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RIGID RESOLUTION OF A FINITELY GENERATED MODULE OVER A REGULAR LOCAL RING

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Abstract. Let M be a finitely generated module over a regular local ring $(R, \mathfrak{n})$, and let $\mathbb{M} = \{M_i\}$ be an $\mathfrak{n}-$stable filtration on M. As a consequence of a recent result by Rossi and Sharifan in [14], we prove that if the i-th Betti numbers of M *and* $(\mathrm{gr}_{\mathbb{M}}(\{M\})$ *coincide with each other, then for each* $j \geq i$ *the* j*-th Betti numbers of them are the same and* $\mathrm{Syz}_i(M)$ *is a Koszul module provided that* $\mathrm{Syz}_i(\mathrm{gr}_{\mathbb{M}}(M)$ *is componentwise linear.*

1. Introduction

Supposing M is a finitely generated module over a regular local ring $(R, \mathfrak{n})$ and if $\mathbb{M} = \{M_i\}$ is an $\mathfrak{n}-$stable filtration on M, then $\mathrm{gr}_{\mathbb{M}}(M) := \bigoplus_{\mathrm{t}\geq 0} \mathrm{M_t}/\mathrm{M_{t+1}}$ will be the corresponding associated graded module. Often by making use of the Hilbert function, mathematicians compare the numerical invariants of M and $\mathrm{gr}_{\mathbb{M}}(M)$ to find reasonable conditions on M for being $\mathrm{gr}_{\mathbb{M}}(M)$ Cohen-Macaulay or having estimated depth. We can deduce more accurate data by comparing the minimal free resolutions of M as an $R-$module and the minimal free resolutions of $\mathrm{gr}_{\mathbb{M}}(M)$ as a $P = \mathrm{gr}_{\mathfrak{n}}(\mathrm{R})-$module.

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For example, consider the linear part of a minimal free resolution of M (see page 6). Röemer [13] has defined the linearity defect of M, written $ld(M)$, as the smallest integer i such that the linear part is exact in homological degrees greater than i. The module $\mathrm{gr}_{\mathfrak{n}}(M) = \bigoplus_{\mathrm{i}\geq 0} \mathfrak{n}^{\mathrm{i}}M/\mathfrak{n}^{\mathrm{i}+1}M$ has a linear $P-$resolution if and only if $ld(M) = 0$. In this case, M is called a Koszul module and its Betti numbers are the same as the ones of $\mathrm{gr}_{\mathfrak{n}}(M)$ (see [9]). Another interesting result is that in the graded setting M is Koszul if and only if M is componentwise linear (see [13, Theorem 3.2.8]).

Here, we deal with the rigidity of the Betti numbers of M by passing through the associated graded module $\mathrm{gr}_{\mathbb{M}}(M)$. Mainly, we generalize the central results of [14] and prove that if $\beta_i(\mathrm{gr}_{\mathbb{M}}(M)) = \beta_{\mathrm{i}}(M)$, for some $i \geq 0$, and $\mathrm{Syz}_i(\mathrm{gr}_{\mathbb{M}}(M))$ is a componentwise linear module, then $\beta_k(\mathrm{gr}_{\mathbb{M}}(M)) = \beta_{\mathrm{k}}(M)$, for each $k \geq i$ (see Theorem 3.3). In particular, under these assumptions, depth $M =$ depth $\mathrm{gr}_{\mathbb{M}}(M)$, the module $\mathrm{Syz}_i(M)$ is Koszul and $ld(M) \leq i$ (see Corollary 3.6).

One of the most important starting points of [14] is a result due to Robbiano (see [12] and also [11]), which says that we can build up an R-free resolution of M from a minimal P-free resolution of $\mathrm{gr}_{\mathbb{M}}(M)$. We should point out that this construction is a very useful element and its applications have appeared in [14, 15]. Also, using properties of this resolution, we are able to give a short proof for Theorem 3.3 by means of [14, Theorem 3.1], which actually shows that $\beta_i(\mathrm{gr}_{\mathbb{M}}(M)) = \beta_{\mathrm{i}}(M)$, for every $i \geq 0$ provided that M and $\mathrm{gr}_{\mathbb{M}}(M)$ have the same minimal number of generators and $\mathrm{gr}_{\mathbb{M}}(M)$ is a componentwise linear module.

The following is the use of Theorem 3.3 to the classical case, a local ring $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ filtered by the $\mathfrak{m}$-adic filtration. In this case, $\mathrm{gr}_{\mathfrak{m}}(\mathrm{A}) = \mathrm{P}/\mathrm{I}^*$, where I^* is a homogeneous ideal of the polynomial ring P generated by the initial forms (w.r.t. the $\mathfrak{n}-$adic filtration) of the elements of I and if we consider the ideal I equipped with the $n-$adic filtration $\mathbb{M} = \{I \cap \mathfrak{n}^i\}$, we have $\mathrm{gr}_{\mathbb{M}}(\mathrm{I}) = \mathrm{I}^*$.

As a consequence of Theorem 3.3, we can see that if $\beta_i(I) = \beta_i(I^*)$, and $\mathrm{Syz}_i(I^*)$ is componentwise linear, then $\beta_k(I) = \beta_k(I^*)$, for each $k \geq i$. It is interesting to compare this result with Conca, et al.'s main theorem in [4] which says that if J is a homogeneous ideal of the polynomial ring P and $\beta_i(J) = \beta_i(\mathrm{Gin}(J))$, then $\beta_k(J) = \beta_k(\mathrm{Gin}(J))$, for each $k \geq i$. In this route, we state an interesting conjecture and some examples (see Discussion 3.9).

2. **Preliminaries**

Throughout the paper, $(R, \mathfrak{n})$ is a regular local ring with infinite residue field k. If $\dim R = n$, then the associated graded ring $\mathrm{gr}_{\mathfrak{n}}(\mathrm{R})$ with respect to the $\mathfrak{n}$-adic filtration is the polynomial ring $P = k[x_1, \cdots, x_n]$. If x is a non-zero element of R, we denote by x^* (or $\mathrm{gr}_{\mathfrak{n}}(\mathrm{x})$) Then initial form of x in P. If $x = 0$, then $x^* = 0$.

Let M be a finitely generated R-module. We say, according to the notation in [16], that a filtration of submodules $\mathbb{M} = \{M_n\}_{n\geq 0}$ on M is called an $\mathfrak{n}$-filtration if $\mathfrak{n}M_n \subseteq M_{n+1}$, for every $n \geq 0$, and a good (or stable) $\mathfrak{n}$-filtration if $\mathfrak{n}M_n = M_{n+1}$, for all sufficiently large n. In the following, a *filtered module* M will always be an R-module equipped with a good $\mathfrak{n}$-filtration $\mathbb{M}$. If $\mathbb{M} = \{M_j\}$ is an $\mathfrak{n}$-filtration of M, Then define

$$\mathrm{gr}_{\mathbb{M}}(M) = \bigoplus_{j\geq 0}(M_j/M_{j+1}),$$

which is a graded $\mathrm{gr}_{\mathfrak{n}}(\mathrm{R})$-module in a natural way. It is called the **associated graded module** to the filtration $\mathbb{M}$.

To avoid triviality, we assume that $\mathrm{gr}_{\mathbb{M}}(M)$ is not zero or equivalently $M \neq 0$. If N is a submodule of M, then by Artin-Rees Lemma, the sequence $\{N \cap M_j \mid j \geq 0\}$ is a good $\mathfrak{n}$-filtration of N. Since

$$(N \cap M_j)/(N \cap M_{j+1}) \simeq (N \cap M_j + M_{j+1})/M_{j+1}, \tag{2.1}$$

$\mathrm{gr}_{\mathbb{M}}(\mathrm{N})$ is a graded submodule of $\mathrm{gr}_{\mathbb{M}}(M)$, denoted by N^*.

The morphism of filtered modules $f : M \to N$ ($f(M_p) \subseteq N_p$ for every p) clearly induces a morphism of graded $\mathrm{gr}_{\mathfrak{n}}(\mathrm{R})$-modules,

$$\mathrm{gr(f)} : \mathrm{gr}_{\mathbb{M}}(\mathrm{M}) \to \mathrm{gr}_{\mathbb{N}}(\mathrm{N}).$$

It is clear that $\mathrm{gr}_{\mathbb{M}}(\dot{})$ is a functor from the category of filtered R-modules into the category of the graded $\mathrm{gr}_{\mathfrak{n}}(\mathrm{R})$-modules. Furthermore, we have a canonical embedding $(\ker f)^* \to \ker(\mathrm{gr(f)})$.

Let $L = \bigoplus_{i=1}^{s} Re_i$ be a free R-module of rank s and $\nu_1, \cdots, \nu_s$ be integers. We define the filtration $\mathbb{L} = \{L_p : p \in \mathbf{Z}\}$ on L as follows

$$L_p := \bigoplus_{i=1}^{s} \mathfrak{n}^{p-\nu_i} e_i = \{(a_1, \cdots, a_s) : a_i \in \mathfrak{n}^{p-\nu_i}\}.$$

We denote the filtered free R-module L by $\bigoplus_{i=1}^{s} R(-\nu_i)$ and we call it *special filtration* on L. If $(\mathbf{F}., \delta.)$ is a complex of finitely generated free

R-modules, a special filtration on $\mathbf{F.}$ is a special filtration on each F_i that makes $(\mathbf{F}., \delta.)$ a complex of filtered modules.

Next, we state a crucial result due to Robbiano which gives a criteria to compare free resolutions of M and $\mathrm{gr}_{\mathbb{M}}(M)$. For the proof and more information, see [14, Theorem 1.8]

Theorem 2.1. *Let M be a filtered R-module and let $(\mathbf{G}., d.)$ be a P-free graded resolution of* $\mathrm{gr}_{\mathbb{M}}(\mathrm{M})$,

$$\mathbf{G}.:\ 0 \to \bigoplus_{i=1}^{\beta_l} P(-a_{li}) \ \xrightarrow{d_l}\ \bigoplus_{i=1}^{\beta_{l-1}} P(-a_{l-1i}) \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} \bigoplus_{i=1}^{\beta_0} P(-a_{0i}) \xrightarrow{d_0} \mathrm{gr}_{\mathbb{M}}(\mathrm{M}) \to 0.$$

Then, we can build up an R-free resolution $(\mathbf{F}., \delta.)$ of M and a special filtration $\mathbb{F}$ on it such that $\mathrm{gr}_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.

We remind that $(\mathbf{F}., \delta.)$ is computed by an inductive process. For each $j \geq 0$, the $R-$free module F_j is defined with the special filtration $F_j = \bigoplus_{i=1}^{\beta_j} R(-a_{ji})$ and the differential map $\delta_j : F_j \to F_{j-1}$ such that $\mathrm{gr}_{\mathbb{F}_j}(F_j) = \mathrm{G}_j$, $\mathrm{gr}_{\mathbb{F}_j}(\delta_j) = \mathrm{d}_j$ and moreover,

$$\ker(d_j) = \mathrm{gr}_{\mathbb{F}_j}(\ker(\delta_j)). \tag{2.2}$$

It is worth saying that the R-free resolution of M,

$$\mathbf{F}.:\ 0 \to R^{\beta_l} \xrightarrow{\delta_l} R^{\beta_{l-1}} \xrightarrow{\delta_{l-1}} \cdots \xrightarrow{\delta_1} R^{\beta_0} \xrightarrow{\delta_0} M \to 0,$$

coming from a minimal free resolution of $\mathrm{gr}_{\mathbb{M}}(\mathrm{M})$, is not necessarily minimal. In particular, $(\mathbf{F}., \delta.)$ is minimal if and only if the Betti numbers of M and $\mathrm{gr}_{\mathbb{M}}(M)$ coincide.

Let $(\mathbf{F}., \delta.)$ be a non-minimal free resolution of an $R-$module M. In [5, Page 6], a method was described to construct the minimal free resolution of M starting from $(\mathbf{F}., \delta.)$. In the following, we explain this with more details.

Remark 2.2. Let

$$F_j = R^{\alpha_j} \xrightarrow{\delta_j} F_{j-1} = R^{\alpha_{j-1}}$$

be part of $(\mathbf{F}., \delta.)$, a free resolution of a module M, and $\mathcal{M}_j = (m_{rs})$ be the matrix of δ_j with respect to the bases $\{e_{j1}, \cdots, e_{j\alpha_j}\}$ of F_j and $\{e_{j-1,1}, \cdots, e_{j-1,\alpha_{j-1}}\}$ of F_{j-1}. Suppose that there exist some non-zero invertible entries in $\mathcal{M}_j$. Let (p,q) be such that $m_{pq} \notin \mathfrak{n}$. Without loss of generality, suppose that $(p,q) = (1,1)$. Let $c = m_{11}$ and replace the basis of F_{j-1} by $e'_{j-1,1} = ce_{j-1,1} + m_{21}e_{j-1,2} + \cdots + m_{\alpha_{j-1}1}e_{j-1,\alpha_{j-1}}$, and $e'_{j-1,i} = e_{j-1,i}$ for $2 \leq i \leq \alpha_{j-1}$.

The matrices of differential maps δ_k change just for δ_j and δ_{j-1}. Since $\delta_j(e_{j1}) = ce_{j-1,1} + m_{21}e_{j-1,2} + \cdots + m_{\alpha_{j-1}1}e_{j-1,\alpha_{j-1}} = e'_{j-1,1}$, the first column of $\mathcal{M}_j$ is replaced with $(1\ 0\ \cdots\ 0)^{tr}$.
Since $\delta_{j-1}(e'_{j-1,1}) = \delta_{j-1}(\delta_j(e_{j1})) = 0$, the first column of $\mathcal{M}_{j-1}$ is replaced with $(0\ \cdots\ 0)^{tr}$. For $s \geq 2$, one can check that the column $(m_{1s}\ \cdots\ m_{rs}\ \cdots\ m_{\alpha_{j-1}s})^{tr}$ of $\mathcal{M}_j$ is replaced with

$$(c^{-1}m_{1s}\ \cdots\ m_{rs} - c^{-1}m_{1s}m_{r1}\ \cdots\ m_{\alpha_{j-1}s} - c^{-1}m_{1s}m_{\alpha_{j-1}1})^{tr}.$$

Now, we consider a subcomplex of $\mathbf{F}.$. Let $H_i = 0$, if $i \neq j-1, j$ and $H_j = F_j|_{e_{j1}}$ and $H_{j-1} = F_{j-1}|_{e'_{j-1,1}}$. Thus, we have found the following trivial subcomplex of $(\mathbf{F}., \delta.)$,

$$\mathbf{H}.: \underbrace{0 \to \cdots \to 0}_{h-j+1} \to R \xrightarrow{id} R \to \underbrace{0 \to \cdots \to 0}_{j},$$

where h is the length of $(\mathbf{F}., \delta.)$. $\mathbf{H}.$ is embedded in $\mathbf{F}.$ in such a way that $\widetilde{\mathbf{F}.} = \mathbf{F}./\mathbf{H}.$ is again a free resolution of M. The matrices of differential maps of $\mathbf{F}./\mathbf{H}.$ are different with those of $(\mathbf{F}., \delta.)$, just for $j-1, j, j+1$.

If we show the matrices of new resolution by $\widetilde{\mathcal{M}}_i$, then delete the first column of $\mathcal{M}_{j-1}$ to obtain $\widetilde{\mathcal{M}}_{j-1}$. Delete the first column and first row of $\mathcal{M}_j$ to get $\widetilde{\mathcal{M}}_j$. Finally, delete the first row of $\mathcal{M}_{j+1}$ to obtain $\widetilde{\mathcal{M}}_{j+1}$.

Continuing in this way, we eventually reach a minimal free resolution.

3. **Rigidity of resolutions**

In this section, we present the main results of the paper. Our interest is to find some conditions such that the tail of a resolution $(\mathbf{F}., \delta.)$ of a filtered module M has a rigid behavior with respect to the Betti numbers of $\mathrm{gr}_{\mathbb{M}}(M)$. We denote by $\mu(\ )$ the minimal number of generators of a module over a local ring (or the minimal number of generators of a graded module over the polynomial ring).

Let $(\mathbf{G}., d.)$ be the minimal free resolution of a graded module M over a polynomial ring (or a module M over a local ring), Set.

$$\mathrm{Syz}_i(M) = \ker(d_{i-1}).$$

Let N be a graded P-module. For $d \in \mathbf{Z}$, write $N_{<d>}$ for the submodule of N which is generated by all homogeneous elements of N with degree d. In the graded case, we may also define the graded Betti numbers; i.e.,

$$\beta_{ij}(N) := \dim_k Tor_i^P(k, N)_j.$$

For the following definition and more information on the topic, see [8, 13, 4].

Definition 3.1. Let N be a graded P-module.

(i) Let $d \in \mathbf{Z}$. Then, N has a d-linear resolution if $\beta_{ij} = 0$, for $j \neq d+i$.

(ii) N is componentwise linear if for all integers d the module $N_{<d>}$ has a d-linear resolution.

Theorem 3.2. ([14], Theorem 3.1.) *Let M be a finitely generated filtered module over a regular local ring $(R, \mathfrak{n})$. Assume:*

(1) $\mu(M) = \mu(\mathrm{gr}_{\mathbb{M}}(M))$.

(2) $\mathrm{gr}_{\mathbb{M}}(M)$ *is a componentwise linear P-module.*

Then, $\beta_i(M) = \beta_i(\mathrm{gr}_{\mathbb{M}}(M))$, for each $i \geq 0$.

What follows is the extension of the above result.

Theorem 3.3. *Let M be a finitely generated filtered module over a regular local ring $(R, \mathfrak{n})$. Assume:*

(1) *for some $i \geq 0$, $\beta_i(M) = \beta_i(\mathrm{gr}_{\mathbb{M}}(M))$.*

(2) $\mathrm{Syz}_i(\mathrm{gr}_{\mathbb{M}}(M))$ *is a componentwise linear P-module.*

Then, $\beta_l(M) = \beta_l(\mathrm{gr}_{\mathbb{M}(\mathrm{M})})$, for each $l \geq i$.

Proof. Denote $\mathrm{gr}_{\mathbb{M}}(M) = M^*$ and let

$$\mathbf{G}.\ 0 \to G_h \overset{d_h}{\to} \cdots \overset{d_{i+1}}{\to} G_i \overset{d_i}{\to} G_{i-1} \to \cdots \to G_0 \overset{d_0}{\to} M^* \to 0$$

be the minimal free resolution of M^*. It is clear that $0 \to G_h \overset{d_h}{\to} \cdots \overset{d_{i+1}}{\to} G_i \overset{d_i}{\to} N^*$ is the minimal free resolution of $N^* = \mathrm{Syz}_i(M^*)$. By Theorem

2.1, we can build up a free resolution $(\mathbf{F}., \delta.)$ for M:

$$0 \to F_h \overset{\delta_h}{\to} \cdots \overset{\delta_{i+1}}{\to} F_i \overset{\delta_i}{\to} F_{i-1} \to \cdots \overset{\delta_1}{\to} F_0 \to M \to 0.$$

Let $N = \ker(\delta_{i-1})$. By construction, $N^* = \mathrm{gr}_{\mathbb{F}_{i-1}}(N)$ and clearly $0 \to F_h \overset{\delta_h}{\to} \cdots \overset{\delta_{i+1}}{\to} F_i \overset{\delta_i}{\to} N$ is a free resolution of N.

By Remark 2.2, we construct the minimal free resolution of M with an inductive process. Let c_1 be the smallest integer less than $i+1$ such that $\mathcal{M}_{c_1}$ has invertible entries and follow the process of Remark 2.2 for $\mathcal{M}_{c_1}$. Continuing this way, the biggest integer that we can choose is $i-1$, because $\beta_i(M) = \beta_i(\mathrm{gr}_{\mathbb{M}}(M))$. In each step k, the matrices of differential maps are different from the ones from the previous step, just for $c_k - 1, c_k$ and $c_k + 1$. So, we get a free resolution $(\widetilde{\mathbf{F}.}, \widetilde{\delta.})$ for M such that there is no invertible entry in the matrices of differential maps for $c \leq i+1$, and moreover the matrices of differential maps are the same as those of $(\mathbf{F}., \delta.)$, for $l > i$.

Since there is no invertible entry in the matrix of the differential map $\delta_{i+1} = \widetilde{\delta_{i+1}}$, we have $\mu(N) = \mu(N^*) = \beta_i(M^*)$. So, by Theorem 3.2, $\beta_j(N) = \beta_j(N^*)$, for each j, which means that there is no invertible entry in the matrices of differential maps $\delta_l = \widetilde{\delta_l}$, for $l \geq i+1$. Therefore, $(\widetilde{\mathbf{F}.}, \widetilde{\delta.})$ is the minimal free resolution of M and $\beta_l(M) = \beta_l(M^*)$, for $l \geq i$. □

An immediate application of the above result is that under the assumption of Theorem 3.3,

$$\mathrm{depth}(M) = \mathrm{depth}(\mathrm{gr}_{\mathbb{M}}(M)) \text{ and } \mathrm{pd}(M) = \mathrm{pd}(\mathrm{gr}_{\mathbb{M}}(M)).$$

The above proof shows that we can find more information about M under the assumption of Theorem 3.3. To denote them, let us remind some notations.

Let $(\mathbf{F}.,\delta.)$ be a minimal R-free resolution of a module M. For all integer i, we have

$$\mathrm{gr}_{\mathfrak{n}}(F_i)(-i) = \bigoplus_{j \geq i} \mathfrak{n}^{j-i} F_i / \mathfrak{n}^{j+1-i} F_i \simeq \mathrm{gr}_{\mathfrak{n}}(\mathrm{R})^{\beta_i(M)}(-i)$$

Following this construction due to Eisenbud, etal. [6], the differential maps δ_i induces a bihomogeneous map,

$$\delta_{i+1}^{lin} : \mathrm{gr}_{\mathfrak{n}}(F_{i+1})(-i-1) \to \mathrm{gr}_{\mathfrak{n}}(F_i)(-i),$$

which can be described by matrices of *linear forms*. Precisely the matrices, say $\mathcal{M}_{i+1}^{lin}$, are obtained by replacing in $\mathcal{M}_{i+1}$, the matrix of δ_{i+1}, all entries of valuation > 1 by 0 and by replacing all the entries of valuation one by their initial forms with respect to the $\mathfrak{n}$-adic filtration. The minimality of $(\mathbf{F}., \delta.)$ ensures that the maps $\{\delta_i^{lin}\}$ are well-defined and form a complex homomorphism denoted by $lin^R(\mathbf{F}.)$, which is not necessarily exact. It is called the *linear part of the resolution.* For the construction of this complex and related results, see [6], as well as [9, 13]. Röemer introduced a measure for the lack of the exactness and defined

$$ld(M) := inf\{j : H_i(lin^R(\mathbf{F}.)) = 0 \text{ for } i \geq j + 1\}.$$

In particular, $ld(M) = 0$ if and only if $lin^R(\mathbf{F}.)$ is exact.

Definition 3.4. A finitely generated R-module M is said to be Koszul if $lin^R(\mathbf{F}.)$ is acyclic, where $\mathbf{F}.$ is the minimal free resolution of M.

Röemer proved in [13, Theorem 3.2.8] that, for graded modules, when $ld(M) = 0$ (meaning Koszul modules), they are equivalently componentwise linear. Herzog and Iyengar proved in [9, Proposition 1.5] that to be *Koszul* is equivalent to the fact that $lin^R(\mathbf{F}.)$ is the minimal free resolution of $\mathrm{gr}_{\mathfrak{n}}(M) = \bigoplus_{\mathrm{j}} \mathfrak{n}^{\mathrm{j}} M/\mathfrak{n}^{\mathrm{j}+1} M$. In particular, this is the case if and only if $\mathrm{gr}_{\mathfrak{n}}(M)$ has a linear resolution as a $\mathrm{gr}_{\mathfrak{n}}(R)$-module.

The following corollary is an immediate consequence of the definition.

Corollary 3.5. *Let M be a finitely generated $R-$module . The following facts hold.*
(i) $ld(M) = \min\{i : \mathrm{Syz}_i(M)$ is a Koszul module$\}$.
(ii) If M is a Koszul module, Then so are all its syzygy modules.

Supposing a filtered $R-$module M, in [14, Theorem 3.6] it is proved that under the assumptions of Theorem 3.2, M itself is Koszul. In a very special situation, using this result we can check the Koszulness of a module by means of a general $\mathfrak{n}$-stable filtration (not necessarily the n-adic filtration). We apply this theory to show that under the assumptions of Theorem 3.3, $\mathrm{Syz}_i(M)$ is a Koszul moldule.

Corollary 3.6. *Let M be a finitely generated filtered module over a regular local ring $(R, \mathfrak{n})$. Assume:*

(1) *for some $i \geq 0$, $\beta_i(M) = \beta_i(\mathrm{gr}_{\mathbb{M}}(\mathrm{M}))$.*

(2) $\mathrm{Syz}_i(\mathrm{gr}_{\mathbb{M}}(M))$ *is a componentwise linear* P*-module.*

Then, $\mathrm{Syz}_i(M)$ *is a Koszul moldule and* $ld(M) \leq i$.

Proof. Let $(\mathbf{G}., d.)$ be the minimal free resolution of $\mathrm{gr}_{\mathbb{M}}(M)$ and $(\mathbf{F}., \delta.)$ be a free resolution of M as described in Theorem 2.1.

Let $N = \ker(\delta_{i-1})$ and $N^* = \ker(d_{i-1})$. By construction, $N^* = \mathrm{gr}_{\mathbb{F}_{i-1}}(N)$ and as we have already seen in the proof of Theorem 3.3, $\mu(N) = \mu(N^*)$. So, by [14, Theorem 3.6], N is Koszul. Notice that we have also shown that $\mathrm{Syz}_i(M)$ has a minimal free resolution with the same differential maps as N. So, $\mathrm{Syz}_i(M)$ is a Koszul module and by Corollary 3.5, $ld(M) \leq i$. □

The following is the application of our results to the classical case, a local ring $(A, \mathfrak{m})$ filtered by the $\mathfrak{m}$-adic filtration. Let I be an ideal of a regular local ring $(R, \mathfrak{n})$ and $A = R/I$. So, $\mathrm{gr}_{\mathfrak{m}}(\mathrm{A}) = \mathrm{P}/\mathrm{I}^*$, where $\mathfrak{m} = \mathfrak{n}/I$ and I^* is the graded ideal generated by the initial forms of I. We recall that if we apply the general theory on filtered modules to $M = I$ and $\mathbb{M} = \{\mathfrak{n}^p \cap I\}$, we obtain $\mathrm{gr}_{\mathbb{M}}(M) = \mathrm{I}^*$. So, by Theorem 2.1, we have

$$\beta_i(R/I) \leq \beta_i(P/I^*). \tag{3.1}$$

Corollary 3.7. *Let* I *be an ideal of a regular local ring* $(R, \mathfrak{n})$*. Assume:*

(1) *for some* $i \geq 0$, $\beta_i(I) = \beta_i(I^*)$

(2) $\mathrm{Syz}_i(I^*)$ *is a componentwise linear* P*-module.*

Then, $\beta_j(I) = \beta_j(I^*)$*, for each* $j \geq i$, $\mathrm{Syz}_i(I)$ *is a Koszul module and* $ld(I) \leq i$.

Example 3.8. Let $R = K[[x_1, \cdots, x_6]]$. Let

$$S = \{x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5 \leq 6, (i_1, i_2, i_3, i_4, i_5) \neq (1, 2, 3, 4, 6)\}.$$

If $I = \langle x_1^2 + x_1x_3x_4x_6, x_1x_2 + x_3x_4x_6, x_3^2\rangle + \langle S\rangle$, then $\mu(I^*) = \mu(I)+1$. Using SINGULAR [7], the minimal free resolution of I^* and I respectively are:

$$0 \to P^{70}(-10) \to P^{381}(-9) \to P(-7) \oplus P^{834}(-8) \to P(-5) \oplus P^{3}(-6) \oplus P^{918}(-7) \to P(-3) \oplus P^{2}(-4) \oplus P^{3}(-5) \oplus P^{508}(-6) \to P^{3}(-2) \oplus P(-4) \oplus P^{113}(-5) \to I^*$$

and

$$0 \to R^{70} \to R^{381} \to R^{835} \to R^{922} \to R^{513} \to R^{116} \to I.$$

So, $\beta_2(I^*) = \beta_2(I)$. $N = \mathrm{Syz}_2(I^*)$ is componentwise linear, because(1) clearly $N_{\langle 5\rangle}$ has linear resolution, (2) the minimal free resolution of $N_{\langle 6\rangle}$ is

$$0 \to P(-11) \to P^{6}(-10) \to P^{15}(-9) \to P^{20}(-8) \to P^{16}(-7) \to P^{9}(-6) \to N_{\langle 6\rangle},$$

and (3) it is easy to check that $N_{\langle 7\rangle}$ also has linear resolution (See [13, lemmas 3.2.2 and 3.2.4]). So, the conditions of Corollary 3.7 hold and $\beta_i(I^*), = \beta_i(I)$ for $i \geq 2$.

Discussion 3.9. The inequality (3.1) suggests an upper bound coming from the homogeneous context. Assume the residue field k of characteristic 0 and let J be a graded ideal of the polynomial ring P. We have a monomial ideal canonically attached to J : the *generic initial ideal* with respect to the revlex order. We denote

$$\mathrm{Gin}(I) := \mathrm{Gin}(I^*).$$

Notice that it is proved in [2], if $R = k[[x_1, \cdots, x_n]]$, then one can define an anti-degree-compatible ordering on the terms of R such that the initial ideal of I, after performing a 'generic change' of coordinates, is a monomial ideal which coincides with $\mathrm{Gin}(I^*)$.This monomial ideal has the same Hilbert function as R/I. Indeed,

$$HF_A(n) = HF_{\mathrm{gr}_{\mathfrak{m}}(\mathrm{A})}(n) = HF_{P/I^*}(n) = HF_{P/\mathrm{Gin}(I)}(n).$$

Nevertheless, since $\beta_i(P/I^*) \leq \beta_i(P/\mathrm{Gin}(I^*))$, then we have

$$\beta_i(R/I) \leq \beta_i(P/I^*) \leq \beta_i(P/\mathrm{Gin}(I)) \qquad (3.2)$$

for every $i \geq 0$.

It is interesting to compare Corollary 3.7 with Conca, etal.'s main result in [4], which says that if J is a homogeneous ideal of the polynomial ring P and $\beta_i(J) = \beta_i(\mathrm{Gin}(J))$ for some i, then $\beta_j(J) = \beta_j(\mathrm{Gin}(J))$, for all $j \geq i$.

Combining the above result with Corollary 3.7 lead us to the following conjecture.

Conjecture 1: *Assume char$(K) = 0$, and let $I \subset R$ be an ideal. Suppose that $\beta_i(I) = \beta_i(\mathrm{Gin}(I))$, for some i. Then,*

$$\beta_k(I) = \beta_k(\mathrm{Gin}(I)) \quad \textit{for all,} \quad k \geq i.$$

To examine this conjecture, it is enough to study the following problem.

Problem 2: *Let J be a homogeneous ideal of the polynomial ring P and $\beta_i(J) = \beta_i(\mathrm{Gin}(J))$, for some i Then, $\mathrm{Syz}_i(J)$ is a componentwise linear module.*

Note that $\beta_i(I) = \beta_i(\mathrm{Gin}(I))$ and using inequality (3.2) we have, in particular, $\beta_i(I^*) = \beta_i(\mathrm{Gin}(I^*))$. So, the conjecture can be followed by Corollary 3.7 and the above problem.

If J is a homogeneous ideal, $\mathrm{Gin}(J)$ is a componentwise linear ideal, and by corollary 3.5, all its syzygy modules are componentwise linear. On the other hand, from the assumption $\beta_i(J) = \beta_i(\mathrm{Gin}(J))$, we can also conclude

$$\beta_{lk}(J) = \beta_{lk}(\mathrm{Gin}(J)),$$

for $l \geq i$ and each k. Thus, the minimal free resolution of $N = \mathrm{Syz}_i(J)$ has the important properties of componentwise linear modules described in [14, Proposition 2.2 and Remak 2.3]. This fact strengthens our given conjecture.

Our next examples are related to Problem 1.

Let J be a graded ideal of the polynomial ring P and suppose that the residue field K is of characteristic zero. Then, $\mu(J) = \mu(\mathrm{Gin}(J))$ if and only if J is a componentwise linear ideal (see [1]). The same result does not hold if we compare the $i-$th Betti numbers for $i > 0$.

Example 3.10. Let $P = K[x_1, \cdots, x_6]$ and

$$J = \langle x_1^2, x_2^2, x_2^2x_3^3, x_1x_3^3, x_2x_3^5x_4, x_2x_3^5x_5, x_2x_3^5x_6, x_2x_3^4x_6 \rangle.$$

Using CoCoA [3], the minimal free resolution of J and $\mathrm{Gin}(J)$ are respectively

$0 \to P(-11) \to P^5(-10) \to P(-7) \oplus P(-8) \oplus P^9(-9) \to P(-4) \oplus P(-5) \oplus P(-6) \oplus P^2(-7) \oplus P^7(-8) \to P^2(-2) \oplus P(-4) \oplus P(-6) \oplus P^2(-7) \to J$

and

$0 \to P(-11) \to P^5(-10) \to P(-6) \oplus P(-7) \oplus P(-8) \oplus P^9(-9) \to P(-3) \oplus P(-4) \oplus P^2(-5) \oplus P^2(-6) \oplus P^2(-7) \oplus P^7(-8) \to P^2(-2) \oplus P(-3) \oplus P(-4) \oplus P(-5) \oplus P(-6) \oplus P^2(-7) \to \mathrm{Gin}(J)$.

It is easy to see that $\mathrm{Syz}_2(J)$ is a componentwise linear module but $\beta_2(J) \neq \beta_2(\mathrm{Gin}(J))$.

For a given graded ideal J and positive integer d, let $J_{\leq d}$ be the ideal generated by homogeneous generators of J whose degrees are less than or equal to d. It is easy to see that $\mu(J) = \mu(\mathrm{Gin}(J))$ implies that for each d, $\mu(\mathrm{Gin}(J)_{\leq d}) = \mu(\mathrm{Gin}(J_{\leq d}))$. the next example shows that the same result does not hold if we compare the $i-$th Betti numbers for $i > 1$.

Example 3.11. Let $P = K[x_1, \cdots, x_5]$ and $I = \langle x_1^2, x_1x_2, x_1x_3, x_1x_4, x_5^2\rangle$. Using CoCoA[3], the minimal free resolution of J and $\mathrm{Gin}(J)$ are respectively

$$\begin{aligned} 0 &\to P(-7) \to P(-5) \oplus P^4(-6) \to P^4(-4) \oplus P^6(-5) \\ &\to P^6(-3) \oplus P^4(-4) \to P^5(-2) \end{aligned}$$

and

$$\begin{aligned} 0 &\to P(-7) \to P(-5) \oplus P^4(-6) \to P^4(-4) \oplus P^6(-5) \\ &\to P^7(-3) \oplus P^4(-4) \to P^5(-2) \oplus P(-3). \end{aligned}$$

So, $\beta_2(J) = \beta_2(\mathrm{Gin}(J))$, but $\beta_2(\mathrm{Gin}(J)_{\leq 2}) \neq \beta_2(\mathrm{Gin}(J_{\leq 2}))$.

The above examples show that Problem 2 is not simple and it needs a more careful study.

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