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Author(s):

S.H. Jafari

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CHARACTERIZATION OF FINITE p-GROUPS BY THE ORDER OF THEIR SCHUR MULTIPLIERS ($t(G) = 7$)

S.H. JAFARI

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ABSTRACT. Let G be a finite p-group of order p^n and $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, where $\mathcal{M}(G)$ is the Schur multiplier of G and $t(G)$ is a nonnegative integer. The classification of such groups G is already known for $t(G) \leqslant 6$. This paper extends the classification to $t(G) = 7$.
Keywords: Schur multiplier, nonabelian tensor square, p-Group.
MSC(2010): Primary: 20D99; Secondary: 19C09.

1. Introduction

The Schur multiplier of groups was introduced by Schur in 1904 [22] to study projective representations of groups. Since then, several works have been done to determine the structure of the Schur multiplier for some classes of groups. Also, one of the main themes of the group theoretical part of research on the Schur multipliers of groups has been to determine all p-groups according to their multipliers.

By a well-known result of Green [9] we know for any group G of order p^n for $n \geqslant 1$, there exists an integer $t(G) \geqslant 0$ such that the Schur multiplier $\mathcal{M}(G)$ has order $p^{\frac{1}{2}n(n-1)-t(G)}$. Those finite p-groups with $t(G) = 0$ or 1 have been classified by Berkovich [1]. The classification has been extended to $t(G) = 2$ by Zhou [24], and to $t(G) = 3$ by Ellis [6]. Later the characterization continued for $t(G) = 4$ or 5 by Khamseh *et al.* [17] (Also by Niroomand [19, 20]), and finally for $t(G) = 6$ by the author [12]. In this article the nonabelian tensor square of groups and the Lazard correspondence is applied to extend the classification to $t(G) = 7$.

Notation:

D_8, Q_8 = Dihedral or Quaternion group of order 8 respectively;

$E^1_{p^3}, E^2_{p^3}$ = Extra special group of order p^3 with exponent p or p^2 respectively, for p odd;

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$E^d_{2^{2m+1}}, E^1_{p^{2m+1}}$ = Central product of m copies of D_8's or $E^1_{p^3}$'s;
$E^q_{2^{2m+1}}$ = Central product of $(m-1)$ copies of D_8's and a single Q_8;
$E^2_{p^{2m+1}}$ = Central product of $(m-1)$ copies of $E^1_{p^3}$'s and a single $E^2_{p^3}$;
$GE^r_{p^{2+r}}$ = Generalized extra special group of order p^{2+r} with exponent p^r and presentation
$\langle x, y, z \mid x^p = y^p = z^{p^r} = 1, [x, y] = z^{p^{r-1}}, [y, z] = [x, z] = 1\rangle$;
$GE^r_{p^{2m+r}}$ = Central product of m copies of $GE^r_{p^{2+r}}$'s ;
T_i = 2-groups as described in Table 1;
Φ_i = Isoclinic families of groups of order p^n, ($n \leqslant 6,\ p \neq 2$) given in [14].

Main Theorem. *Let G be a group of order p^n and $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$. Then $t(G) = 7$ if and only if G is isomorphic with one of the following groups:*

$n = 5$: $C_{p^3} \times (C_p)^2,\ D_8 \times C_4,\ T_6,\ T_7,\ T_{21},\ T_{22},\ T_{23},\ T_{24},\ \Phi_2(22) \times C_p,\ \Phi_2(221)c,\ \Phi_2(221)d,\ \Phi_3(211)a \times C_p,\ \Phi_3(211)b_r \times C_p,\ \Phi_3(2111)c,\ \Phi_4(2111)a,\ \Phi_4(2111)b,\ \Phi_4(2111)c,\ \Phi_6(1^5)$ *for* $p > 3$, $\Phi_7(2111)a,\ \Phi_7(2111)b_r,\ \Phi_7(2111)c,\ \Phi_9(1^5),\ \Phi_{10}(1^5)$ *for* $p > 3$;
$n = 6$: $E^1_{p^3} \times E^1_{p^3},\ \Phi_{11}(1^6),\ \Phi_{13}(1^6),\ \Phi_{15}(1^6)$;
$n = 7$: $Q_8 \times (C_2)^4,\ T_4 \times (C_2)^3,\ E^d_{2^5} \times (C_2)^2,\ E^q_{2^5} \times (C_2)^2,\ E^d_{2^7},\ E^q_{2^7},\ E^1_{p^5} \times (C_p)^2,\ E^2_{p^5} \times (C_p)^2,\ E^2_{p^3} \times (C_p)^4,\ \Phi_2(211)b \times (C_p)^3,\ GE^2_{p^6} \times C_p,\ E^1_{p^7},\ E^2_{p^7}$;
$n = 8$: $C_{p^2} \times (C_p)^6,\ D_8 \times (C_2)^5$;
$n = 9$: $E^1_{p^3} \times (C_p)^6$.

2. Preliminaries

Let G and H be groups acting upon each other in a compatible way, that is,

$$^{^gh}g' = {}^g({}^h({}^{g^{-1}}g')) \text{ and } {}^{^hg}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for $g, g' \in G$ and $h, h' \in H$, and acting upon themselves by conjugation. The nonabelian tensor product $G \otimes H$ of G and H is a group generated by the symbols $g \otimes h$, where $g \in G$ and $h \in H$, with defining relations

$$gg' \otimes h = ({}^gg' \otimes {}^gh)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)\ ({}^hg \otimes {}^hh')$$

where $g, g' \in G$ and $h, h' \in H$. In the case $G = H$ and the actions are conjugation, $G \otimes G$ is called nonabelian tensor square of G, this concept of the nonabelian tensor product of groups was introduced by Brown and Loday in [3].

It is shown in [2] that for all $g, g', h, h' \in G$,

$$[g, h] \otimes h' = (g \otimes h)\ {}^{h'}(g \otimes h)^{-1} \quad \text{and} \quad g' \otimes [g, h] = {}^{g'}(g \otimes h)(g \otimes h)^{-1}.$$

Also by [2, Proposition 9], given a central subgroup Z of any group G, there is an exact sequence

$$(G \otimes Z) \times (Z \otimes G) \xrightarrow{\iota} G \otimes G \longrightarrow G/Z \otimes G/Z \longrightarrow 1, \tag{2.1}$$

where $\iota(g \otimes z, z' \otimes g') = (g \otimes z)(z' \otimes g')$ for all $z, z' \in Z$ and $g, g' \in G$. Besides, if $Z \subseteq G'$, the above equations imply that the sequence

$$Z \otimes G \longrightarrow G \otimes G \longrightarrow G/Z \otimes G/Z \longrightarrow 1 \tag{2.2}$$

is exact.

In the next theorem, the order of the nonabelian tensor square of a group G is expressed in terms of the orders of G, $\mathcal{M}(G)$ and $\mathcal{M}(G^{ab})$, where G^{ab} is the derived factor of G.

Theorem 2.1 ([13, Lemma 2.3])**.** *Let G be a d-generator finite p-group.*

(i) *If $p > 2$, then $|G \otimes G| = |G||\mathcal{M}(G)||\mathcal{M}(G^{ab})|$.*

(ii) *If $p = 2$ and $G^{ab} \cong \prod_{i=1}^{d} C_{2^{e_i}}$ where $1 \leqslant e_i \leqslant e_j$, for all i, j with $1 \leqslant i \leqslant j \leqslant d$, then $|G \otimes G| = 2^k|G||\mathcal{M}(G)||\mathcal{M}(G^{ab})|$, for some nonnegative integer $k \leqslant d$.*

We recall a bound for the order of the Schur multiplier of finite p-groups.

Theorem 2.2 ([7])**.** *Let G be a d-generator group of order p^n. Suppose that G^{ab} has order p^m and exponent p^e, and that the central quotient $G/Z(G)$ is a δ-generator group. Then*

$$|\mathcal{M}(G)| \leqslant p^{d(m-e)/2+(\delta-1)(n-m)-max\{0,\delta-2\}}.$$

Finally we recall the Lazard correspondence which is used in proof of main theorem. It has been known since the 1950s that the Baker–Campbell–Hausdorff formula gives an isomorphism between the category of nilpotent Lie rings with order p^n and nilpotency class c and the category of finite p-groups with order p^n and nilpotency class c, provided $p > c$. This is known as the Lazard correspondence. We will apply an effective version of it [4] to transform questions on order of the Schur multipliers of p-groups (of class $< p$) to the same questions on Lie rings.

3. Proof of the main theorem

Throughout the rest of the paper we always assume that G is a d-generator group of order p^n for $n \geqslant 1$ with

$$|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)} \tag{3.1}$$

Note that as $\frac{1}{2}n(n-1) - t(G) \geqslant 0$, the equality $t(G) = 7$ holds only for $n \geqslant 5$.

We will also assume that $G/Z(G)$ is a δ-generator group, $|G'| = p^c$ and the Frattini subgroup $\Phi(G)$ is of order p^a so that $a = n - d$. Ellis [6] established the following inequalities:

$$2(t(G) - c(d + 1 - \delta)) \geqslant a^2 - a \text{ and } a \geqslant c \geqslant 0,\ d \geqslant \delta \tag{3.2}$$

$$2(t(G) - c) \geqslant a^2 - a \tag{3.3}$$

Lemma 3.1. *If $t(G) = 7$, then the possible values of (c, a) are $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$ or $(3,3)$.*

Proof. It is readily obtained by inequality (3.3). Also as

$$c \leqslant a \leqslant \sqrt{2t(G) - c}$$

which is given in [7], so that the possibilities with $a = 4$ do not occur. □

We will prove the main theorem by a series of propositions, each with a statement detailing the groups satisfying the possible values of (c, a).

Proposition 3.2. *If $c = 0$, then $G \cong C_{p^3} \times (C_p)^2$ or $C_{p^2} \times (C_p)^6$.*

Proof. Suppose that $(c, a) = (0, 0)$. Then $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)}$. Therefore $t(G) = 0$. If $(c, a) = (0, 1)$, then $n \geqslant 2$ and $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)}$. So equation (3.1) holds if and only if $n = 8$ and $G \cong C_{p^2} \times (C_p)^6$, similarly for the case $(0, 2)$ we observe that $t(G) = 7$ if and only if $G \cong C_{p^3} \times (C_p)^2$. Also (3.1) does not hold when $(c, a) = (0, 3)$. □

Proposition 3.3. *If $c = 1$, then G is isomorphic with one of the following groups:*
$D_8 \times C_4$, T_6, T_7, T_{24}, $\Phi_2(22) \times C_p$, $\Phi_2(221)c$, $\Phi_2(221)d$, $Q_8 \times (C_2)^4$, $T_4 \times (C_2)^3$, $E^d_{2^5} \times (C_2)^2$, $E^q_{2^5} \times (C_2)^2$, $E^d_{2^7}$, $E^q_{2^7}$, $E^1_{p^5} \times (C_p)^2$, $E^2_{p^5} \times (C_p)^2$, $E^2_{p^3} \times (C_p)^4$, $\Phi_2(211)b \times (C_p)^3$, $GE^2_{p^6} \times C_p$, $E^1_{p^7}$, $E^2_{p^7}$, $D_8 \times (C_2)^5$, $E^1_{p^3} \times (C_p)^6$.

Proof. Assume $(c, a) = (1, 1)$. By a result of [18, Lemma 2.1] we have $t(G) = 7$ if and only if G is isomorphic with the groups $E^1_{p^7}$, $E^2_{p^7}$, $E^d_{2^7}$, $E^q_{2^7}$, $Q_8 \times (C_2)^4$, $T_4 \times (C_2)^3$, $E^d_{2^5} \times (C_2)^2$, $E^q_{2^5} \times (C_2)^2$, $E^1_{p^5} \times (C_p)^2$, $E^2_{p^5} \times (C_p)^2$, $E^2_{p^3} \times (C_p)^4$, $\Phi_2(211)b \times (C_p)^3$, $GE^2_{p^6} \times C_p$, $D_8 \times (C_2)^5$ or $E^1_{p^3} \times (C_p)^6$.

Suppose $(c, a) = (1, 2)$. Then $G^{ab} \cong C_{p^2} \times (C_p)^{n-3}$ and Theorem 2.2 together with our hypothesis imply that $n = 5$. According to the main results of [13], let $|G \otimes G| = p^{n(n-1)-l}$ for some non-negative integer l. So it follows from Theorem 2.1 and [13, Theorem 3.1] that $l = 9$ when p is odd, and $6 \leqslant l \leqslant 9$ when p is even. Hence $G \cong T_6$, T_7, $D_8 \times C_4$, $\Phi_2(22) \times C_p$ or $\Phi_2(221)c$.

Suppose $(c, a) = (1, 3)$. Then $G^{ab} \cong (C_{p^2})^2 \times (C_p)^{n-5}$ and by Theorem 2.2 we have $n = 5$. Therefore Theorem 2.1 and the Remark on [12, p. 541] imply that $G = \Phi_2(221)d$ for $p > 2$ and $G = T_{24}$ for $p = 2$. □

Proposition 3.4. *If $(c, a) = (2, 2)$, then G is isomorphic with one of the following groups:*
T_{21}, T_{22}, $\Phi_3(211)a \times C_p$, $\Phi_3(211)b_r \times C_p$, $\Phi_3(2111)c$, $\Phi_4(2111)a$, $\Phi_4(2111)b$, $\Phi_4(2111)c$, $\Phi_7(2111)a$, $\Phi_7(2111)b_r$, $\Phi_7(2111)c$, $E^1_{p^3} \times E^1_{p^3}$, $\Phi_{13}(1^6)$, $\Phi_{15}(1^6)$.

Proof. Suppose $(c, a) = (2, 2)$. Then inequality (3.2) forces $d = \delta$, $\delta + 1$ or $\delta + 2$. We will proceed by investigating each of these possibilities.

Case 1. Suppose $d = \delta$. Then it follows from Theorem 2.2 that $n \leqslant 8$.

Suppose $n = 8$. Then G must have exponent p. We claim that there is no such group that satisfying $t(G) = 7$. By sequence (2.2), the upper bound of $|G \otimes G|$ given in [11, Theorem 2.3], and by Theorem 2.1 it is sufficient to find a central subgroup Z of order p such that $|Im(Z \otimes G \to G \otimes G)| \leqslant p^5$.

We have by [23] that our group G must be a descendant of the following algebras:

$$\langle a, b \mid class\ 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle \oplus \langle f\rangle,\ \langle a, b, c, d, e, f \mid class\ 1\rangle$$

Let G have class three, and let the 6 generators of G be a, b, c, d, e, f. Assume that modulo the $\gamma_3(G)$, the derived subgroup is generated by $[b, a]$ and that c, d, e, f are central.

Then we can assume that $\gamma_3(G)$ is generated by $[b, a, a]$ and that $[b, a, b] = 1$. All commutators of c, d, e, f with any of a, b, c, d, e, f are powers of $[b, a, a]$. Replacing c, d, e, f by elements of the form $c[b, a]^k, d[b, a]^l, e[b, a]^m, f[b, a]^n$ for suitable k, l, m, n (as necessary), we can assume that $[c, b] = [d, b] = [e, b] = [f, b] = 1$.

So c, d, e, f generate a group which is either abelian, or has class 2 with derived subgroup $\langle b, a, b\rangle$. In other words $\langle c, d, e, f\rangle$ is either abelian or extra-special. Thus c, d, e, f satisfy one of the following three sets of relations:

(i) c, d, e, f all commute,
(ii) $[c, d] = [b, a, a], [c, e] = [c, f] = [d, e] = [d, f] = [e, f] = 1$,
(iii) $[c, d] = [e, f] = [b, a, a], [c, e] = [c, f] = [d, e] = [d, f] = 1$.

In case (i), we can assume that c, d, e, f all commute with a, or that $[c, a] = [b, a, a]$ and $[d, a] = [e, a] = [f, a] = 1$.

In case (ii), replacing a by $ac^r d^s$ for suitable r, s we can assume that $[c, a] = [d, a] = 1$. We either have $[e, a] = [f, a] = 1$, or $[b, e] = [b, a, a], [f, a] = 1$.

In case (iii), replacing a by $ac^r d^s e^t f^u$ for suitable r, s, t, u we can assume that $[c, a] = [d, a] = [e, a] = [f, a] = 1$.

Therefore for $p > 3$, just the group

$$\begin{aligned}\langle a, b, c, d, e, f \mid [b, a, b] &= [a, c] = [a, d] = [a, e] = [a, f] = [b, c] = [b, d] = [b, e]\\ &= [b, f] = [c, e] = [c, f] = [d, e] = [d, f] = [e, f] = 1, [c, d] = [e, f] = [b, a, a]\rangle\end{aligned}$$

satisfies $Z(G) \leqslant G'$. It has a PC presentation of the form

$$\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \mid [a_2, a_1] = a_7,\ [a_3, a_4] = [a_5, a_6] = [a_7, a_1] = a_8\rangle$$

in which all other commutators are trivial. It is readily obtained that $a_8 \otimes a_5 = 1$. So one can choose $Z = \langle a_8\rangle$. Note that as a_8 is central it follows that the relation $[a_7, a_1] = a_8$ holds if one turns the Lie ring presentation into its Lazard correspondence group presentation. Also if two elements of a Lie ring commute, then so too do the Lazard corresponding elements in the Lazard correspondence group. If $p = 3$ then there is no group of exponent 3 which satisfies our condition by computing the order of Schur multiplier of groups of order 3^8 in GAP [8]. If $p = 2$, as a group of exponent 2 would be abelian, then it is a contrary to $c = 2$.

As $Z(G) \leqslant G'$ if G has class two, then for an odd prime p it may be generated by PC generators $a_1, a_2, \ldots, a_8$ such that $G' = \langle a_7, a_8 \rangle$ and some of the PC relations must be $[a_2, a_1] = a_7$, and $[a_5, a_2] = [a_5, a_1] = 1$. On the other hand it is readily seen that $a_7 \otimes a_5 = 1$, as desired. As well as the above case, for $p = 2$ the same result follows.

Suppose $n = 7$. Then by the same argument as above we claim that there is no group G that satisfies $t(G) = 7$. So it suffices to find a central subgroup Z of order p such that $|Im(Z \otimes G \to G \otimes G)| \leqslant p^3$. As $d = 5$, it is clear that G must be a descendant of the algebras 5.1, 6.2 or 6.3 given in [21].

Suppose G is a descendant of 5.1, then it has class two and hence the presentation of the given Lie ring is the same as its Lazard correspondence group. It is readily seen that for all cases discussed in [21] except case 4, we have $[a, c] \otimes e = [a, c] \otimes d = 1$. Therefore taking $Z = \langle [a, c] \rangle$ it follows that these groups do not satisfy $t(G) = 7$. For case 4, if G has exponent p then $G \cong H \times C_p$ for some subgroup H. Hence by a simple calculation we must have $t(H) = 5$ contrary to the results of [17]. If the exponent of G is greater than p, then we have $[a, b] \otimes e = [a, b] \otimes a = 1$, $[a, b] \otimes e = [a, b] \otimes b = 1$ or $[a, b] \otimes e = [a, b] \otimes c = 1$, as required.

Suppose G is a descendant of the algebra 6.2. For $p > 3$, as $\gamma_3(G) = \langle a^{p^2} \rangle$ by a similar method discussed above and since either $a^{p^2} \otimes a = a^{p^2} \otimes e = 1$ or $a^{p^2} \otimes d = a^{p^2} \otimes e = 1$, the desired result holds.

Finally, suppose G is a descendant of the algebra 6.3. In any case given in [21] one can show that either $[b, a, a] \otimes c = [b, a, a] \otimes d = 1$ or $[b, a, a] \otimes e = [b, a, a] \otimes d = 1$. So choosing $Z = \langle [b, a, a] \rangle$ establishes our claim. Note that for $p \leqslant 3$ we use GAP to deduce that there is no group G with $t(G) = 7$.

Suppose $n = 6$. Then G must belong to one of the isoclinic families of $\Phi_{12}, \Phi_{13}, \Phi_{15}$ or Φ_{22} given in James's classification of p-groups of order less than or equal to p^6, $p \neq 2$ (see [14]). In the family Φ_{12}, if $G = E^i_{p^3} \times E^i_{p^3}$ for i =1 or 2, then a simple computation implies the group G satisfies our condition if and only if $i = 1$. In all other groups of the family Φ_{12} we have $\gamma_i \otimes \alpha_j = \gamma_i \otimes \beta_j = 1$ when $i \neq j$. Also if $\alpha_1^p = \gamma_1$, then $\gamma_1 \otimes \alpha_1 = 1$ and if $\alpha_1^p = \gamma_2$ or $\alpha_1^p = \gamma_1\gamma_2$, then $\gamma_2 \otimes \beta_2 = 1$. By putting $Z = \langle \gamma_1 \rangle$ or $\langle \gamma_2 \rangle$ in sequence (2.2) and applying the bound given in [11, Theorem 2.3], we have $|G \otimes G| \leqslant p^{19}$. Hence Theorem 2.1 yields $t(G) \geqslant 8$.

For any group G in the family Φ_{13} we have $\beta_2 \otimes \alpha_3 = \beta_2 \otimes \alpha_4 = 1$. Because α_3 commutes with α_2 and α_4, and α_4 commutes with α_1 and α_3. Also for any group G except the group $\Phi_{13}(1^6)$ if either $\beta_2 = \alpha_2^p$ or $\beta_2 = \alpha_3^p$, then $\beta_2 \otimes \alpha_2 = 1$ and if $\beta_2 = \alpha_1^p$ then $\beta_2 \otimes \alpha_1 = 1$. In addition for the groups $\Phi_{13}(2211)e_r$ and $\Phi_{13}(21^4)a$ we observe $\beta_2 \otimes \alpha_2 = \beta_1 \otimes \alpha_3 = \alpha_1^p \otimes \alpha_3 = (\alpha_1 \otimes \alpha_3)^p (\alpha_1 \otimes \beta_2)^{\frac{1}{2}p(p-1)} = \alpha_1 \otimes \alpha_3^p = 1$, and for the group $\Phi_{13}(21^4)d$ similarly $\beta_2 \otimes \alpha_2 = 1$. Thus the central subgroup $Z = \langle \beta_2 \rangle$ together with exact sequence (2.2) and Theorem 2.1 imply that $t(G) \geqslant 8$.

Now we show that the remaining group of this family satisfies our condition. Let $G = \Phi_{13}(1^6)$. We first use the method of [4] to determine a presentation of its Lazard correspondence Lie ring L_p, which has the same order and nilpotency class for $p \geqslant 3$. As the class of G is two, the correspondence Lie ring has the same presentation. Also the Lie ring L_p may be regarded as a Lie algebra over the field $\mathbb{Z}_p$. So computing the multiplier as in Hardy and Stitzinger [10] yields that the dimension of the multiplier of L_p is 8. Also the Schur multipliers of L_p and G are isomorphic by [5]. Therefore $t(G) = 7$ as claimed.

In the family Φ_{15} for the group $\Phi_{15}(2211)a$ set $Z = \langle \beta_1 \rangle$. It is readily seen that $\beta_1 \otimes \alpha_1 = \beta_1 \otimes \alpha_4 = 1$ and $\beta_1 \otimes \alpha_3 = \alpha_1^p \otimes \alpha_3 = (\alpha_1 \otimes \alpha_3)^p (\alpha_1 \otimes \beta_2)^{\frac{1}{2}p(p-1)} = \alpha_1 \otimes \alpha_3^p = 1$. Hence $t(G) \geqslant 8$ by the same argument as above. In group $\Phi_{15}(2211)b_{r,s}$ we have $\beta_2 \otimes \alpha_2 = \beta_2 \otimes \alpha_3 = 1$. Also $\beta_1 \otimes \alpha_1 = (\beta_2 \otimes \alpha_1)^{-r}$, $\beta_1 \otimes \alpha_3 = (\beta_2 \otimes \alpha_3)^{-r}$, $\beta_1 \otimes \alpha_4 = (\beta_2 \otimes \alpha_4)^{-r}$ and $\beta_1 \otimes \alpha_2 \in \langle \beta_2 \otimes \alpha_1 \rangle$. By taking $Z = Z(G)$ we conclude that $|Im(Z \otimes G \to G \otimes G)| \leqslant p^2$ and consequently $t(G) \geqslant 9$. In the groups $\Phi_{15}(2211)c$, $\Phi_{15}(2211)d_r$ and $\Phi_{15}(21^4)$, since $\beta_1 \otimes \alpha_1 = \beta_1 \otimes \alpha_2 = \beta_1 \otimes \alpha_3 = 1$, one can deduce that these groups do not satisfy our condition.

We claim that for the remaining group $G = \Phi_{15}(1^6)$ of this family the condition $t(G) = 7$ holds. As for the group $\Phi_{13}(1^6)$ above, the presentation of G and its Lazard correspondence Lie ring are the same. So again computing the multiplier as in [10] yields that the dimension of the multiplier of L_p is 8, as desired.

Finally in the family Φ_{22} by the same manner we observe that for all groups of this family $\alpha_3 \otimes \alpha_1 = \alpha_3 \otimes \alpha = \alpha_3 \otimes \beta_1 = 1$. GAP shows that for groups of order 2^6 there is no group G with $t(G) = 7$. This result is independent of the values of c, a, d and δ, so we will not repeat it here.

Suppose $n = 5$. Then our group G must belong to one of the isoclinic families of Φ_4 or Φ_7.

In the family Φ_4 for the group $\Phi_4(2111)a$ take $Z = \langle \beta_2 \rangle$, and for the groups $\Phi_4(2111)b$ and $\Phi_4(2111)c$ take $Z = \langle \beta_1 \rangle$. Then obviously $G/Z \cong E^1_{p^3} \times C_p$ and $|Im(Z \otimes G \to G \otimes G)| = 1$. Therefore $G \otimes G \cong G/Z \otimes G/Z$ and we conclude that $t(G) = 7$ for all of these groups. However, for other groups of this family taking either $Z = \langle \beta_1 \rangle$ or $Z = \langle \beta_2 \rangle$ it follows that $|G \otimes G| \leqslant |G/Z \otimes G/Z||Im(Z \otimes G \to G \otimes G)| \leqslant p^{9+1}$. Hence none of the remaining groups satisfy our condition.

Suppose G belongs to Φ_7. Then for the group $G = \Phi_7(2111)a$ it is clear that $\alpha_3 \otimes \alpha = \alpha_3 \otimes \alpha_1 = \alpha_3 \otimes \beta = 1$ and so $G \otimes G \cong G/Z \otimes G/Z$, where $Z = \langle \alpha_3 \rangle$ is the center of G. Thus Theorem 2.1 yields $|\mathcal{M}(G)| = p^3$ and we obtain $t(G) = 7$. For the group $G = \Phi_7(2111)b_r$ we have $\alpha_3^r \otimes \alpha = \alpha_1^p \otimes \alpha = (\alpha_1 \otimes \alpha)^p = \alpha_1 \otimes \alpha^p = 1$, when $p > 3$. Also $\alpha_3 \otimes \alpha_1 = \alpha_3 \otimes \beta = 1$. So again the nonabelian tensor squares of G and central factor group of G are isomorphic and the required result holds.

Note that for $p = 3$, GAP shows $t(G) = 7$. Similarly for the group $\Phi_7(2111)c$ we conclude that $t(G) = 7$ and finally if $G = \Phi_7(1^5)$, then $t(G) = 6$ by results of [12]. When $p = 2$ the groups T_{21} and T_{22}, which are described in Table 1, satisfy our condition by GAP.

Case 2. Suppose $d = \delta + 1$. Then Theorem 2.2 yields $n \leqslant 7$.

Suppose $n = 7$. The calculations in Case 1 show that there is no group with $t(G) = 7$.

Suppose $n = 6$. Then G must belong to one of the families of Φ_4 or Φ_7. In the first family as $d = 4$ put $Z = \langle \beta_2 \rangle$ in the groups $G = \Phi_4(2211)g$, $\Phi_4(2211)h$ and $\Phi_4(2211)i$. Also, we observe that $\beta_2 \otimes \alpha_2 = \beta_2 \otimes \alpha_1 = \beta_2 \otimes \alpha = \beta_2 \otimes \gamma = 1$. Therefore $|Im(Z \otimes G \to G \otimes G)| = 1$, and $|G \otimes G| \leqslant p^{18}$ by [11, Theorem 2.3], whence $t(G) \geqslant 9$. For the group $G = \Phi_4(21^4)d$, using its central subgroup $Z = \langle \beta_1 \rangle$ we conclude that $|Im(Z \otimes G \to G \otimes G)| = 0$. So by the same argument as above one can obtain $t(G) \geqslant 9$. Also if $G \cong H \times C_p$ for some subgroup H, we must have $t(H) = 5$, contrary to the results of [17]. In the family Φ_7 only the group $G = \Phi_7(21^4)d$ has four generators. Obviously if $Z = \langle \alpha_3 \rangle$ then $|Im(Z \otimes G \to G \otimes G)| \leqslant p$ and $|G \otimes G| \leqslant p^{19}$, from which follows that $t(G) \geqslant 8$.

Suppose $n = 5$. Then G must be in the family Φ_3. As $d = 3$, taking $Z = \langle \alpha_3 \rangle$ in group $G = \Phi_3(2111)c$ we have $|Im((Z \otimes G) \times (G \otimes Z) \to G \otimes G)| = 1$. Hence $|G \otimes G| = p^{11}$ by sequence (2.1) and consequently $t(G) = 7$ by using Theorem 2.1. Assume $G \cong H \times C_p$ where H is a subgroup of G, then it follows by a direct computation that $t(H) = 5$ which implies that $H = \Phi_3(211)a$ or $\Phi_3(211)b_r$ by the results of [17]. Therefore for these groups we have $t(G) = 7$. When $p = 2$, $t(T_{23}) = 7$ from GAP.

Case 3. Suppose $d = \delta + 2$. Then $n = 6$ by Theorem 2.2 and our group G must belong to the family Φ_3. Since $d = 4$, the group G can be expressed as the direct product of its subgroups and one can easily check that none of these groups satisfy our condition. □

Proposition 3.5. *There exists no group with* $(c, a) = (2, 3)$*.*

Proof. Suppose $(c, a) = (2, 3)$. Then $d = n - 3$ and it follows from the equation (3.2) and Theorem 2.2 that $n = 5$ and $d = \delta = 2$. So our group G must belong to one of the families Φ_3 or Φ_8. For groups of the first family, substituting α_3 by $[\alpha_2, \alpha]$ (or α_3^r by $[\alpha_2, \alpha]^r$) and using the fact that $\alpha_2 \otimes \alpha_2 = 1$, we observe that $\alpha_3 \otimes \alpha_1 = 1$. Also by putting $Z = \langle \alpha_3 \rangle$, from [13, Theorem 3.1], we know that the group $G/Z \otimes G/Z$ has order at most p^7. Hence $|G \otimes G| \leqslant p^8$ and $t(G) \geqslant 8$. For the only group in family Φ_8, set $Z = \langle \beta^p \rangle$. As $\beta^p \otimes \alpha_1 = 1$ we have again that $|G \otimes G| \leqslant p^8$, and so $t(G) \neq 7$. As well as $p > 2$, there is no group for $p = 2$ by GAP. □

Proposition 3.6. *If* $(c, a) = (3, 3)$*, then* G *is isomorphic with one of the groups* $\Phi_6(1^5)$ *for* $p > 3$*,* $\Phi_9(1^5)$*,* $\Phi_{10}(1^5)$ *for* $p > 3$ *or* $\Phi_{11}(1^6)$*.*

Proof. Suppose $(c, a) = (3, 3)$. Clearly $d = \delta = n - 3$ from the equation (3.2) and Theorem 2.2 yields $n \leqslant 6$.

Suppose $n = 6$. The condition $t(G) = 7$ forces $|G \otimes G| = p^{17}$ by 2.1. Therefore $G = \Phi_{11}(1^6)$ by [11, Theorem 3.1].

Suppose $n = 5$. Then we must have $|G \otimes G| = p^9$ by 2.1. So again [11, Theorem 3.1] implies that $G = \Phi_6(1^5)$ for $p > 3$, $\Phi_9(1^5)$ or $\Phi_{10}(1^5)$ for $p > 3$. It follows from GAP that the condition $t(G) = 7$ does not hold when p is even. □

Table 1.

Name	Relations	SmallGroup	(c, a)
T_4	$a^4 = b^2 = c^2 = 1,\ [b, c] = a^2,\ [a, b] = [a, c] = 1$	$(16, 13)$	$(1, 1)$
T_6	$a^4 = b^4 = c^2 = 1,\ [a, b] = a^2,\ [a, c] = [b, c] = 1$	$(32, 23)$	$(1, 2)$
T_7	$b^2 = c^4 = 1,\ [b, c] = a^2,\ [a, b] = [a, c] = 1$	$(32, 24)$	$(1, 2)$
T_{21}	$a^4 = b^2 = c^2 = 1,\ [a, c] = a^2,\ [a, b] = [b, c, c] = 1$	$(32, 28)$	$(2, 2)$
T_{22}	$b^4 = c^2 = 1,\ [c, a] = a^{-2},\ [b, c]b^2 = a^2,\ [a, b] = 1$	$(32, 31)$	$(2, 2)$
T_{23}	$a^8 = b^2 = c^2 = 1,\ [a, b] = a^{-2},\ [a, c] = [b, c] = 1$	$(32, 39)$	$(2, 2)$
T_{24}	$a^4 = b^4 = 1,\ [a, b]^2 = 1,\ [a, b, a] = [a, b, b] = 1$	$(32, 2)$	$(1, 3)$

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(Seid Hadi Jafari) Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

E-mail address: s.hadijafari@mshdiau.ac.ir