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ε-WEAKLY CHEBYSHEV SUBSPACES AND QUOTIENT SPACES

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ABSTRACT. It will be determined under what conditions ε-quasi Chebyshev and ε-weakly Chebyshev subspaces are transmitted to and from quotient spaces.

1. Introduction and Preliminaries

Let X be a (complex or real) Banach space, $\varepsilon > 0$ be given and let W be a subspace of X. A point $y_0 \in W$ is said to be a ε-approximation for $x \in X$ if

$$\|x - y_0\| \leq d(x, W) + \varepsilon.$$

For $x \in X$, put

$$P_{W,\varepsilon}(x) = \{y \in W : \|x - y\| \leq d(x, W) + \varepsilon\}$$

and

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\}.$$

It is clear that $P_{W,\varepsilon}(x)$ is a non-empty, bounded and convex subset of X. Also, $P_{W,\varepsilon}(x)$ is closed for all $x \in X$, if W is closed.

We know that a subspace W of a Banach space X is called proximinal (respectively, quasi-Chebyshev or Weakly-Chebyshev)

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if $P_W(x)$ is non-empty (and respectively, compact or weakly compact) for all $x \in X$.

Recently, these types of subspaces are investigated (see [3]-[8]). There are some works on sum and quotient of proximinal subspaces of Banach spaces (see [1],[2]). Also, the first author has defined ε-quasi Chebyshev and ε-weakly Chebyshev subspaces of Banach spaces (see [9],[10]).

Definition 1.1. Let X be a Banach space. A subspace W is called ε-quasi Chebyshev (ε-weakly Chebyshev) in X if $P_{W,\varepsilon}(x)$ is a compact (weakly compact) set in X for each $x \in X$.

Note that, every ε-quasi Chebyshev subspace is ε-weakly Chebyshev. But, the converse is not true, (see [10]).

Let M be a proximinal subspace of a Banach space X, and let $f \in M^{\perp}$ be arbitrary. Define the linear functional T_f on X/M by $T_f(x+M) = f(x)$ for all $x + M \in X/M$. Since M is proximinal, $T_f \in (X/M)^*$ and $\|T_f\| \leq \|f\|$.

We conclude this section by a list of known lemmas needed in the proof of the main results.

Lemma 1.2. [12; Theorem 4]. *Let W be a subspace of a normed linear space X, $x \in X \backslash \overline{W}$ and $g_0 \in W$. Then, $g_0 \in P_W(x)$ if and only if $\|x - g_0\| = \|x - g_0\|_{W^{\perp}}$, where $\|x - g_0\|_{W^{\perp}} = sup\{|f(x - g_0)| : \|f\| \leq 1, f \in W^{\perp}\}$.*

Lemma 1.3. [9; Theorem 2.11]. *Let W be a closed subspace of a Banach space X and $\varepsilon > 0$ be given. Then, W is ε-quasi Chebyshev subspace of X if and only if W is finite dimensional.*

Lemma 1.4. [10; Theorem 2.7]. *Let W be a closed subspace of a Banach space X and $\varepsilon > 0$ be given. Then, W is ε-weakly Chebyshev subspace of X if and only if W is reflexive.*

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let M be a proximinal subspace of a Banach space X, $\varepsilon > 0$ be given and let W be a ε-quasi Chebyshev subspace of X such that M is a subspace of W. Then, $\frac{W}{M}$ is ε-quasi Chebyshev in $\frac{X}{M}$.*

Proof. Let $\pi : X \longrightarrow \frac{X}{M}$ be the canonical map. Since $P_{\frac{W}{M},\varepsilon}(x + M) = \pi(P_{W,\varepsilon}(x))$ for all $x \in X$ and π is continuous, $\frac{W}{M}$ is a ε-quasi Chebyshev subspace of $\frac{X}{M}$. □

Theorem 2.2. *Let M be a finite dimensional subspace of a Banach space X, $\varepsilon > 0$ be given and let W be a closed subspace of X. Then the followings are equivalent:*

(a) $M + W$ is ε-quasi Chebyshev in X.

(b) $\frac{M+W}{M}$ is ε-quasi Chebyshev in $\frac{X}{M}$.

Proof. It is an immediate consequence of Lemma 1.2 and the relation $dim(M + W) = dim(\frac{M+W}{M}) + dim(M)$, which implies that $M+W$ is finite dimensional if M and $\frac{M+W}{M}$ are finite dimensional. □

Corollary 2.3. *Let M be a finite dimensional subspace of a Banach space X, $\varepsilon > 0$ be given and let W be a closed subspace of X such that M is a subspace of W. If $\frac{W}{M}$ is ε-quasi Chebyshev in $\frac{X}{M}$,* *then W is ε-quasi Chebyshev in X.*

The following example shows that finite dimensionality of M can not omit in Corollary 2.3.

Example 2.4. Let $X = c_0$, $W = \{\{x_n\}_{n\geq 1} \in X : x_1 = 0\}$ and $M = \{\{x_n\}_{n\geq 1} \in X : x_1 = x_2 = 0\}$. Then, it is easy to show that M and W are proximinal subspaces of X. Since $dim\frac{W}{M} = 1$, by Lemma 1.2, $\frac{W}{M}$ is ε-quasi Chebyshev in $\frac{X}{M}$, but W is not ε-quasi Chebyshev in X.

Theorem 2.5. *Let M be a proximinal subspace of a Banach space X, $\varepsilon > 0$ be given and let W be a ε-weakly Chebyshev subspace of X such that M is a subspace of W. Then, $\frac{W}{M}$ is ε-weakly Chebyshev in $\frac{X}{M}$.*

Proof. It is an immediate consequence of Lemma 1.3 and the well known fact ([11]) that a reflexive space has every its quotient spaces reflexive. $\square$

Lemma 2.6. *Let X be a Banach space and S a finite dimensional subspace of a X such that $\frac{X}{S}$ is reflexive. Then, X is reflexive.*

Proof. It is well known that X is reflexive if and only if the unit ball B of X is weakly compact. As a consequence of the Eberlein-Šmulian Theorem B is weakly compact if and only if every sequence $\{a_n\}_{n\geq 1}$ in B has a weakly convergent subsequence. Let $\{a_n\}_{n\geq 1}$ be a sequence in B. Since S is finite dimensional, S is complemented subspace of X. That is, there exists a projection $P : X \longrightarrow S$. That means that P is a linear bounded and onto map with $P^2 = P$. We note that $W := ker(P)$ is a complement of S. That is W is closed, $S + W = X$ and $S \cap W = \{0\}$.

We note that $\{a_n + S\}_{n\geq 1}$ belongs to the unit ball of $\frac{X}{S}$. Thus, from reflexivity of $\frac{X}{S}$ we obtain a subsequence $\{a_{n_k} + S\}_{n\geq 1}$ converging weakly to an element $a + S \in \frac{X}{S}$. Since S is finite dimensional and $m_k := P(a_{n_k} - a) \in S$ and $\|m_k\| \leq 2\|P\|$ we can suppose, without loss of generality, that there exists $m_0 \in S$ with $\|m_k - m_0\| \longrightarrow 0$.

Now, we can consider the subset of X^* defined by

$$V = \{f \in X^* : f(x) = f(P(x)) \text{ for all } x \in X\}.$$

It is easy to check that V is a closed subspace of X which is a complement of $S^\perp$ (in fact, V=(Range$(I - P))^\perp$). that means, V is closed, $V \cap S^\perp = \{0\}$ and $V + S^\perp = X^*$. As is well known we can identify $S^\perp$ with $(\frac{X}{S})^*$. Let $f \in X^*$ be arbitrary. We can split f in the form $f = f_1 + f_2$ with $f_1 \in S^\perp$ and $f_2 \in V$. Therefore, we have

$$f(a_{n_k}) = f_1(a_{n_k}) + f_2(a_{n_k}) = T_{f_1}(a_{n_k} + S) + f_2(a_{n_k}).$$

We have that $T_{f_1}(a_{n_k} + S) \longrightarrow T_{f_1}(a + S) = f_1(a)$.

On the other hand,

$$\begin{aligned} f_2(a_{n_k}) &= f_2(a_{n_k} - a) + f_2(a) = f_2(P(a_{n_k} - a)) + f_2(a) \\ &= f_2(m_k) + f_2(a) \longrightarrow f_2(m_0) + f_2(a). \end{aligned}$$

Therefore, from $f_1(m_0) = 0$ we get

$$f(a_{n_k}) \longrightarrow f_1(a) + f_2(a) + f_2(m_0) = (f_1 + f_2)(a + m_0) = f(a + m_0).$$

Hence, $\{a_{n_k}\}_{k\geq 1}$ converges weakly to $a + m_0$. □

Corollary 2.7. *Let M be a finite dimensional subspace of a Banach space X, $\varepsilon > 0$ be given and let W be a closed subspace of X such that M is a subspace of W. If $\frac{W}{M}$ is ε-weakly Chebyshev in $\frac{X}{M}$, then W is ε-weakly Chebyshev in X.*

The following example shows that finite dimensionality of M can not omit in Corollary 2.7.

Example 2.8. Let $X = \ell^\infty$, $W = \ell^1$, $M = \{\{x_n\}_{n\geq 1} \in W : x_1 = 0\}$ and let $\varepsilon > 0$ be given. By Lemma 1.3, W is not ε-weakly

Chebyshev in X. Since $dim\frac{W}{M} = 1$, $\frac{W}{M}$ is ε-weakly Chebyshev in $\frac{X}{M}$.

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