T|)}\|Tx_0\| = \frac{1}{r(|T|)}\|y\|.$$

This shows that S is a scalar multiple of an isometry and so it is an OP map. Now let $x \in H$ and $y \in M$. Then,

$$\begin{aligned}
\langle Tx, y\rangle &= \langle x, T^*y\rangle \\
&= \langle T^*S^*x, T^*y\rangle \\
&= \langle S^*x, TT^*y\rangle \\
&= r(|T|)^2\langle SS^*x, STT^*y\rangle \\
&= r(|T|)^2\langle SS^*x, T^*y\rangle \\
&= r(|T|)^2\langle S^*x, S^*T^*y\rangle \\
&= \langle r(|T|)^2 S^*x, y\rangle.
\end{aligned}$$

Hence,

$$\langle Tx - r(|T|)^2S^*x, y\rangle = 0 \quad (y \in M).$$

Therefore, $Tx = r(|T|)^2S^*x$. Since x is arbitrary,

$$T = r(|T|)^2S^* \quad \text{or equivalently} \quad S = \frac{1}{r(|T|)^2}T^*.$$

Now we have,

$$\left(\frac{1}{r(|T|)}T\right)\left(\frac{1}{r(|T|)}T\right)^* = T\left(\frac{1}{r(|T|)^2}T^*\right) = TS = I_M.$$

Also,

$$\left(\frac{1}{r(|T|)}T\right)^*\left(\frac{1}{r(|T|)}T\right) = \left(\frac{1}{r(|T|)^2}T^*\right)T = ST = I_H.$$

Thus, $\frac{1}{r(|T|)}T$ is a unitary map between Hilbert spaces H and M, which means that T is a scalar multiple of a unitary.

□

Theorem 3.3. *Let $T \in OP(H)$. Then T is normal if and only if T is surjective.*

Proof. Let T be a surjective OP operator. Theorem 3.2 shows that T is normal because $\mathrm{ran}(T) = H$.

For the converse, let T be normal and $M = \mathrm{ran}(T)$. By Theorem 3.2, there is a unitary operator $U : H \to M$ such that $T = \gamma U$ for some $\gamma > 0$ and $UU^* = I_M$, $U^*U = I_H$. Let $U^\star$ be the adjoint of the operator U as an element of $B(H)$. Clearly, $U^\star|_M = U^*$. Since $H = M \oplus M^\perp$, we can write for every $x \in H:\ x = Ty + z$ for some $y \in H,\ z \in M^\perp$. Now we have

$$UU^\star x = U\left(U^\star(Ty + z)\right) = U(U^\star Ty + U^\star z). \tag{3.1}$$

We have $U^\star z = 0$, because $z \in M^\perp = \mathrm{ran}(T)^\perp$ and $T = \gamma U$, so $z \in \mathrm{ran}(U)^\perp = \ker(U^\star)$. Now, equation 3.1 implies that

$$UU^\star x = UU^\star(Ty) = UU^*(Ty) = Ty.$$

Hence, for every $x \in H$, we have

$$UU^\star x = \begin{cases} x, & x \in M; \\ 0, & x \in M^\perp. \end{cases}$$

On the other hand, $U^\star Ux = U^*Ux = x$. Since $T = \gamma U$ is normal, U is normal. Therefore, $UU^\star x = U^\star Ux$. But it is impossible unless, $M^\perp = \{0\}$. Therefore, $\mathrm{ran}(T) = H$, i.e., T is surjective. □

Theorem 3.4. *Let $T \in OP(H)$. Then, the following statements are equivalent:*
- (i) $T^* \in OP(H)$.
- (ii) *T is normal.*
- (iii) *T is surjective.*

Proof. This immediately follows from Theorem 2.2 and Theorem 3.3. □

Corollary 3.5. *Let $T \in OP(H)$ and one of the following conditions holds:*
- (i) $T^* \in OP(H)$;
- (ii) *T is normal;*
- (iii) *T is surjective.*

Then,
1. $\mathrm{sp}(T) \subseteq \Gamma_{r(T)} = \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$.
2. *If $\mathrm{sp}(T) \neq \Gamma_{r(T)}$, then there exists a self adjoint operator S such that $T = \exp(\alpha + iS)$ where $\alpha = \log(r(T))$.*

Proof. Applying Theorem 3.4, together with one of the conditions (i)-(iii), we may assume that T is a normal and invertible operator in $B(H)$. Since T is normal and OP, by Corollary 2.4, $\|Tx\| = r(T)\|x\|$ for each $x \in H$. Also, T^{-1} is OP and normal. Thus $\|T^{-1}x\| = r(T^{-1})\|x\|$. It is straightforward for OP operators to check that $r(T^{-1}) = \frac{1}{r(T)}$. We know that $\mathrm{sp}(T^{-1}) = \{\lambda^{-1} : \lambda \in \mathrm{sp}(T)\}$. Therefore, we have

$$r(T^{-1}) = \sup\{|\lambda^{-1}| : \lambda \in \mathrm{sp}(T)\} = \frac{1}{\inf\{|\lambda| : \lambda \in \mathrm{sp}(T)\}}$$

(note that $\inf\{|\lambda| : \lambda \in \mathrm{sp}(T)\} \neq 0$, because T is invertible and so the compact set $\mathrm{sp}(T)$ does not contain 0).
On the other hand,

$$r(T^{-1}) = \frac{1}{r(T)} = \frac{1}{\sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}}$$

whence, $\inf\{|\lambda| : \lambda \in \mathrm{sp}(T)\} = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}$, which yields that $\mathrm{sp}(T) \subseteq \Gamma_{r(T)}$.

Now, if $\mathrm{sp}(T) \neq \Gamma_{r(T)}$, then $r(T)^{-1}\mathrm{sp}(T) \neq \Gamma_1$. Since T is normal, by the spectral theorem, $r(T)^{-1}\mathrm{sp}(T) = \mathrm{sp}(r(T)^{-1}T) \neq \Gamma_1$. As we observed in the proof of Theorem 3.2, $r(T)^{-1}T$ is a unitary operator. Hence, applying Lemma 3.1, there is a self adjoint operator S such that $r(T)^{-1}T = \exp(iS)$ which implies that $T = r(T)\exp(iS)$. Thus, we have $T = \exp(\alpha + iS)$, where $\alpha = \log(r(T))$. □

In what follows, $N(H)$ and $F(H)$ are the set of all normal operators and finite-rank operators in $B(H)$, respectively.

Corollary 3.6. *Let H be an infinite dimensional Hilbert space. For any $T \in B(H)$:*

(i) *If $T \in OP(H) \cap F(H)$, then $T \in B(H)\setminus N(H)$ and $T^* \in B(H)\setminus OP(H)$.*
(ii) *If $T \in OP(H) \cap N(H)$, then $T \in B(H) \setminus F(H)$.*

Proof. (i) Since H is an infinite-dimensional space, if $T \in OP(H) \cap F(H)$, then T can not be a surjective operator and so, by Theorem 3.4, it can not be normal and T^* is not OP. A similar argument proves (ii). □

As we see in Corollary 3.5, the spectrum of every normal OP operator is contained in Γ_r, for some $r > 0$. It is natural to ask if there is a normal OP operator T such that $\mathrm{sp}(T) = \Gamma_r$. The following proposition gives an affirmative answer to this question. In fact, we show that the cardinal number of the set of such operators is greater than that of the real numbers.

Recall that a diagonalisable operator T on a separable infinite dimensional Hilbert space H is of the form $T(e_n) = \lambda_n e_n$ in which (λ_n) is a bounded sequence of complex numbers and (e_n) is an orthonormal basis for H. Moreover, $\|T\| = \sup_n\{|\lambda_n|\}$ and the spectrum of T is the closure of the set $\{\lambda_n : n = 1, 2, \ldots\}$ (cf. [14, Example 1.4.3]).

Proposition 3.7. *Let $r \in \mathbb{R}^+$ and H be a separable Hilbert space. Then, there exists an operator $T \in OP(H)$ such that* $\mathrm{sp}(T) = \Gamma_r$.

Proof. Let (e_n) is an orthonormal basis for H. Since Γ_r is a compact subset of $\mathbb{C}$, we can choose a dense sequence (λ_n) in Γ_r. Define the diagonalisable operator $T : H \to H$ with $T(e_n) = \lambda_n e_n$. Since the spectrum of T is the closure of the set $\{\lambda_n : n = 1, 2, \ldots\}$, $\mathrm{sp}(T) = \Gamma_r$. Now, we claim that T is SOP.

Let $x, y \in H$. Then

$$x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n \ , \ y = \sum_{m=1}^{\infty}\langle y, e_m\rangle e_m;$$

$$\langle x, y\rangle = \sum_{n=1}^{\infty}\langle x, e_n\rangle\langle e_n, y\rangle \ .$$

Since T is continuous, we have

$$Tx = T\left(\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n\right) = \sum_{n=1}^{\infty}\langle x, e_n\rangle T(e_n) = \sum_{n=1}^{\infty}\langle x, e_n\rangle \lambda_n e_n$$

and in the same way

$$Ty = \sum_{n=1}^{\infty}\langle y, e_n\rangle \lambda_n e_n.$$

Therefore,

$$\begin{aligned}
\langle Tx, Ty\rangle &= \langle \sum_{n=1}^{\infty}\langle x, e_n\rangle \lambda_n e_n, \sum_{m=1}^{\infty}\langle y, e_m\rangle \lambda_m e_m\rangle \\
&= \sum_{n=1}^{\infty} |\lambda_n|^2 \langle x, e_n\rangle\langle e_n, y\rangle \\
&= r^2 \sum_{n=1}^{\infty}\langle x, e_n\rangle\langle e_n, y\rangle \\
&= r^2\langle x, y\rangle,
\end{aligned}$$

which implies that T is SOP. □

Remark 3.8. Let T be a diagonalisable operator in $B(H)$ with respect to a bounded sequence (λ_n) dominated by a positive number M and an orthonormal basis (e_n) for H. As we observed in the proof of Proposition 3.7, we have

$$\begin{aligned}
\langle Tx, Ty\rangle &= \langle \sum_{n=1}^{\infty}\langle x, e_n\rangle \lambda_n e_n, \sum_{m=1}^{\infty}\langle y, e_m\rangle \lambda_m e_m\rangle \\
&= \sum_{n=1}^{\infty} |\lambda_n|^2 \langle x, e_n\rangle\langle e_n, y\rangle \\
&\leq M \sum_{n=1}^{\infty}\langle x, e_n\rangle\langle e_n, y\rangle \\
&= M\langle x, y\rangle.
\end{aligned}$$

This shows that T is OP and so it is SOP.

Corollary 3.9. *Every compact normal operator $T \in B(H)$ is SOP.*

Proof. From [14, Theorem 2.4.4], every compact normal operator $T \in B(H)$ is diagonalisable. Now, Remark 3.8 implies that T is SOP. □

Note. By Corollary 3.9, every compact normal operator $T \in B(H)$ is SOP and so it is surjective by Theorem 3.4. If H is an infinite dimensional space, T can not be a finite-rank operator. This shows that for an infinite dimensional

Hilbert space H, one can bring forward a more exact expression of [14, Theorem 2.4.4] as follows:

Theorem 3.10. *If* $T \in (K(H) \setminus F(H)) \cap N(H)$*, then* T *is diagonalisable.*

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(Esmaeil Ansari-Piri) Department of Mathematics, University of Guilan, P.O. Box 1914, Rasht, Iran.

E-mail address: eansaripiri@gmail.com

(Reza Ganj Bakhsh Sanati) Institute of Higher Education of ACECR(Academic Center of Education and Culture Research), Rasht branch, Iran.

E-mail address: reza_sanaaty@gmail.com

(Morteza Kardel) Department of Mathematics, University Campus 2, University of Guilan, P.O. Box 1914, Rasht, Iran.
(As a faculty member of (and supported by) Islamic Azad University, Zabol branch, Zabol, Iran.)

E-mail address: mkardelmath@gmail.com