SOME SEPARATION AXIOMS VIA MODIFIED θ-OPEN SETS

M. CALDAS, S. JAFARI AND T. NOIRI

Abstract. In this paper, we introduce some separation axioms by utilizing modified θ-open sets and obtain a characterization of Hausdorff spaces due to Dickman and Porter [?] as a corollary of the results.

1. Introduction

Veličko [?] introduced the notion of θ-open sets and the θ-closure operator to study H-closed spaces. Janković [?] investigated several separation axioms by using θ-open sets and the θ-closure operator. Dickman and Porter [?] investigated the relationships among θ-closed sets, Hausdorff spaces and H-closed spaces. Among others, they showed that a topological space $X$ is Hausdorff if and only if for each $x \in X$ the singleton $\{x\}$ is θ-closed. Recently, the following modifications of θ-open sets are introduced and studied: semi-θ-open [?], pre-θ-open [?] and semipre-θ-open [?].

In this paper, by using the notion of $m$-θ-open sets [?], we introduce and investigate separation axioms $m\theta-D_0$, $m\theta-D_1$ and $m\theta-D_2$ analogous to $D_0$, $D_1$ and $D_2$ due to Tong [?]. As a corollary of...
our results, we obtain a characterization of Hausdorff spaces stated above.

2. Preliminaries

In what follows \((X, \tau)\) and \((Y, \sigma)\) (or \(X\) and \(Y\)) denote topological spaces. Let \(A\) be a subset of \(X\). We denote the interior, the closure and the complement of a set \(A\) by \(\text{Int}(A)\), \(\text{Cl}(A)\) and \(X \setminus A\) or \(A^c\) respectively. A point \(x \in X\) is called the \(\theta\)-cluster point of \(A\) if \(A \cap \text{Cl}(U) \neq \emptyset\) for every open set \(U\) of \(X\) containing \(x\). The set of all \(\theta\)-cluster points of \(A\) is called the \(\theta\)-closure of \(A\), denoted by \(\text{Cl}_{\theta}(A)\).

A subset \(A\) is called \(\theta\)-closed if \(A = \text{Cl}_{\theta}(A)\). The complement of a \(\theta\)-closed set is called \(\theta\)-open. We denote the collection of all \(\theta\)-open sets of \((X, \tau)\) by \(\tau_{\theta}\).

**Definition 2.1.** Let \((X, \tau)\) be a topological space. A subset \(A\) of \(X\) is said to be

1. semi-open [?] if \(A \subset \text{Cl}(\text{Int}(A))\),
2. preopen [?] if \(A \subset \text{Int}(\text{Cl}(A))\),
3. \(\beta\)-open [?] or semi-preopen [?] if \(A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))\).

The family of all semi-open (resp. preopen, \(\beta\)-open, semi-preopen) sets in \(X\) is denoted by \(SO(X)\) (resp. \(PO(X), \beta(X), SPO(X)\)).

**Definition 2.2.** The complement of a semi-open (resp. preopen, \(\beta\)-open, semi-preopen) set is said to be semi-closed [?] (resp. preclosed [?], \(\beta\)-closed [?], semi-preclosed [?]).

**Definition 2.3.** The intersection of all semi-closed (resp. preclosed, \(\beta\)-closed, semi-preclosed) sets of \(X\) containing \(A\) is called the semi-closure [?] (resp. preclosure [?], \(\beta\)-closure [?] or semi-preclosure [?]) of \(A\) and is denoted by \(s\text{Cl}(A)\) (resp. \(p\text{Cl}(A), \beta\text{Cl}(A), s\text{pCl}(A)\)).

**Definition 2.4.** Let \((X, \tau)\) be a topological space. A subset \(A\) of \(X\) is said to be

1. semi-\(\theta\)-open [?] if for each \(x \in A\) there exists a semi-open set \(U\)
such that \( x \in U \subset s\text{Cl}(U) \subset A \),
(2) pre-\( \theta \)-open \([2] \) if for each \( x \in A \) there exists a preopen set \( U \)
such that \( x \in U \subset p\text{Cl}(U) \subset A \),
(3) semipre-\( \theta \)-open \([2] \) for each \( x \in A \) there exists a semipre-open
set \( U \) such that \( x \in U \subset sp\text{Cl}(U) \subset A \).

3. \( m \)-spaces

**Definition 3.1.** A subfamily \( m \) of the power set \( P(X) \) of a
nonempty set \( X \) is called an \( m \)-structure on \( X \) if \( m \) satisfies the
following:
(1) \( \emptyset \in m \) and \( X \in m \),
(2) \( \cup_{\alpha \in \Delta} A_\alpha \in m \) whenever \( A_\alpha \in m \) for each \( \alpha \in \Delta \).

We call the pair \( (X, m) \) an \( m \)-space. Each member of \( m \) is said
to be \( m \)-open and the complement of an \( m \)-open set is said to be
\( m \)-closed.

**Remark 3.2.** It should be noted that condition (2) in Definition
3.1 is called property (B) by Maki et al. in \([2] \). In this paper, we
always assume the property (B) on \( m \)-structures.

**Remark 3.3.** Let \( (X, \tau) \) be a topological space. Then the families
\( \tau_\theta, \tau, SO(X), PO(X), \beta(X) \) are all \( m \)-structures on \( X \). It is well-
known that \( \tau_\theta \) is a topology for \( X \).

**Definition 3.4.** Let \( X \) be a nonempty set and \( m \) an \( m \)-structure
on \( X \). For a subset \( A \) of \( X \), the \( m \)-closure of \( A \) and the \( m \)-interior
of \( A \) are defined in \([2] \) as follows:
(1) \( m\text{Cl}(A) = \cap \{ F \mid A \subset F, X \setminus F \in m \} \),
(2) \( m\text{Int}(A) = \cup \{ U \mid U \subset A, U \in m \} \).

**Remark 3.5.** Let \( (X, \tau) \) be a topological space and \( A \) a subset of
\( X \). If \( m = \tau \) (resp. \( SO(X), PO(X), \beta(X) \)), then we have
(1) \( m\text{Cl}(A) = \text{Cl}(A) \) (resp. \( s\text{Cl}(A), p\text{Cl}(A), \beta\text{Cl}(A) \)),
(2) \( m\text{Int}(A) = \text{Int}(A) \) (resp. \( s\text{Int}(A), p\text{Int}(A), \beta\text{Int}(A) \)).
(cf. Remark 3.10 below for the $\theta$-closure $Cl_\theta(A)$ of $A$)

**Lemma 3.6.** (eg. Maki et al. [?]) Let $m$ be an $m$-structure on a nonempty set $X$. For subsets $A$ and $B$ of $X$, the following properties hold:

1. $mCl(X \setminus A) = X \setminus mInt(A)$ and $mInt(X \setminus A) = X \setminus mCl(A)$,
2. $mCl(\emptyset) = \emptyset$, $mCl(X) = X$, $mInt(\emptyset) = \emptyset$ and $mInt(X) = X$,
3. If $A \subset B$, then $mCl(A) \subset mCl(B)$ and $mInt(A) \subset mInt(B)$,
4. $A \subset mCl(A)$ and $mInt(A) \subset A$,
5. $mCl(mCl(A)) = mCl(A)$ and $mInt(mInt(A)) = mInt(A)$.

**Lemma 3.7.** (Popa and Noiri [?]) Let $m$ be an $m$-structure on a nonempty set $X$. Then $x \in mCl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing $x$.

**Lemma 3.8.** Let $m$ be an $m$-structure on a nonempty set $X$. Then for a subset $A$ of $X$ the following properties hold:

1. $A \in m$ if and only if $A = mInt(A)$,
2. $A$ is $m$-closed if and only if $A = mCl(A)$,
3. $mCl(A)$ is $m$-closed and $mInt(A)$ is $m$-open.

**Proof.** This is an immediate consequence of Lemmas 3.6 and 3.7. □

**Definition 3.9.** Let $A$ be a subset of an $m$-space $(X, m)$. A point $x \in X$ is called

1. an $m_\theta$-adherent point of $A$ if $mcl(U) \cap A \neq \emptyset$ for every $U \in m$ containing $x$,
2. An $m_\theta$-interior point of $A$ if $x \in U \subset mCl(U) \subset A$ for every $U \in m$ containing $x$.

The set of all $m_\theta$-adherent points of $A$ is called the $m_\theta$-closure [?] of $A$ and is denoted by $mCl_\theta(A)$. If $A = mCl_\theta(A)$, then $A$ is called $m$-$\theta$-closed. The complement of an $m$-$\theta$-closed set is said to be $m$-$\theta$-open. The set of all $m_\theta$-interior points of $A$ is called the $m_\theta$-interior of $A$ and is denoted by $mInt_\theta(A)$.

This is clear that union of $m$-$\theta$-open sets in $X$ is $m$-$\theta$-open.
Remark 3.10. (Noiri and Popa [?]). Let $A$ and $B$ be subsets of an $m$-space $(X, m)$. Then the following properties hold:
(1) $X \setminus \text{Cl}_m(A) = \text{Int}_m(X \setminus A)$ and $X \setminus \text{Int}_m(A) = \text{Cl}_m(X \setminus A)$,
(2) $A$ is open if and only if $A = \text{Int}_m(A)$,
(3) $A \subseteq \text{Cl}_m(A) \subseteq \text{Cl}_m(A)$ and $\text{Int}_m(A) \subseteq \text{Int}_m(A) \subseteq A$,
(4) If $A \subseteq B$, then $\text{Cl}_m(A) \subseteq \text{Cl}_m(B)$ and $\text{Int}_m(A) \subseteq \text{Int}_m(B)$,
(5) If $A$ is $m$-open, then $\text{Cl}_m(A) = \text{Cl}_m(A)$,
(6) $\text{Cl}_m(A)$ is $m$-closed and $\text{Int}_m(A)$ is $m$-open.

4. Some New Separation Axioms

Tong [?] defined a subset $A$ of a topological space $(X, \tau)$ to be a $D$-set if there are two open sets $U$, $V$ in $X$ such that $U \neq X$ and $A = U \setminus V$. Analogously, we define $m\theta$-$D$-sets as follows:

Definition 4.1. Let $(X, m)$ be an $m$-space. A subset $A \subseteq X$ is called an $m\theta$-Difference set (in short $m\theta$-$D$-set) if there are two $m\theta$-open sets $U$, $V$ in $X$ such that $U \neq X$ and $A = U \setminus V$.

It is true that every $m\theta$-open set $U \neq X$ is an $m\theta$-$D$-set since $U = U \setminus \emptyset$.

Definition 4.2. An $m$-space $(X, m)$ is said to be
(1) $m\theta$-$D_0$ (resp. $m\theta$-$D_1$) if for $x, y \in X$ such that $x \neq y$ there exists an $m\theta$-$D$-set of $X$ containing $x$ but not $y$ or (resp. and) an $m\theta$-$D$-set containing $y$ but not $x$,
(2) $m\theta$-$D_2$ if for $x, y \in X$ such that $x \neq y$ there exist disjoint $m\theta$-$D$-sets $G$ and $E$ such that $x \in G$ and $y \in E$,
(3) $m\theta$-$T_0$ (resp. $m\theta$-$T_1$) if for $x, y \in X$ such that $x \neq y$ there exists an $m\theta$-open set $U$ of $X$ containing $x$ but not $y$ or (resp. and) an $m\theta$-open set $V$ of $X$ containing $y$ but not $x$,
(4) $m\theta$-$T_2$ if for $x, y \in X$ such that $x \neq y$ there exist disjoint $m\theta$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

Remark 4.3. Let $(X, \tau)$ be a topological space. If $m = \tau$, then $m\theta$-$T_1 = \theta$-$T_1$ [?] and $m\theta$-$T_2 = \theta$-$T_2$ [?].
Theorem 4.4. For an $m$-space $(X, m)$, the following properties hold:

1. If $(X, m)$ is $m\theta$-$T_i$, then it is $m\theta$-$T_{i-1}$ for $i = 1, 2$.
2. If $(X, m)$ is $m\theta$-$T_i$, then it is $m\theta$-$D_i$ for $i = 0, 1, 2$.
3. If $(X, m)$ is $m\theta$-$D_i$, then it is $m\theta$-$D_{i-1}$ for $i = 1, 2$.

Proof. This is obvious from Definition 4.2. □

Theorem 4.5. For an $m$-space $(X, m)$, the following statements are true:

1. $(X, m)$ is $m\theta$-$D_0$ if and only if $(X, m)$ is $m\theta$-$T_0$.
2. $(X, m)$ is $m\theta$-$D_1$ if and only if $(X, m)$ is $m\theta$-$D_2$.

Proof. The sufficiency for (1) and (2) follows from Theorem 4.4. □

Necessity for (1). Let $(X, m)$ be $m\theta$-$D_0$ so that for any pair of distinct points $x$ and $y$ of $X$ at least one belongs to an $m\theta$-$D$-set $O$. Therefore, we choose $x \in O$ and $y \notin O$. Suppose $O = U \setminus V$ for $U \neq X$ and $m\theta$-open sets $U$ and $V$. This implies that $x \in U$. For the case that $y \notin O$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the $m$-space $(X, m)$ is $m\theta$-$T_0$ since $x \in U$ and $y \notin U$. For (ii), the $m$-space $(X, m)$ is also $m\theta$-$T_0$ since $y \in V$ but $x \notin V$.

Necessity for (2). Suppose that $(X, m)$ is $m\theta$-$D_1$. It follows from the definition that for any distinct points $x$ and $y$ in $X$ there exist $m\theta$-$D$-sets $G$ and $E$ such that $G$ containing $x$ but not $y$ and $E$ containing $y$ but not $x$. Let $G = U \setminus V$ and $E = W \setminus D$, where $U$, $V$, $W$ and $D$ are $m\theta$-open. By the fact that $x \notin E$, we have two cases, i.e. either $x \notin W$ or both $W$ and $D$ contain $x$. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or (ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U \setminus V$ that $x \in U \setminus (V \cup W)$, and also it follows from $y \in W \setminus D$ that $y \in W \setminus (U \cup D)$. Thus we have $U \setminus (V \cup W)$ and $W \setminus (U \cup D)$ which are disjoint. If (ii) is the case, it follows that $x \in U \setminus V$ and $y \in V$ since $y \in U$ and $y \in V$. Therefore $(U \setminus V) \cap V = \emptyset$. If $x \in W$ and $x \in D$, we have $y \in W \setminus D$ and $x \in D$. Hence $(W \setminus D) \cap D = \emptyset$. This shows that
X is $m\theta-D_2$.

**Definition 4.6.** An $m$-space $(X, m)$ is said to be $m$-$T_2$ [2] if for $x, y \in X$ such that $x \neq y$ there exist disjoint $m$-open sets $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$.

**Remark 4.7.** Let $(X, \tau)$ be a topological space. If $m = \tau$, then $m-T_2 =$ Hausdorff.

**Theorem 4.8.** For an $m$-space $(X, m)$, the following properties are equivalent:

1. $(X, m)$ is $m$-$T_2$;
2. $(X, m)$ is $m\theta$-$T_1$;
3. $(X, m)$ is $m\theta$-$T_0$.

**Proof.** (1) $\Rightarrow$ (2): Let $x$ and $y$ be any pair of disjoint points of $X$. For any $z \in X \setminus \{y\}$, by (1) there exist $m$-open sets $U_y$ and $U_z$ such that $y \in U_y$, $z \in U_z$ and $U_y \cap U_z = \emptyset$. We have $m\text{Cl}(U_y) \cap U_z = \emptyset$ and $z \in U_z \subseteq m\text{Cl}(U_y) \subseteq X \setminus U_y \subseteq X \setminus \{y\}$. Therefore, $X \setminus \{y\}$ is an $m\theta$-$\theta$-open set containing $x$. Quite similarly, we can show that $X \setminus \{x\}$ is an $m\theta$-$\theta$-open set containing $y$. Therefore, $(X, m)$ is $m\theta$-$T_1$.

(2)$\Rightarrow$ (3): This is obvious.

(3)$\Rightarrow$ (1): Suppose that $(X, m)$ is $m\theta$-$T_0$. For any pair of distinct points $x, y$, there exists an $m\theta$-$\theta$-open set $U$ such that (1) $x \notin U$ and $y \notin U$ or (2) $x \notin U$ and $y \in U$. In case (1), there exists $U_x \in m$ such that $x \in U_x \subseteq m\text{Cl}(U_x) \subseteq U$. Therefore, $y \in X \setminus U \subset X \setminus m\text{Cl}(U_x)$, $X \setminus m\text{Cl}(U_x)$ is $m$-open and $U_x \cap (X \setminus m\text{Cl}(U_x)) = \emptyset$. In case (2), similarly, we obtain $U_y \in m$ such that $y \in U_y$, $x \in X \setminus m\text{Cl}(U_y)$ and $U_y \cap (X \setminus m\text{Cl}(U_y)) = \emptyset$. This shows that $(X, m)$ is $m$-$T_2$. □

**Corollary 4.9.** For an $m$-space $(X, m)$, the following properties are equivalent:

1. $(X, m)$ is $m\theta$-$D_2$;
2. $(X, m)$ is $m\theta$-$D_1$;
3. $(X, m)$ is $m\theta$-$D_0$;
(4) $(X, m)$ is $m\theta$-$T_0$;
(5) $(X, m)$ is $m\theta$-$T_1$;
(6) $(X, m)$ is $m$-$T_2$.

**Proof.** This is an immediate consequence of Theorems 4.4, 4.5 and 4.8. □

**Corollary 4.10.** (Dickman and Porter[?]). For a topological space $(X, \tau)$, the following properties are equivalent:
(1) $(X, \tau)$ is Hausdorff;
(2) $(X, \tau)$ is $\theta$-$T_1$;
(3) For each $x \in X$, $Cl_\theta(\{x\}) = \{x\}$.

**Proof.** This is an immediate consequence of Corollary 4.9. □

**Remark 4.11.** For a topological space $(X, \tau)$, by Corollary 4.9 we have the following diagram:

$$
\begin{array}{c}
T_2 \leftrightarrow \theta-D_2 \\
\uparrow \quad \uparrow \\
\theta-T_1 \leftrightarrow \theta-D_1 \\
\uparrow \quad \uparrow \\
\theta-T_2 \leftrightarrow \theta-D_0
\end{array}
$$

5. Quasi-$\theta$-continuous functions

**Definition 5.1.** A function $f : (X, m_X) \to (Y, m_Y)$, where $X$ and $Y$ are nonempty sets with $m$-structures $m_X$ and $m_Y$, respectively, is said to be
(1) $\theta$-$M$-continuous [?] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(m_X Cl(U)) \subset m_Y Cl(V)$. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $\theta$-$M$-continuous if it has the property at each point $x \in X$.
(2) quasi-$\theta$-$M$-continuous if for each $x \in X$ and each $m\theta$-open set $V$ containing $f(x)$, there is an $m\theta$-open set $U$ containing $x$ such that $f(U) \subset V$, equivalently, if $f^{-1}(V)$ is $m\theta$-open in $(X, \tau)$ for every $m\theta$-open set $V$ of $(Y, \sigma)$. 
Remark 5.2. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. If \( m_x = \tau \), \( m_Y = \sigma \) and \( f : (X, m_X) \to (Y, m_Y) \) is \( \theta \)-M-continuous (resp. quasi-\( \theta \)-M-continuous), then \( f \) is \( \theta \)-continuous [?] (resp. quasi-\( \theta \)-continuous [?]).

Remark 5.3. (1) A function \( f : (X, \tau) \to (Y, \sigma) \) is quasi-\( \theta \)-continuous if and only if \( f : (X, \tau_\theta) \to (Y, \sigma_\theta) \) is continuous.

(2) It is shown in [?] that every \( \theta \)-continuous function is quasi-\( \theta \)-continuous but the converse is not true.

Theorem 5.4. If \( f : (X, m_X) \to (Y, m_Y) \) is a quasi-\( \theta \)-M-continuous surjection and \( A \) is an \( m\theta-D \)-set in \( Y \), then \( f^{-1}(A) \) is an \( m\theta-D \)-set in \( X \).

Proof. Let \( A \) be an \( m\theta-D \)-set in \( Y \). Then there are \( m\theta \)-open sets \( U \) and \( V \) in \( Y \) such that \( A = U \setminus V \) and \( U \neq Y \). By the quasi-\( \theta \)-M-continuity of \( f \), \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( m\theta \)-open in \( X \). Since \( U \neq Y \) and \( f \) is surjective, we have \( f^{-1}(U) \neq X \). Hence \( f^{-1}(A) = f^{-1}(U) \setminus f^{-1}(V) \) is an \( m\theta-D \)-set. \( \Box \)

We prove now another characterization of \( m\theta-D_1 \) spaces.

Theorem 5.5. An \( m \)-space \((X, m_X)\) is \( m\theta-D_1 \) if and only if, for each pair of distinct points \( x \) and \( y \) in \( X \), there exists a quasi-\( \theta \)-M-continuous surjection \( f \) of \((X, m_X)\) onto an \( m\theta-D_1 \) space \((Y, m_Y)\) such that \( f(x) \neq f(y) \).

Proof. Necessity. For every pair of distinct points of \( X \), it suffices to take the identity function on \( X \).

Sufficiency. Let \( x \) and \( y \) be any pair of distinct points in \( X \). By hypothesis, there exists a quasi-\( \theta \)-M-continuous surjection \( f \) of a space \( X \) onto an \( m\theta-D_1 \) space \( Y \) such that \( f(x) \neq f(y) \). Therefore, by Theorem 4.5 there exist disjoint \( m\theta-D \)-sets \( A_x \) and \( B_y \) in \( Y \) such that \( f(x) \in A_x \) and \( f(y) \in B_y \). Since \( f \) is quasi-\( \theta \)-M-continuous and surjective, by Theorem 5.4, \( f^{-1}(A_x) \) and \( f^{-1}(B_y) \)
are disjoint $m\theta$-$D$-sets in $X$ containing $x$ and $y$, respectively. Hence by Theorem 4.5, $X$ is $m\theta$-$D_1$ space. □

**Theorem 5.6.** If $(Y, m_Y)$ is $m\theta$-$D_1$ and $f : (X, m_X) \to (Y, m_Y)$ is a quasi-$\theta$-$M$-continuous injection, then $(X, m_X)$ is $m\theta$-$D_1$.

**Proof.** Let $x$ be any point of $X$. Since $(Y, m_Y)$ is $m\theta$-$D_1$, $(Y, m_Y)$ is $m\theta$-$T_1$ and hence $\{f(x)\}$ is $m\theta$-closed in $(Y, \sigma)$ by Corollary 4.9. Since $f$ is a quasi-$\theta$-$M$-continuous injection, $\{x\} = f^{-1}(\{f(x)\})$ is $m$-$\theta$-closed in $(X, m_X)$. Therefore $(X, m_X)$ is $m\theta$-$D_1$. □

**Corollary 5.7.** If $f : (X, m_X) \to (Y, m_Y)$ is a quasi-$\theta$-$M$-continuous injection and $(Y, m_Y)$ is $m$-$T_2$, then $(X, m_X)$ is $m$-$T_2$.

**Proof.** This is an immediate consequence of Corollary 4.9 and Theorem 5.6. □

As corollaries of Theorem 5.5 and 5.6, we obtain the following:

**Corollary 5.8.** A topological space $(X, \tau)$ is $\theta$-$D_1$ if and only if for each pair of distinct points $x$ and $y$ in $X$, there exists a quasi-$\theta$-continuous surjection $f$ of $X$ onto a $\theta$-$D_1$ space $(Y, \sigma)$ such that $f(x) \neq f(y)$.

**Corollary 5.9.** If $(Y, \sigma)$ is $\theta$-$D_1$ (resp. Hausdorff) and $f : (X, \tau) \to (Y, \sigma)$ is a quasi-$\theta$-continuous injection, then $(X, \tau)$ is $\theta$-$D_1$ (resp. Hausdorff).

**References**


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M. Caldas
Departamento de Matematica Aplicada
Universidade Federal Fluminense
Rua Mario Santos Braga
s/n 24020-140
Niteroi
RJ Brasil
e-mail: gmamccs@vm.uff.br

S. Jafari
Department of Mathematics and Physics
Roskilde University
Postbox 260
4000 Roskilde
Denmark
e-mail: sjafari@ruc.dk

T. Noiri
Yatsushiro College of Technology
2627 Hiryama shinmachi
Yatsushiro-shi
Kumamoto-ken
866-8501 Japan
e-mail: noiri@as.yatsushiro-nct.ac.jp