ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 7, pp. 2517–2530

Title:

Nonlinear Picone identities to Pseudo p-Laplace operator and applications

Author(s):

T. Feng and M. Yu

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 7, pp. 2517–2530
Online ISSN: 1735-8515

NONLINEAR PICONE IDENTITIES TO PSEUDO P-LAPLACE OPERATOR AND APPLICATIONS

T. FENG AND M. YU*

(Communicated by Asadollah Aghajani)

Abstract. In this paper, we derive a nonlinear Picone identity to the pseudo p-Laplace operator, which contains some known Picone identities and removes a condition used in many previous papers. Some applications are given including a Liouville type theorem to the singular pseudo p-Laplace system, a Sturmian comparison principle to the pseudo p-Laplace equation, a new Hardy type inequality with weight and remainder term, a nonnegative estimate of the functional associated to pseudo p-Laplace equation.

Keywords: Nonlinear Picone identity, pseudo p-Laplace equation, pseudo p-Laplace system.
MSC(2010): Primary: 35J25; Secondary: 35J66.

1. Introduction and main results

If the membrane is fixed out of elastic strings in a rectangular fashion, its deformation energy corresponds to the pseudo p-Laplace operator (Belloni-Ferone-Kawohl [7], Belloni-Kawohl [8], Cianchi-Salani [12], Demengel [13], Emamizadeh-Rezapour [17])

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right), \tag{1.1}$$

which can be also derived from the optimal transport problem with congestion effects (Brasco-Carlier [11]), some reinforced materials (Lions [23], Tang [29]), the dynamics of fluids in the anisotropic media when the conductivities of the media are different in each direction (Antontsev-Díaz-Shmarev [4], Bear [6]) and the image processing (Weickert [31]). Hence, the pseudo p-Laplace operator not only has the widespread practical background in the natural science, but also has the important theoretical value in the mathematics. Note

Article electronically published on December 30, 2017.
Received: 7 October 2016, Accepted: 10 December 2017.
*Corresponding author.

that if $p=2$, then (1.1) becomes the classical Laplace operator. It is worthy to note that the pseudo p-Laplace operator belongs to a kind of anisotropic Laplace operator (Boccardo-Marcellini-Sbordone [9], Fragalà-Gazzola-Kawohl [18], Stroffolini [28])

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right), p_i>1,$$

which differents from the usual p-Laplace operator

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

where $|\nabla u|=\left(\sum\limits_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}$. The pseudo p-Laplace operator can be deduced from the anisotropic Laplace operator when $p_i=p=const$. We can also deduce the pseudo p-Laplace operator from a kind of differential operator

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left(\sum_{k=1}^{n}\left|\frac{\partial u}{\partial x_k}\right|^{q}\right)^{\frac{p-q}{q}}\left|\frac{\partial u}{\partial x_i}\right|^{q-2}\frac{\partial u}{\partial x_i}\right)$$

when $p=q$. Moreover, we point out this kind of differential operator can be derived from the so-called Finsler p-Laplace operator (Belloni-Ferone-Kawohl [7], Belloni-Kawohl [8], Cianchi-Salani [12])

$$Qu=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big((H(\nabla u))^{p-1}H_{\xi_i}(\nabla u)\Big),$$

where $H(\xi)=\left(\sum\limits_{k=1}^{n}|\xi_k|^q\right)^{\frac{1}{q}}$.

Let us mention some pioneering works for elliptic problems related the pseudo p-Laplace operator. For the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)=\lambda u^{q-1}, & x\in\Omega,\\ u\geqslant 0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where constants $\lambda>0, q>1, \Omega\subset\mathbb{R}^n\,(n\geqslant 2)$ is a smooth bounded domain, Fragalà-Gazzola-Kawohl [18] established existence results in the subcritical case by the minimax methods, and obtained nonexistence results in the at least critical case in domains with a geometric property by modifying the standard notion of starshapedness. Note that the classical model of anisotropic functional

$$\int_{\Omega}\left(\left|\frac{\partial u}{\partial x_1}\right|^{p_1}+\left|\frac{\partial u}{\partial x_2}\right|^{p_2}+\cdots+\left|\frac{\partial u}{\partial x_n}\right|^{p_n}\right)dx,$$

whose minimizers are the solutions of some homogeneous anisotropic elliptic problems. Moussa [25] proved that the Schwarz rearrangement does not decrease the energy functional for the pseudo p-Laplace operator, namely

$$\sum_{i=1}^{n}\int_{\Omega^*}\left|\frac{\partial u^*}{\partial x_i}\right|^p dx \geqslant \sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^p dx,$$

where $u \in W_0^{1,p}(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^n$, u^* and Ω^* denote the Schwarz rearrangement of u and Ω, respectively. Marcellini [24] obtained $H_{loc}^{1,\infty}(\Omega)$ regularity of minimizers of the anisotropic functional

$$I(u) = \int_\Omega f\left(\nabla u(x)\right)dx$$

with

$$m\sum_{i=1}^{n}|\xi_i|^{p_i} \leqslant f(\xi) \leqslant M\left(1+\sum_{i=1}^{n}|\xi_i|^{p_i}\right)$$

and

$$m|\lambda|^2 \leqslant \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j \leqslant M\left(1+\sum_{i=1}^{n}|\xi_i|^{p_i-2}\right)|\lambda|^2$$

for every $\lambda, \xi \in \mathbb{R}^n$, positive constants m , M and $2 \leqslant p_i < \frac{2n}{n-2}$. Furthermore, for the integral functional

$$I(u) = \int_\Omega f\left(x, \nabla u(x)\right)dx,$$

where f satisfies the anisotropic coerciveness condition $f(x,\xi) \geqslant m\sum\limits_{i=1}^{n}|\xi_i|^{p_i}$, Boccardo-Marcellini [9] obtained $L^\infty(\Omega)$ regularity result. It is shown that the so-called growth conditions are necessary for the local regularity of minimizers in Giaquinta [19]. Moreover, between the radially decreasing symmetrals of minimizers of anisotropic variational problems and minimizers of suitably symmetrized problems

$$\begin{cases} \min \int_\Omega f\left(x, \nabla u(x)\right)dx, \\ u = u_0 \quad on\ \partial\Omega, \end{cases}$$

where u_0 is a prescribed boundary datum, f is a Carathéodory function and satisfies $f(x,\xi) \geqslant \Phi(\xi)$ for a finite-valued n-dimensional Young function Φ, Alberico-Cianchi [1] established a pointwise inequality and derived a priori sharp estimates for norms of the relevant minimizers.

Recently, there are a large number of papers and an increasing interest about elliptic problems related to the pseudo p-Laplace operator. For the following eigenvalue problem

$$\text{(Q)} \qquad \begin{cases} Qu + \lambda_p|u|^{p-2}u = 0, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases}$$

where

$$\lambda_p(\Omega) = \inf\left\{\int_\Omega H(\nabla u)^p dx; u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p dx = 1\right\}.$$

In Belloni-Ferone-Kawohl [7], the positivity of the first eigenfunction, simplicity of the first eigenvalue, Faber-Krahn and Payne-Rayner type inequalities are derived and the symmetry of positive solutions of more general equations $Qu = f(u)$ are also shown under some suitable conditions. In Cianchi-Salani [12], a symmetry result is established for solutions to the overdetermined elliptic problem $-Qu = 1$ and $p = 2$ in variational form, which extended Serrin's theorem in the isotropic radial case. Belloni-Kawohl [8] obtained viscosity solutions of the problem (Q) as $p \to \infty$. Moreover, Demengel [13] proved the Lipschitz interior regularity for the viscosity and weak solutions of the pseudo p-Laplace equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = (p-1)f,$$

where $f \in C\left(\overline{B_1}\right)$. For the following eigenvalue problem with Robin boundary condition

$$\begin{cases} -\sum\limits_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + \alpha f|u|^{p-2}u = \lambda|u|^{p-2}u, & in\ \Omega, \\ (1-\beta)\sum\limits_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)\nu_i + \beta|u|^{p-2}u = 0, & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\alpha \geqslant 0$, $0 < \beta < 1$ are constants, f is a measurable function and the vector $(\nu_1, \cdots, \nu_n) =: \nu$ denotes the unit outward normal on $\partial\Omega$, Emamizadeh-Rezapour [17] proved an optimization of the principal eigenvalue and gave the existence and optimality conditions.

As is well known, the Picone identity is an important tool in the analysis of partial differential equations. In this paper, we provide a nonlinear Picone identity to the pseudo p-Laplace operator, which contains some known Picone identities and removes a condition used in many previous papers. As applications, we give a Liouville type theorem to singular pseudo p-Laplace systems, a Sturmian comparison principle to pseudo p-Laplace equation, a new Hardy type inequality with weight and remainder term, and a nonnegative estimate of the functional associated to pseudo p-Laplace equation. Before giving the main result of this paper, we first briefly review the research development of nonlinear Picone identities (see [2, 3, 10, 14, 20, 21, 22, 26, 27] and related references for linear Picone identities).

Recently, a nonlinear Picone identity for the Laplace operator was presented by Tyagi [30]: If v and u are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$\alpha|\nabla u|^2 - \frac{|\nabla u|^2}{f'(v)} + \left(\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}}\right)^2$$
$$= \alpha|\nabla u|^2 - \nabla\left(\frac{u^2}{f(v)}\right)\nabla v,$$

where $f(y) > 0, 0 < y \in \mathbb{R}$, and $f'(y) \geq \frac{1}{\alpha}$ for some $\alpha > 0$. Bal [5] generalized the result of Tyagi with α=1 to the p-Laplace operator: If v and u are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$|\nabla u|^p - \frac{pu^{p-1}|\nabla v|^{p-2}\nabla v \cdot \nabla u}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2}$$
$$= |\nabla u|^p - \nabla\left(\frac{u^p}{f(v)}\right)|\nabla v|^{p-2}\nabla v,$$

where $f(y) > 0, 0 < y \in \mathbb{R}$, and $f'(y) \geq (p-1)\left[f(y)^{\frac{p-2}{p-1}}\right], p > 1$. For related results we refer to Dwivedi-Tyagi [15] (see also [16]).

The main result in this paper is

Theorem 1.1. *Let $v > 0$ and u be two differentiable functions in the domain $\Omega \subset \mathbb{R}^n$, and denote*

$$R(u,v) = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i},$$

$$L(u,v) = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^n p\frac{|u|^{p-2}u}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} + \sum_{i=1}^n \frac{|u|^p f'(v)}{[f(v)]^2}\left|\frac{\partial v}{\partial x_i}\right|^p,$$

where $f(v) > 0$ and $f'(v) \geq (p-1)[f(v)]^{\frac{p-2}{p-1}}, p > 1$. Then

$$R(u,v) = L(u,v) \tag{1.2}$$

and $L(u,v) \geq 0$; moreover, $L(u,v) = 0$ a.e. in Ω if and only if $u = c_1 v + c_2$ a.e. in Ω for some constants c_1, c_2.

Remark 1.2. We mention that the assumption $u \geq 0$ was required in many previous papers. But we remove it here. If $p = 2$, then (1.2) is the result in Tyagi [30] with α=1; if $p = 2$ and $f(v) = v$, then (1.2) is the result in Allegretto [2]; if $p > 2$ and $f(v) = v^{p-1}$, then (1.2) is the one in Jaros [21] with p=r. We also note that the "if and only if "condition for $L(u,v) = 0$ is different from the known results.

This paper is organized as follows: The proof of Theorem 1.1 is given in Section 2; Section 3 is devoted to applications.

2. **Proof of Theorem 1.1**

Proof of Theorem 1.1. We first prove $R(u,v) = L(u,v)$. Via a direct calculation,

$$\begin{aligned} R(u,v) &= \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \\ &= \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^{n}\left(p\frac{|u|^{p-2}u}{f(v)}\frac{\partial u}{\partial x_i} - \frac{|u|^p f'(v)}{[f(v)]^2}\frac{\partial v}{\partial x_i}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i} \\ &= \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^{n}p\frac{|u|^{p-2}u}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n}\frac{|u|^p f'(v)}{[f(v)]^2}\left|\frac{\partial v}{\partial x_i}\right|^p \\ &= L(u,v). \end{aligned}$$

Next we verify that $L(u,v) \geq 0$. In fact, in virtue of

$$u\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} \leq |u|\left|\frac{\partial v}{\partial x_i}\right|\left|\frac{\partial u}{\partial x_i}\right|,$$

it yields

$$\begin{aligned} L(u,v) &\geq \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p - \sum_{i=1}^{n}p\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\left|\frac{\partial u}{\partial x_i}\right| + \sum_{i=1}^{n}\frac{|u|^p f'(v)}{[f(v)]^2}\left|\frac{\partial v}{\partial x_i}\right|^p \\ &= \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p + \sum_{i=1}^{n}(p-1)\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}} \\ &\quad - \sum_{i=1}^{n}p\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\left|\frac{\partial u}{\partial x_i}\right| \\ &\quad - \sum_{i=1}^{n}(p-1)\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}} + \sum_{i=1}^{n}\frac{|u|^p f'(v)}{[f(v)]^2}\left|\frac{\partial v}{\partial x_i}\right|^p \\ &:= I + II, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} I &= \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p + \sum_{i=1}^{n}(p-1)\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}} \\ &\quad - \sum_{i=1}^{n}p\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\left|\frac{\partial u}{\partial x_i}\right|, \end{aligned}$$

$$II = -\sum_{i=1}^{n}(p-1)\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}} + \sum_{i=1}^{n}\frac{|u|^p f'(v)}{[f(v)]^2}\left|\frac{\partial v}{\partial x_i}\right|^p.$$

Let us recall Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \tag{2.2}$$

for $a \ge 0, b \ge 0$, where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the equality holds if and only if $a = b^{\frac{1}{p-1}}$. Taking $a = \left|\frac{\partial u}{\partial x_i}\right|$ and $b = \frac{u^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}$ in (2.2), it gives

$$p\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\left|\frac{\partial u}{\partial x_i}\right| \le p\left[\frac{1}{p}\left|\frac{\partial u}{\partial x_i}\right|^p + \frac{p-1}{p}\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}}\right] \tag{2.3}$$

and thus $I \ge 0$ by (2.3). Noting the assumptions for $f(v)$, it implies

$$\begin{aligned} II &\ge \sum_{i=1}^{n} \frac{|u|^p (p-1)\left[f(v)\right]^{\frac{p-2}{p-1}}}{\left[f(v)\right]^2}\left|\frac{\partial v}{\partial x_i}\right|^p - \sum_{i=1}^{n}(p-1)\left(\frac{|u|^{p-1}}{f(v)}\left|\frac{\partial v}{\partial x_i}\right|^{p-1}\right)^{\frac{p}{p-1}} \\ &= 0. \end{aligned}$$

Hence $L(u,v) \ge 0$ by (2.1).

The proof process above also shows that $L(u,v) = 0$ a.e. in Ω if and only if

$$\left|\frac{\partial u}{\partial x_i}\right| = \left(\frac{1}{f(v)}\right)^{\frac{1}{p-1}} |u| \left|\frac{\partial v}{\partial x_i}\right|, f(v) > 0, \tag{2.4}$$

$$f'(v) = (p-1)\left[f(v)\right]^{\frac{p-2}{p-1}}, \tag{2.5}$$

$$u\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} = |u|\left|\frac{\partial v}{\partial x_i}\right|\left|\frac{\partial u}{\partial x_i}\right| \tag{2.6}$$

a.e. in Ω.

If $u = c_1 v + c_2$, for constants c_1, c_2, then u satisfies (2.4), (2.5) and (2.6), and so $L(u,v) = 0$.

Conversely, if $L(u,v)(x_0) = 0, x_0 \in \Omega$, we consider two cases $u(x_0) \ne 0$ and $u(x_0) = 0$, separately.

(a) If $u(x_0) \ne 0$, then it follows (2.4), (2.5) and (2.6). We will obtain $u = c_1 v + c_2$ by solving (2.4), (2.5) and (2.6). In fact, it follows by (2.5) that

$$\frac{f'(v)}{\left[f(v)\right]^{\frac{p-2}{p-1}}} = (p-1),$$

and by integrating,

$$(p-1)\left[f(v)\right]^{\frac{1}{p-1}} = (p-1)v + c_3,$$

which implies

$$f(v) = (v + c_0)^{p-1}; \tag{2.7}$$

substituting (2.7) into (2.4),

$$(v+c_0)\left|\frac{\partial u}{\partial x_i}\right| = |u|\left|\frac{\partial v}{\partial x_i}\right|; \tag{2.8}$$

finally, putting (2.8) into (2.6), it gives

$$(v+c_0)\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i} = 0, a.e..$$

In virtue of $f(v) = (v+c_0)^{p-1} > 0$, we have

$$\frac{(v+c_0)\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}}{(v+c_0)^2} = \frac{\partial}{\partial x_i}\left(\frac{u}{v+c_0}\right) = 0, a.e.,$$

which implies

$$u = c_1 v + c_2.$$

(b) If $u(x_0) = 0$, we denote $S = \{x \in \Omega \,|u(x) = 0\}$ and then $\frac{\partial u}{\partial x_i} = 0$ a.e. in S. Thus

$$\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\right) = 0,$$

which shows that $u = c_1 v + c_2$. The proof of Theorem 1.1 is ended. □

3. Applications

Throughout this section, we always assume

$$v > 0, f(v) > 0, \quad and \quad f'(v) \geq (p-1)\left[f(v)\right]^{\frac{p-2}{p-1}} \quad for \quad p > 1.$$

We will give several applications of Theorem 1.1. The first is a Liouville theorem.

Proposition 3.1. *Let $(u, v) \in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ be the solution to the Dirichlet problem of a singular pseudo p-Laplace system*

$$\begin{cases} -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f(v), & x \in \Omega, \\ -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right) = \frac{[f(v)]^2}{|u|^{p-1}}, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ u = 0, v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Then $u = c_1 v + c_2$ a.e. in Ω.

Proof. For any $\phi, \varphi \in W_0^{1,p}(\Omega)$, it gets by (3.1) that

$$\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}dx = \int_\Omega f(v)\phi dx, \tag{3.2}$$

$$\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\frac{\partial \varphi}{\partial x_i}dx=\int_{\Omega}\frac{[f(v)]^2}{|u|^{p-1}}\varphi dx. \tag{3.3}$$

Choosing $\phi = |u|$ and $\varphi = \frac{|u|^p}{f(v)}$ in (3.2) and (3.3), respectively, we have

$$\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p}dx=\int_{\Omega}f(v)\,|u|\,dx=\sum_{i=1}^{n}\int_{\Omega}\frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}dx.$$

Combining it and (1.2), it follows

$$\begin{aligned}\int_{\Omega}L(u,v)dx&=\int_{\Omega}R(u,v)dx\\&=\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p}dx-\sum_{i=1}^{n}\int_{\Omega}\frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}dx\\&=0,\end{aligned}$$

and the conclusion is true. □

Remark 3.2. Here we do not use the condition $u > 0$ in Ω.

The next is a Sturmian comparison principle to a singular pseudo p-Laplace equation.

Proposition 3.3. *Let $f_1(x)$ and $f_2(x)$ be two continuous functions with $f_1(x) < f_2(x)$. Assume that $u \in W_0^{1,p}(\Omega)$ satisfies the Dirichlet problem*

$$\begin{cases}-\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)=f_1(x)u^{p-1}, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases} \tag{3.4}$$

Then any nontrivial solution of the equation

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right)=f_2(x)f(v), x\in\Omega, \tag{3.5}$$

must change sign.

Proof. Assume that the solution v in (3.5) does not change sign and without loss generality, let $v > 0$ in Ω. We have by (3.4), (3.5) and (1.2) that

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}dx \\ &= \int_\Omega \left(f_1(x) - f_2(x)\right)|u|^p dx \\ &< 0, \end{aligned}$$

which is a contradiction. This accomplishes the proof. □

Proposition 3.4. *Suppose that there exists a constant $k > 0$ and a function $h(x)$ such that for a differentiable function $v > 0$ in Ω,*

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right) \geq kh(x)f(v). \tag{3.6}$$

Then

$$\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx \geq k\int_\Omega h(x)u^p dx \tag{3.7}$$

for any $0 \leq u \in C_0^1(\Omega)$.

Proof. By (3.6) and (1.2), we have

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_\Omega \frac{|u|^p}{f(v)}\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right)dx \\ &\leq \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx - k\int_\Omega h(x)|u|^p dx, \end{aligned}$$

which implies (3.7). □

Proposition 3.4 can help us establish a new Hardy type inequality with weight and remainder term.

Proposition 3.5. *Suppose that* $u \in C_0^1(\mathbb{R}^n \backslash \{0\}), 1 < p < n$. *Then*

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n \backslash \{0\}} \left| \frac{\partial u}{\partial x_i} \right|^p dx \geq \left(\frac{n-p}{p} \right)^{p-1} \frac{(n+p)(1-p)}{p} \int_{\mathbb{R}^n \backslash \{0\}} \frac{\sum_{i=1}^{n} |x_i|^p}{|x|^{2p}} |u|^p dx$$

$$+ \left(\frac{n-p}{p} \right)^{p-1} (p-1) \int_{\mathbb{R}^n \backslash \{0\}} \frac{\sum_{i=1}^{n} |x_i|^{p-2}}{|x|^{2p-2}} |u|^p dx. \tag{3.8}$$

Proof. Let $v = |x|^\beta$, where $\beta = \frac{p-n}{p}$, hence

$$\frac{\partial v}{\partial x_i} = \beta |x|^{\beta-2} x_i,$$

$$\left| \frac{\partial v}{\partial x_i} \right|^{p-2} = |\beta|^{p-2} |x|^{(\beta-2)(p-2)} |x_i|^{p-2},$$

$$\left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} = \beta |\beta|^{p-2} |x|^{\beta p - \beta - 2p + 2} |x_i|^{p-2} x_i$$

and

$$\begin{aligned} &-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) \\ &= \left[\left(\frac{n-p}{p} \right)^{p-1} \frac{(n+p)(1-p)}{p} \frac{\sum_{i=1}^{n} |x_i|^p}{|x|^{2p}} \right] v^{p-1} \\ &+ \left[\left(\frac{n-p}{p} \right)^{p-1} (p-1) \frac{\sum_{i=1}^{n} |x_i|^{p-2}}{|x|^{2p-2}} \right] v^{p-1}. \end{aligned} \tag{3.9}$$

Consequently, taking $f(v) = v^{p-1}$ in (3.9), we know that (3.6) in Proposition 3.4 is satisfied and hence (3.8) is followed by (3.7). □

Finally, we consider a pseudo p-Laplace equation with weight

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a(x) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right) - a(x) f(v) = 0, \tag{3.10}$$

where $a(x)$ is a positive weight. Let

$$W_{loc}^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega') \ |for \ any \ \Omega' \subset\subset \Omega \right\}.$$

A function $0 < v \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ is called a super-solution to (3.10) if

$$\int_\Omega a(x) \sum_{i=1}^n \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}\frac{\partial \varphi}{\partial x_i}dx - \int_\Omega a(x) f(v)\varphi dx \geq 0 \tag{3.11}$$

for any $0 \leq \varphi \in W^{1,p}_c(\Omega) \cap C(\Omega)$, where

$$W^{1,p}_c(\Omega) := \left\{u \in W^{1,p}_{loc}(\Omega), supp\ u \subset \Omega\right\}.$$

Proposition 3.6. *Let $0 < v \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ be a super-solution of (3.10). Then*

$$I(u) := \int_\Omega a(x) \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx - \int_\Omega a(x)|u|^p dx \geq 0 \tag{3.12}$$

for any $u \in W^{1,p}_c(\Omega) \cap C(\Omega)$.

Proof. Taking $\varphi = \frac{|u|^p}{f(v)}$ in (3.11), it follows

$$\int_\Omega a(x) \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}dx - \int_\Omega a(x)|u|^p dx \geq 0. \tag{3.13}$$

Noting that

$$\begin{aligned}&\int_\Omega a(x) \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{|u|^p}{f(v)}\right)\left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}dx \\ &= \int_\Omega a(x) \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx - \int_\Omega a(x) R(u,v)dx,\end{aligned}$$

we substitute it into (3.13) to obtain

$$\int_\Omega a(x) \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx - \int_\Omega a(x)|u|^p dx \geq \int_\Omega a(x) R(u,v)dx.$$

Since $a(x) > 0$ and $R(u,v) \geq 0$, it yields (3.12). □

Acknowledgement

This work is supported by National Natural Science Foundation of China (Grant No. 11701453).

References

[1] A. Alberico and A. Cianchi, Comparison estimates in anisotropic variational problems, *Manuscripta Math.* **126** (2008), no. 4, 481–503.

[2] W. Allegretto, Sturmianian theorems for second order systems, *Proc. Amer. Math. Soc.* **94** (1985), no. 2, 291–296.

[3] W. Allegretto and Y. Huang, A Picone's identity for the p-laplacian and applications, *Nonlinear Anal.* **32** (1998), no. 7, 819–830.

[4] S.N. Antontsev, J.I. Díaz and S. Shmarev, Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics, Springer Science & Business Media, 2012.

[5] K. Bal, Generalized Picone's identity and its applications, *Electron. J. Differential Equations* **2013** (2013), no. 243, 1–6.

[6] J. Bear, Dynamics of Fluids in Porous Media, American Elsevier, New York, 1972.

[7] M. Belloni, V. Ferone and B. Kawohl, Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators, *Zeit. Angew. Math. Phys. (ZAMP)* **54** (2003), no. 5, 771–783.

[8] M. Belloni and B. Kawohl, The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$, *ESAIM Control Optim. Calc. Var.* **10** (2004), no. 1, 28–52.

[9] L. Boccardo, P. Marcellini and C. Sbordone, L^{∞}-regularity for variational problems with sharp nonstandard growth conditions, *Boll. Un. Mat. Ital. A (7)* **4** (1990), no. 2, 219–225.

[10] G. Bognar and O. Dosly, Picone-type identity for pseudo p-laplace with variable power, *Electron. J. Differential Equations* **2012** (2012), no. 174, 8 pages.

[11] L. Brasco and G. Carlier, On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds, *Adv. Calc. Var.* **7** (2014), no. 3, 379–407.

[12] A. Cianchi and P. Salani, Overdetermined anisotropic elliptic problems, *Math. Ann.* **345** (2009), no. 4, 859–881.

[13] F. Demengel, Lipschitz interior regularity for the viscosity and weak solutions of the pseudo p-laplace equation, *Adv. Differential Equations* **21** (2016), no. 3-4, 373–400.

[14] J. Dou, Picone inequalities for p-sub-Laplacian on the Heisenberg group and its Applications, *Commun. Contemp. Math.* **12** (2010), no. 2, 295–307.

[15] G. Dwivedi and J. Tyagi, Remarks on the qualitative questions for biharmonic operators, *Taiwanese J. Math.* **19** (2015), no. 6, 1743–1758.

[16] G. Dwivedi and J. Tyagi, Picone's identity for biharmonic operators on Heisenberg group and its applications, *NoDEA Nonlinear Differential Equations Appl.* **23** (2016), no. 2, Article 10, 26 pages.

[17] B. Emamizadeh and M.Z. Rezapour, Optimization of the principal eigenvalue of the pseudo p-Laplace operator with Robin boundary conditions, *Internat. J. Math.* **23** (2012), no. 12, 1250127, 17 pages.

[18] I. Fragalà, F. Gazzola and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004), no. 5, 715–734.

[19] M. Giaquinta, Growth conditions and regularity, a counterexample, *Manuscripta Math.* **59** (1987), no. 2, 245–248.

[20] J. Jaroš, Picone's identity for a Finsler p-Laplacian and comparison of nonlinear elliptic equations, *Math. Bohem.* **139** (2014), no. 3, 535–552.

[21] J. Jaroš, Caccioppoli estimates through an anisotropic Picone's identity, *Proc. Amer. Math. Soc.* **143** (2015), no. 3, 1137–1144.

[22] J. Jaroš, A-harmonic Picone's identity with applications", *Ann. Mat. Pura Appl. (4)* **194** (2015), no. 3, 719–729.

[23] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris, 1969.

[24] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Rat. Mech. Anal.* **105** (1989), no. 3, 267–284.

[25] M. Moussa, Schwarz rearrangement does not decrease the energy for the pseudo p-Laplacian operator, *Bol. Soc. Parana. Mat. (3)* **29** (2011), no. 1, 49–53.

[26] P. Niu, H. Zhang and Y. Wang, Hardy type and Rellich type inequalities on the Heisenberg group, *Proc. Amer. Math. Soc.* **129** (2001), no. 12, 3623–3630.

[27] M. Picone, Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine", *Ann. Scuola Norm. Sup. Pisa CI. Sci.* **11** (1910) 1–144.
[28] B. Stroffolini, Some remarks on the regularity of anisotropic variational problems, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* **17** (1993), no. 5, 229–239.
[29] Q. Tang, Regularity of minimizer of non-isotropic integrals of the calculus of variations, *Ann. Mat. Pura Appl.* **164** (1993), no. 1, 77–87.
[30] J. Tyagi, A nonlinear Picone's identity and its applications, *Appl. Math. Lett.* **26** (2013), no. 6, 624–626.
[31] J. Weickert, Anisotropic Diffusion in Image Processing, Teubner, Stuttgart, 1998.

(Tingfu Feng) Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, P.R. China.
E-mail address: `ftfml@mail.nwpu.edu.cn`

(Mei Yu) Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, P.R. China.
E-mail address: `yumei@nwpu.edu.cn`