ON THE CODERIVATIONS OF SOME HOPF ALGEBRAS

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ABSTRACT. In this paper we specify the form of coderivations of polynomials on \mathbb{R} and we define the continuous extended coderivations of $C(\mathbb{R})$. We also introduce a cocommutative bialgebra H such that the set of coderivations of H is zero.

1. Introduction and Preliminaries

Let K be a field. For a given vector space V over K, we denote the linear dual space of V over a field K by $V^* = Hom_K(V, K)$. We refer to [7] for definition of a coalgebra. If (C, Δ, ε) is a coalgebra over K and (A, M, U) is a finite- dimensional algebra over K, then C^* and A^* have respectively algebra and coalgebra structures (See[7,Propositions 1.1.1 and 1.1.2]). We will use Sweedler's \sum -notation and conventions extensively; for example, we will write $\Delta(d) = \sum_{(d)} d_{(1)} \otimes d_{(2)}$, for $d \in C$, etc.

In the first section of this paper we first prove that if D is a coderivation of C then D^* is a derivation of C^* and if D is a derivation of a finite dimensional algebra A then D^* is a coderivation of A^* . We then prove that the space of coderivations of a coalgebra C has a lie algebra structure with lie bracket, $[D_1, D_2] =$

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 $D_1 o D_2 - D_2 o D_1$ where D_1 and D_2 are coderivations of C. Furthermore we specify the form of coderivations of polynomials on R.

In the second section we prove that $C(\mathbf{R}) \otimes C(\mathbf{R})$ can be identified with a dense subset of $C(\mathbb{R} \times \mathbb{R})$. Furthermore we define the continuous extended coderivations of $C(\mathbf{R})$ and propose a conjecture about continuous extended coderivations of $C(\mathbb{R})$.

In the third section we prove if $\overline{D}: C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ is a continuous extended coderivation of $C^{\infty}(\mathbb{R})$ then $\overline{D}(f) = \sum_{k=0}^{\infty} b_k \frac{d^k}{dX^k}(f)$,

for any $f \in C^{\infty}(\mathbb{R})$.

Finally, in the fourth section we prove that the space of coderivations of the group algebra K[G] is zero, where G is a group.

2. Coderivations of Coalgebras

Definition 2.1. Suppose (C, Δ, ε) is a coalgebra over a field K. A coderivation is an endomorphism $D: C \to C$ of a coalgebra C which satisfies

 $\Delta oD = (1_C \otimes D + D \otimes 1_C) o\Delta.[5]$

We prove Propositions (2.2 and 2.3) below which are stated in |5|.

Proposition 2.2. Suppose (C, Δ, ε) is a coalgebra and $D \in End_K(C)$ is a coderivation of C. Then $D^* \in End_K(C^*)$ is a derivation of C^* .

Proof. Since (C, Δ, ε) is a coalgebra, (C^*, M, U) is an algebra such that

 $M = \Delta^* o \rho$, where ρ is the linear injection $\rho : C^* \otimes C^* \to (C \otimes C)^*$ given by $\langle \rho(a^* \otimes b^*), c \otimes d \rangle = \langle a^*, c \rangle \langle b^*, d \rangle$ for all $a^*, b^* \in C^*$ and $c, d \in C$.

Hence $M = \Delta^*|_{c^* \otimes c^*}$ is a product in the dual algebra C^* . Since D is a coderivation of C, hence $\Delta oD = (1_C \otimes D + D \otimes 1_C)o\Delta$. Taking transposes we have

 $D^*oM = M * (1_{C^*} \otimes D^* + D^* \otimes 1_{C^*})$. Thus D^* is a derivation of C^* . \Box

Let A be any algebra. A cofinite ideal of A is an ideal J of A such that the quotient space $\frac{A}{J}$ is finite- dimensional. Define $A^{\circ} = \{g \in A^* | ker(g) \text{ contains a cofinite ideal of } A\}$. Then A° is a subspace of A^* ; it is obvious that if A is finite dimensional then $A^{\circ} = A^*$.

If (A, M, U) is any algebra then $(A^{\circ}, \Delta^{\circ}, \varepsilon^{\circ})$ is a coalgebra such that $\Delta^{\circ} = M^*|_{A^{\circ}}$; moreover, for any $f \in A^{\circ}$, $\varepsilon^{\circ}(f) = f(1)$ [7, Proposition 6.0.2].

Proposition 2.3. Suppose that (A, M, U) is any algebra and D: $A \to A$ is a derivation of A. Then $D^{\circ} = D^*|_{A^{\circ}}$ defines a coderivation $D^{\circ} : A^{\circ} \to A^{\circ}$ of the dual coalgebra of A. In particular if A is finite dimensional algebra then $D^* \in End(A^*)$ is a coderivation of A^* .

Proof. We first show that $D^*(A^\circ) \subseteq A^\circ$. Let $f \in A^\circ$. There exists a cofinite ideal I of A such that $I \subseteq ker(f)$. Put $J = D^{-1}(I) \cap I$. It is clear that J is an ideal of A. The pullback of a cofinite subspace is a cofinite subspace and that the intersection of two cofinite subspace is cofinite. Hence J is a cofinite ideal of A.

Since $D^*(A^\circ) \subseteq A^\circ$, hence $D^\circ = D^*|_{A^\circ}$ defines a linear map $D^\circ: A^\circ \to A^\circ$ of the dual coalgebra of A. Since D is a derivation, so $DoM = Mo(1_A \otimes D + D \otimes 1_A)$. Therefore $\Delta^\circ oD = (1_{A^\circ} \otimes D^\circ + D^\circ \otimes 1_{A^\circ})o\Delta^\circ$ and D° is a coderivation. \Box

Proposition 2.4. Let COD(C) be the space of all coderivations of coalgebra C. Then COD(C) is a Lie algebra under the associative braket for endomorphisms of C.

Proof. [5, Lemma 3(b)]. \Box

We denote the space of continuous functions on \mathbb{R} by $C(\mathbb{R})$ and the space of polynomial functions on \mathbb{R} by $\mathbb{R}[X] = \{F(X) \in C(\mathbb{R}) \mid F(X) = \sum_{i=1}^{m} a_i X^i\}$.

$$F(X) = \sum_{i=0}^{\infty} a_i X^i \}$$

By [3, Proposition III.1.4], there exists an isomorphism of coalgebras

$$\mathbf{R}[X] \otimes \mathbf{R}[X] \cong \mathbf{R}[X,Y] . \tag{*}$$

Now we define the linear maps Δ, ε and S as follows. For all $f \in \mathbb{R}[X]$ (note that f is continuous) and $a, b \in \mathbb{R}$, by (*) we have

$$\Delta : \mathbf{R}[X] \to \mathbf{R}[X] \otimes \mathbf{R}[X] \text{ by } \Delta(f)(a,b) = f(a+b)$$

$$\varepsilon : \mathbf{R}[X] \to \mathbf{R} \text{ by } \varepsilon(f) = f(0) \text{ and}$$

$$S : \mathbf{R}[X] \to \mathbf{R}[X] \text{ by } S(f)(a) = f(-a)$$

It is easy to show that $\mathbb{R}[X]$ has a cocommutative Hopf algebra structure, [1, page 83] and [4, page 25].

Lemma 2.5. If
$$F(X) = \sum_{i=0}^{m} a_i X^i \in Poly(\mathbb{R})$$
 then
$$\Delta(F(X)) = \sum_{i=0}^{m} \sum_{k=0}^{i} a_i \binom{i}{k} X^k \otimes X^{i-k}$$

Proof. Since $\Delta(X) = X \otimes 1 + 1 \otimes X$ and Δ is an algebra map, hence

 $\Delta(X^n) = \sum_{k=0}^n {n \choose k} X^k \otimes X^{n-k}, \ (n = 0, 1, 2, \dots) \ . \ \text{Since } \Delta \text{ is also a linear map,}$

$$\Delta(F(X)) = \sum_{i=0}^{m} \sum_{k=0}^{i} a_i \binom{i}{k} X^k \otimes X^{i-k}. \Box$$

Let *H* be a cocommutative Hopf algebra and $\mathcal{L} = \{h \in H | \Delta(h) = h \otimes 1 + 1 \otimes h\}$ be the space of primitives of *H*. Let $C_{\mathcal{L}}(H) = \{\phi \in End(H) | (\phi.g)(1) \in \mathcal{L} \text{ for all } g \in H\}$ where $(\phi.g)(h)$ is defined as $\sum_{(g)} S(g_{(1)})\phi(g_{(2)}h)$ for all $h \in H$. [5]

Proposition 2.6. In a cocommutative Hopf algebra H, $COD(H) = C_{\mathcal{L}}(H)$.

Proof. [5, Theorem 1]. \Box

Lemma 2.7. Let $f \in \mathbb{R}[X]$. $f \in \mathcal{L}$ if and only if f = cX where c = f(1).

Proof. Let $f \in \mathcal{L}$, so $\Delta(f) = f \otimes 1 + 1 \otimes f$. Thus by the definition of Δ , f(a+b) = f(a) + f(b) for all $a, b \in \mathbb{R}$. Since f is a continuous function, hence f = cX. It is clear that c = f(1). The converse is trivial. \Box

Proposition 2.8. $D \in COD(Poly(\mathbb{R}))$ if and only if

$$D(X^{n}) = \sum_{k=0}^{n} {n \choose k} [D(X-1)^{k}](1) X^{n-k+1} \text{ for } n = 0, 1, 2, \dots$$

Proof. Put f = D(1). Since $\Delta(1) = 1 \otimes 1$ it follows that f = S(1)f = f(1). By Proposition 2.6, this element is primitive and by Lemma 2.7 it follows that f = f(1)X.

We now compute $(Dg)(1) = \sum_{(g)}^{N} S(g_{(1)})D(g_{(2)})$ for g = X using the fact that $\Delta(X) = X \otimes 1 + 1 \otimes X$.

$$S(X)f + S(1)D(X) = -X^2f(1) + D(X)$$

Let $h(X) = -X^2 f(1) + D(X)$. By Propositon 1.6, h(X) is primitive and by Lemma 1.7, h(X) = h(1)X. Thus

$$-X^{2}f(1) + D(X) = h(X)$$

= $h(1)X$
= $-Xf(1) + X[D(X)](1)$

and hence

$$D(X) = X^{2} f(1) - X f(1) + X[D(X)](1) .$$

By induction it is easy to show that $D(X^n) = \sum_{k=0}^n {n \choose k} [D(X - 1)^k](1)X^{n-k+1}$, for n = 0, 1, ...

Theorem 2.9. $D \in COD(\mathbb{R}[X])$, if and only if

$$D(F) = \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k}(F),$$

where $b_k = \frac{[D(X-1)^k](1)}{k!}$, for all $F \in \mathbf{R}[X]$.

Proof. Let $F = \sum_{n=0}^{m} a_n X^n \in \mathbf{R}[X]$, then:

$$D(F) = \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k} (\sum_{n=0}^m a_n X^n)$$

=
$$\sum_{k=0}^{\infty} \sum_{n=k}^m b_k X a_n {n \choose k} k! X^{n-k}$$

=
$$\sum_{k=0}^{\infty} \sum_{n=k}^m a_n {n \choose k} X^{n-k+1} [D(X-1)^k] (1). \quad (*)$$

On the other hand by Proposition 2.8 we have:

$$D(F) = \sum_{n=0}^{m} a_n D(X^n) = \sum_{n=0}^{m} \sum_{k=0}^{n} a_n {n \choose k} X^{n-k+1} [D(X-1)^k](1). \quad (**)$$

Since the coefficients of X^n in (*) and (**) are equal for all $n \ge 1$, hence (*) = (**). The converse is true by Proposition 2.6. \Box

3. On the Continuous Extended Coderivations of $C(\mathbb{R})$

The seminorms $P_k(f) = \sup\{|f(x,y)| : (x,y) \in [-k,k] \times [-k,k]\}$ induce the metric $d(f,g) = \sum_{k=1}^{\infty} \frac{2^{-k}P_k(f-g)}{1+P_k(f-g)}$ in the space $C(\mathbb{R} \times \mathbb{R})$, [6, Chapter 1, Ex. 18].

Lemma 3.1. $C(\mathbb{R}) \otimes C(\mathbb{R})$ can be identified with a dense subset of $C(\mathbb{R} \times \mathbb{R})$.

Proof. Let
$$Z = \sum_{i=1}^{n} f_i \otimes g_i$$
 belong to $C(\mathbb{R}) \otimes C(\mathbb{R})$. Define
 $\varphi : C(\mathbb{R}) \times C(\mathbb{R}) \to C(\mathbb{R} \times \mathbb{R})$

by $\varphi(f,g)(x,y) = f(x)g(y)$, where f and g are in $C(\mathbb{R})$ and x, yare in \mathbb{R} . It is clear that φ is a bilinear map. Now by the universal mapping property, there exists a unique linear map $F : C(\mathbb{R}) \otimes$ $C(\mathbb{R}) \to C(\mathbb{R} \times \mathbb{R})$ such that $F(Z)(x,y) = \sum_{i=1}^{n} f_i(x)g_i(y)$. We will show that the image of $C(\mathbb{R}) \otimes C(\mathbb{R})$ is dense in $C(\mathbb{R} \times \mathbb{R})$. Let $f \in$ $C(\mathbb{R} \times \mathbb{R})$ and $\varepsilon > 0$ be given. We choose N > 0 sufficiently large. By the Stone-Weierstrass theorem there exist continuous functions f_i and g_i , $i = 1, 2, \ldots, n$, such that $|f(x,y) - \sum_{i=1}^{n} f_i(x)g_i(y)| < \frac{\varepsilon}{2N}$, for any $(x,y) \in [-N,N] \times [-N,N]$. Setting $Z = \sum_{i=1}^{n} f_i \otimes g_i$, we have:

$$d(f - F(Z)) = \sum_{k=1}^{\infty} \frac{2^{-k} P_k(f - F(Z))}{1 + P_k(f - F(Z))}$$

$$< \sum_{k=1}^{N} \frac{2^{-k} P_k(f - F(Z))}{1 + P_k(f - F(Z))} + \frac{\varepsilon}{2}$$

$$\leq N P_N(f - F(Z)) + \frac{\varepsilon}{2}$$

$$= N \sup\{|f(x, y) - F(Z)(x, y)| : (x, y) \in [-N, N]\}$$

$$\times [-N, N]\} + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \cdot$$

It remains to show that the map F is one-to-one. The proof follows by induction. Suppose that $f \otimes g \neq 0$. Then $f \neq 0$ and $g \neq 0$. Thus there exist a and b such that $f(a) \neq 0$ and $g(b) \neq 0$. We conclude that $F(f \otimes g)(a, b) = f(a)g(b) \neq 0$ and hence $F(f \otimes g) \neq 0$. Now we assume that the assertion is true for k. Let $\sum_{i=1}^{k+1} f_i(x)g_i(y) = 0$. If $f_{k+1} \equiv 0$ then the proof is complete. If $f_{k+1} \not\equiv 0$ then there exists $x_0 \in \mathbb{R}$ such that $f_{k+1}(x_0) \neq 0$. For every $y \in \mathbb{R}$, we have $g_{k+1}(y) = -\sum_{i=1}^k \frac{f_i(x_0)}{f_{k+1}(x_0)}g_i(y)$. Hence $\sum_{i=1}^k \left[f_i(x) - \frac{f_{k+1}(x)f_i(x_0)}{f_{k+1}(x_0)}\right]g_i(y) = 0$. By induction, $\sum_{i=1}^{k+1} f_i \otimes g_i =$ $\sum_{i=1}^k \left[f_i - \frac{f_{k+1}f_i(x_0)}{f_{k+1}(x_0)}\right] \otimes g_i = 0$, and the proof is complete. \Box

We denote the closure of $C(\mathbb{R}) \otimes C(\mathbb{R})$ in $C(\mathbb{R} \times \mathbb{R})$ by $\overline{C(\mathbb{R})} \otimes C(\mathbb{R})$ and by Lemma 3.1, $\overline{C(\mathbb{R})} \otimes C(\mathbb{R})$ can be identified with $C(\mathbb{R} \times \mathbb{R})$.

Now we define the linear maps $\overline{\Delta}, \overline{\varepsilon}$ and \overline{S} as follows. For all $f \in C(\mathbb{R})$ and $a, b \in R$

$$\overline{\Delta}: C(\mathbf{\mathbb{R}}) \to \overline{C(\mathbf{\mathbb{R}}) \otimes C(\mathbf{\mathbb{R}})} \text{ by } \overline{\Delta}(f) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \widehat{f}_{ni} \text{ where}$$
$$\overline{\Delta}(f)(a,b) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(a) \widehat{f}_{ni}(b) = f(a+b)$$
$$\overline{\varepsilon}: C(\mathbf{\mathbb{R}}) \to \mathbf{\mathbb{R}} \text{ by } \overline{\varepsilon}(f) = f(0) \text{ and}$$

 $\overline{\varepsilon}: C(\mathbf{\mathbb{R}}) \to \mathbf{\mathbb{R}} \text{ by } \overline{\varepsilon}(f) = f(0) \text{ and} \\ \overline{S}: C(\mathbf{\mathbb{R}}) \to C(\mathbf{\mathbb{R}}) \text{ by } \overline{S}(f)(a) = f(-a)$

Proposition 3.2. The linear map $\overline{\Delta} : C(\mathbb{R}) \to \overline{C(\mathbb{R}) \otimes C(\mathbb{R})}$ is continuous with respect to the topology induced by the identification in Lemma 3.1.

Proof. The seminorms $P_k(f) = Sup\{|f(x)|: x \in [-k, k]\}$ induce the metric $d(f,g) = \sum_{k=1}^{\infty} \frac{2^{-k}P_k(f-g)}{1+P_k(f-g)}$ in the space $C(\mathbb{R})$. Let $\{f_n\}$ be a sequence in $C(\mathbb{R})$ such that f_n converges to zero w.r.t. themetric d. We first show that $\lim_{n\to\infty} P_k(f_n) = 0$ for all k. Since $\sum_{k=1}^{\infty} \frac{2^{-k}P_k(f_n)}{1+P_k(f_n)}$ is uniformly convergent, we have $0 = \lim_{n\to\infty} d(f_n, 0) =$ $\sum_{k=1}^{\infty} 2^{-k} \lim_{n\to\infty} (\frac{P_k(f_n)}{1+P_k(f_n)})$. Hence $\lim_{n\to\infty} P_k(f_n) = 0$, for all k.

On the other hand,

$$0 \le P_k(\overline{\Delta}(f_n)) = \sup\{|\overline{\Delta}(f_n)(a,b)| : (a,b) \in [-k,k] \times [-k,k]\} \\ = \sup\{|f_n(a+b)| : (a,b) \in [-k,k] \times [-k,k]\} \\ \le \sup\{|f_n(t)| : t \in [-2k,2k]\} = P_{2k}(f_n) .$$

Hence $\lim_{n\to\infty} P_k(\overline{\Delta}(f_n)) = 0$, for all k. Now since $\overline{\Delta}$ is a linear map, if f_n converges to zero w.r.t. the metric d, then to prove the theorem it suffices to show that $\overline{\Delta}(f_n)$ converges to zero w.r.t. the metric d. Let $\varepsilon > 0$ be given. Since $\sum_{k=1}^{\infty} \frac{2^{-k} P_k(\overline{\Delta}(f_n))}{1 + P_k(\overline{\Delta}(f_n))}$ is uniformly convergent, there exists $N(\epsilon) > 0$ such that

$$\sum_{k=1}^{\infty} \frac{2^{-k} P_k(\overline{\Delta}(f_n))}{1 + P_k(\overline{\Delta}(f_n))} < \left[\sum_{k=1}^{N(\epsilon)} 2^{-k} P_k(\overline{\Delta}(f_n))\right] + \epsilon/2$$
$$\leq \left[\sum_{k=1}^{N(\epsilon)} 2^{-k} P_{2k}(f_n)\right] + \epsilon/2 .$$

Hence $\lim_{n \to \infty} d(\overline{\Delta}(f_n), 0) = 0.$

Lemma 3.3. The linear map $\overline{S}: C(\mathbb{R}) \to C(\mathbb{R})$ is continuous.

Proof. Let $\{f_n\}$ be a sequence of $C(\mathbb{R})$ such that f_n converges to zero w.r.t. the metric d (in Proposition 3.2). We show that $\lim_{n\to\infty} P_k(\overline{S}(f_n)) = 0$, for all k.

$$P_k(\overline{S}(f_n)) = \sup\{|\overline{S}(f_n)(x)| : x \in [-k, k]\}$$

=
$$\sup\{|f_n(-x)|x \in [-k, k]\} = P_k(f_n).$$

In the proof of Proposition 3.2, we showed that $\lim_{n\to\infty} P_k(f_n) = 0$ for all k; thus $\lim_{n\to\infty} P_k(\overline{S}(f_n)) = 0$, for all k. By a proof similar to that of Proposition 3.2 we can show that $\overline{S}(f_n)$ converges to zero w.r.t. the metric d. \Box

Lemma 3.4. The linear maps $\overline{\Delta}, \overline{\varepsilon}$ and \overline{S} satisfy the following properties.

For
$$\overline{\Delta}(f) = \sum_{i=0}^{m_n} f_{ni} \otimes \hat{f}_{ni}, \ f, g \in C(\mathbb{R}) \ and \ a \in \mathbb{R}.$$

I) $\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\Delta}(f_{ni}) \otimes \hat{f}_{ni} = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \overline{\Delta}(\hat{f}_{ni})$
II) $\overline{\Delta}(f)\overline{\Delta}(g) = \overline{\Delta}(fg)$
III) $\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\varepsilon}(f_{ni})\hat{f}_{ni} = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}\overline{\varepsilon}(\hat{f}_{ni}) = f$
IV) $\overline{\varepsilon}(fg) = \overline{\varepsilon}(f)\overline{\varepsilon}(g)$
V) $\left(\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{S}(f_{ni})\hat{f}_{ni}\right)(a) = \left(\lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}\overline{S}(\hat{f}_{ni})\right)(a) = f(0) = \varepsilon(f)$
VI) $\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{S}(f_{ni}) \otimes \overline{S}(\hat{f}_{ni}) = (\overline{\Delta}o\overline{S})(f).$

Proof. (I) Note that $\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\Delta}(f_{ni}) \otimes \hat{f}_{ni} \in \overline{C(\mathbb{R}) \otimes C(\mathbb{R})} \otimes C(\mathbb{R})$ and by Lemma 3.1, $\overline{C(\mathbb{R}) \otimes C(\mathbb{R})} \otimes C(\mathbb{R})$ can be identified by $C(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$.

Similarly $\lim_{n\to\infty} \sum_{i=0}^{m_n} f_{ni} \otimes \overline{\Delta}(\hat{f}_{ni}) \in \overline{C(\mathbf{R}) \otimes \overline{C(\mathbf{R})} \otimes C(\mathbf{R})}$ and by Lemma 3.1, $\overline{C(\mathbf{R}) \otimes \overline{C(\mathbf{R})} \otimes C(\mathbf{R})}$ can be identified by $C(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$. Now we have:

$$\left(\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\Delta}(f_{ni}) \otimes \hat{f}_{ni}\right) (a, b, c) = \lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\Delta}(f_{ni})(a, b) \hat{f}_{ni}(c)$$
$$= \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(a+b) \hat{f}_{ni}(c)$$
$$= \overline{\Delta}(f)(a+b, c)$$
$$= f(a+b+c)$$
$$= \left(\lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \overline{\Delta}(\hat{f}_{ni})\right) (a, b, c)$$

where $a, b, c \in \mathbb{R}$.

II) The proof is obvious.

III) Note that $\lim_{n\to\infty}\sum_{i=0}^{m_n}\overline{\varepsilon}(f_{ni})\hat{f}_{ni}\in \overline{C(\mathbb{R})}=C(\mathbb{R})$, because $C(\mathbb{R})$ is complete w.r.t. the metric d in Proposition 3.2, [6, page 27]. Now

we have:

$$\left(\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{\varepsilon}(f_{ni}) \widehat{f}_{ni}\right) (a) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(0) \widehat{f}_{ni}(a)$$
$$= \overline{\Delta}(f)(0, a)$$
$$= f(a)$$
$$= \overline{\Delta}(f)(a, 0)$$
$$= \lim_{n \to \infty} \left(\sum_{i=0}^{m_n} f_{ni} \overline{\varepsilon}(\widehat{f}_{ni})\right) (a)$$

where $a \in \mathbb{R}$.

IV) The proof is obvious. V)

$$\left(\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{S}(f_{ni}) \hat{f}_{ni}\right) (a) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(-a) \hat{f}_{ni}(a)$$
$$= \overline{\Delta}(f)(-a, a)$$
$$= f(0)$$
$$= \overline{\Delta}(f)(a, -a)$$
$$= \left(\lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \overline{S}(\hat{f}_{ni})\right) (a)$$

where $a \in \mathbb{R}$. VI)

$$\left(\lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{S}(f_{ni}) \otimes \overline{S}(\hat{f}_{ni})\right)(a,b) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(-a)\hat{f}_{ni}(-b)$$
$$= \overline{\Delta}(f)(-a,-b)$$
$$= f(-a-b)$$
$$= \overline{S}(f)(a+b)$$
$$= \overline{\Delta}(\overline{S}(f))(a,b)$$

where $a, b \in \mathbb{R}$. \Box

Definition 3.5. The linear map $\overline{D} : C(\mathbb{R}) \to C(\mathbb{R})$ is called an extended coderivation of $C(\mathbb{R})$ if

$$\overline{\Delta}o\overline{D}(f) = \lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{D}(f_{ni}) \otimes \hat{f}_{ni} + f_{ni} \otimes \overline{D}(\hat{f}_{ni})$$

where $\overline{\Delta}(f) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \hat{f}_{ni}$ and $f \in C(\mathbb{R})$.

We denote the set of all continuous extended coderivations of $C(\mathbb{R})$ by $\overline{COD}(C(\mathbb{R}))$.

Lemma 3.6. If $\overline{D} : C(\mathbb{R}) \to C(\mathbb{R})$ is defined by $\overline{D}(f) = Xf$, for all $f \in C(\mathbb{R})$. Then \overline{D} is a continuous extended coderivation.

Proof. Let $\{f_n\}$ be a sequence in $C(\mathbb{R})$ such that f_n converges to zero w.r.t. the metric d (in Proposition 3.2). We show that $\lim_{n\to\infty} P_k(\overline{D}(f_n)) = 0$, for all k.

$$P_k(\overline{D}(f_n)) = \sup\{|\overline{D}(f_n)(x)| : x \in [-k,k]\}$$

=
$$\sup\{|xf_n(x)| : x \in [-k,k]\} \le kP_k(f_n) .$$

In the proof of Proposition 3.2, we showed that $\lim_{n\to\infty} P_k(f_n) = 0$ for all k; thus $\lim_{n\to\infty} P_k(\overline{D}(f_n)) = 0$, for all k. By a proof similar to that of Proposition 3.2 we can show that $\overline{D}(f_n)$ converges to zero w.r.t. the metric d. Now we have:

$$\begin{split} \overline{\Delta}(\overline{D}(f))(a,b) &= (Xf)(a+b) \\ &= af(a+b) + bf(a+b) \\ &= a\overline{\Delta}(f)(a,b) + b\overline{\Delta}(f)(a,b) \\ &= \lim_{n \to \infty} \sum_{i=0}^{m_n} (af_{ni}(a))\hat{f}_{ni}(b) + \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(a)(b\hat{f}_{ni}(b)) \\ &= \lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{D}(f_{ni})(a)\hat{f}_{ni}(b) + f_{ni}(a)\overline{D}(\hat{f}_{ni})(b) \end{split}$$

where $\overline{\Delta}(f) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \hat{f}_{ni}$ and $a, b \in \mathbb{R}$.

where $\Delta(f) = \lim_{n \to \infty} \sum_{i=0}^{n} f_{ni} \otimes f_{ni}$ and $a, b \in \mathbb{R}$. Thus \overline{D} is a continuous extended coderivation of $C(\mathbb{R})$.

Thus \overline{D} is a continuous extended coderivation of $C(\mathbb{R})$. \Box

Let
$$C_L(C(\mathbf{\mathbb{R}})) = \{\varphi \in End(C(\mathbf{\mathbb{R}})) | \lim_{n \to \infty} \sum_{i=0}^{m_n} \overline{S}(f_{ni})\varphi(\hat{f}_{ni})$$

$$= \left[\lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni}(-1)\varphi(\hat{f}_{ni})(1)\right] X ,$$
$$\text{where}\overline{\Delta}(f) = \lim_{n \to \infty} \sum_{i=0}^{m_n} f_{ni} \otimes \hat{f}_{ni}, \text{ for all } f \in C(\mathbf{\mathbb{R}})\}$$

Conjecture. $COD(C(\mathbb{R})) = C_L(C(\mathbb{R})).$

4. On the Coderivations of $C^{\infty}(\mathbb{R},\mathbb{R})$

For C^r manifolds M and N, we denote the set of C^r maps from M to N by $C^r(M, N)$. At first we assume r is finite.

The weak or "compact-open C^r " topology on $C^r(M, N)$ is generated by the sets defined as follows. Let $f \in C^r(M, N)$. Let $(\phi, U), (\psi, V)$ be charts on M, N; let $K \subset U$ be a compact set such that $f(K) \subset V$; let $0 < \epsilon \leq \infty$.

Define a *weak subbasic neighborhood*

$$N^{r}(f;(\phi,U),(\psi,V),K,\epsilon)$$
(1)

to be the set of C^r maps $g: M \to N$ such that $g(K) \subset V$ and

$$||D^{k}(\psi f \phi^{-1})(x) - D^{k}(\psi g \phi^{-1})(x)|| < \epsilon,$$

for all $x \in \phi(K), k = 0, \dots, r$. This means that the local representations of f and g, together with their first k derivatives, are within ε at each point of K.

The weak topology on $C^r(M, N)$ is generated by sets (1); it defines the topological space $C_w^r(M, N)$. A neighborhood of f is thus any set containing the intersection of a finite number of sets of type (1). We now define the spaces $C_w^{\infty}(M, N)$. The weak topology on $C^{\infty}(M, N)$ is simply the union of the topologies induced by the inclusion maps $C^{\infty}(M, N) \to C_w^r(M, N)$ for r finite. (See[2, chapter 2]).

We denote the $C^{\infty}(\mathbb{R},\mathbb{R})$ by $C^{\infty}(\mathbb{R})$. We identify $C^{\infty}(\mathbb{R}) \otimes C^{\infty}(\mathbb{R})$ with a dense subset of $C^{\infty}(\mathbb{R} \times \mathbb{R})$ (as in Lemma 3.1)

and we denote the closure of $C^{\infty}(\mathbb{R}) \otimes C^{\infty}(\mathbb{R})$ in $C^{\infty}(\mathbb{R} \times \mathbb{R})$ by $\overline{C^{\infty}(\mathbb{R}) \otimes C^{\infty}(\mathbb{R})}$.

Proposition 4.1. Let $\overline{\Delta}_{\infty} : C^{\infty}(\mathbb{R}) \to \overline{C^{\infty}(\mathbb{R})} \otimes \overline{C^{\infty}(\mathbb{R})}$ be defined by $\overline{\Delta}_{\infty}(f)(a,b) = f(a+b)$, where $f \in C^{\infty}(\mathbb{R})$ and $a, b \in \mathbb{R}$; then $\overline{\Delta}_{\infty}$,

is a continuous linear map w.r.t. the weak topology on $C^{\infty}(\mathbb{R})$.

Proof. Let $f \in C^{\infty}(\mathbb{R})$. We must show that for $N^{\infty}(\overline{\Delta}_{\infty}(f), K \times K, \epsilon)$ there exists $W^{\infty}(f, K', \delta)$ such that $\overline{\Delta}_{\infty}(g) \in N^{\infty}(\overline{\Delta}_{\infty}(f), K \times K, \epsilon)$, for all $g \in W^{\infty}(f, K', \delta)$. Put K' = K + K and $\delta = \epsilon$. Let $g \in W^{\infty}(f, K + K, \delta)$. Hence $\|D^k(f)(x) - D^k(g)(x)\| < \epsilon$, for all $x \in K + K, k = 0, 1, \cdots$. Let $i + j = k = 0, 1, \cdots$ and $(x, y) \in K \times K$ then

$$\begin{aligned} &||\frac{\partial^{i+j}}{\partial x^{i}\partial y^{j}}(\overline{\Delta}_{\infty}f)(x,y) - \frac{\partial^{i+j}}{\partial x^{i}\partial y^{j}}(\overline{\Delta}_{\infty}g)(x,y)|| \\ &= ||\frac{\partial^{i+j}}{\partial x^{i}\partial y^{j}}(f(x+y)) - \frac{\partial^{i+j}}{\partial x^{i}\partial y^{j}}(g(x+y))|| \\ &= ||\frac{\partial^{i+j}}{\partial (x+y)^{i+j}}(f(x+y)) - \frac{\partial^{i+j}}{\partial (x+y)^{i+j}}(g(x+y))| \\ &= ||(D^{k}f)(t) - (D^{k}g)(t)|| < \epsilon, \text{ because } t \in K + K. \end{aligned}$$

Thus $\overline{\Delta}_{\infty}(g) \in N^{\infty}(\overline{\Delta}_{\infty}(f), K \times K, \epsilon).$ \Box

Lemma 4.2. Let $S_{\infty} : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ be defined by $S_{\infty}(f)(a) = f(-a), f \in C^{\infty}(\mathbb{R})$ and $a \in \mathbb{R}$; then S_{∞} is a continuous map w.r.t. the weak topology on $C^{\infty}(\mathbb{R})$.

Proof. Let $f \in C^{\infty}(\mathbb{R})$. We must show that for $N^{\infty}(S_{\infty}(f), K, \epsilon)$, there exists $W^{\infty}(f, K', \delta)$ such that $S_{\infty}(g) \in N^{\infty}(S_{\infty}(f), K, \epsilon)$, for all $g \in W^{\infty}(f, K', \delta)$. Put K' = -K and $\delta = \epsilon$. \Box

Theorem 4.3. Let $\overline{D} : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ be an extended coderivation (Def. 3.5) of $C^{\infty}(\mathbb{R})$. If \overline{D} is continuous w.r.t. the weak topology on $C^{\infty}(\mathbb{R})$, then $\overline{D}(f) = \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k}(f)$, for all $f \in C^{\infty}(\mathbb{R})$.

Proof. By [2, page 40, Ex.4] polynomials are dense in $C_w^{\infty}(\mathbb{R}, \mathbb{R})$. Let

 $f\in C^\infty({\rm I\!R}).$ Then there exists a sequence $\{P_n\}$ of polynomials such that

 $f = \lim_{n \to \infty} P_n$. By Theorem 2.9, we conclude that

$$\overline{D}(f) = \lim_{n \to \infty} \overline{D}(P_n) = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k}(P_n).$$

On the other hand since the operator $f \to \frac{d^k}{dX^k}(f)$ is continuous w.r.t. to the weak topology on $C^{\infty}(\mathbb{R})$ [6, page 39, Ex.17], hence:

$$\sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k}(f) = \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k} (\lim_{n \to \infty} P_n)$$
$$= \lim_{n \to \infty} \sum_{k=0}^{\infty} b_k X \frac{d^k}{dX^k} (P_n)$$
$$= \overline{D}(f) . \Box$$

Now let $a_0y + a_1y' + \dots + a_{n-1}y^{(n-1)} + a_ny^{(n)} = g(x)$ be a linear differential equation with $g(x) \in C^{\infty}(\mathbb{R})$. Let \overline{D} be a continuous extended coderivation on $C^{\infty}(\mathbb{R})$, then by Theorems 2.9 and 4.3, $\overline{D}(y) = x \sum_{k=0}^{\infty} b_k y^{(k)}$ where $y \in C^{\infty}(\mathbb{R})$. then $\frac{1}{x}\overline{D}(y) = b_0y + b_1y' + \dots + b_ny^{(n)} + \dots$.

In this method we put $b_i = a_i$, for all $i \ge 0$, thus $\frac{1}{x}\overline{D}(y) = g(x)$. For example if y is analytic i.e., $y = \sum_{n=0}^{\infty} c_n x^n$, then

$$\overline{D}(y) = \overline{D}(\sum_{n=0}^{\infty} c_n x^n) = \sum_{n=0}^{\infty} c_n \overline{D}(x^n) \cdot$$

Therefore $\sum_{n=0}^{\infty} c_n \overline{D}(x^n) = xg(x)$ and we can find the c_n 's by the solving this system of equations.

5. An Example of a Cocommutative Bialgebra H such that the Set of Coderivations on H is Zero

Let $R^* = \mathbb{R} - \{0\}$ and $\mathbb{R}^*[X]$ be the space of polynomial functions on \mathbb{R}^* .

 $\mathbb{R}^*[X]$ is a monoid algebra. Let $f \in \mathbb{R}^*[X]$ and $a \in \mathbb{R}^*$. Define $\Delta(f)(a,b) = f(a,b), \varepsilon(f) = f(1)$ and $S(f)(a) = f(\frac{1}{a})$.

Lemma 5.1. The coproduct $\Delta : \mathbb{R}^*[X] \to \mathbb{R}^*[X] \otimes \mathbb{R}^*[X]$ has the property: $\Delta(X^n) = X^n \otimes X^n, n = 0, 1, 2, \dots$

Proof. Let $a, b \in \mathbb{R}^*$. Then $[\Delta(X^n)](a, b) = X^n(ab) = (ab)^n = a^n b^n$. On the other hand $[\Delta(X^n)](a \otimes b) = \sum_{(X^n)} X^n_{(1)}(a) X^n_{(2)}(b)$ because

$$\Delta(X^n) \in \mathbf{R}^*[X] \otimes \mathbf{R}^*[X] \; .$$

We conclude that $\Delta(X^n) = X^n \otimes X^n$. \Box

Lemma 5.2. Let \mathcal{L} be the set of primitive elements on $\mathbb{R}^*[X]$. Then $\mathcal{L} = \{0\}$.

Proof. Let $F(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{R}^*[X]$. We have $\Delta(F(X)) = \Delta(\sum_{i=0}^{m} a_i X^i) = \sum_{i=0}^{m} a_i (X^i \otimes X^i).$ If $F(X) \in \mathcal{L}$ then $\Delta(F(X)) = F(X) \otimes 1 + 1 \otimes F(X) = \sum_{i=0}^{m} a_i (X^i \otimes X^i)$.

 $1 + 1 \otimes X^i$). Since $\{1, X, X^2, \dots, X^m\}$ is a basis, hence $a_i = 0$ for all $i = 0, 1, \dots, m$. Thus F(X) = 0. We conclude that $\mathcal{L} = \{0\}$. \Box

For any group G, let H = K[G] be the group algebra of G. If we define

 $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, then H is a cocommutative bialgebra.

Proposition 5.3. The bialgebra H = K[G] has no coderivations except 0.

Proof. Let $D: H \to H$ be a coderivation and g be any element of the basis H. We have $\Delta(D(g)) = g \otimes D(g) + D(g) \otimes g$. If $D(g) = \sum_{i=1}^{n} k_i g_i$, for some $k_i \in K$ and g_i belonging to the basis of H, then $\sum_{i=1}^{n} k_i (g_i \otimes g_i) = \sum_{i=1}^{n} k_i (g \otimes g_i + g_i \otimes g)$. If n > 1, then $k_i = 0$, for any $1 \leq i \leq n$. However if n = 1 and $g \neq g_1$ then $k_1 = 0$. Because $\varepsilon(D(g)) = k_1$ and since $(\varepsilon \otimes I)o\Delta = I$, we conclude that $k_1 = 0$. \Box

Remark 5.4. Zero is the only element of the set of coderivations on $\mathbb{R}^*[X]$.

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