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LIMITS IN MODIFIED CATEGORIES OF INTEREST

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Abstract. We prove completeness of the category of crossed modules in a modified category of interest. We define pullback crossed modules and pullback cat^1-objects that are both obtained by pullback diagrams with extra structures on certain arrows. These constructions unify many corresponding results for the cases of groups, commutative algebras and can also be adapted to various algebraic structures.
Keywords: Modified category of interest, crossed module, cat^1-object, limit.
MSC(2010): Primary: 18D05; Secondary: 17A30, 18A30, 18A35.

1. Introduction

The notion of category of interest was introduced to unify various properties of algebraic structures. The main idea is due to Higgins [13] which was improved by Orzech [19]. As indicated in [8, 9, 15–17, 19], algebraic categories are the main examples of category of interest. On the other hand, the categories of cat^1-objects of Lie (associative, Leibniz, etc.) algebras are not category of interest. Because of this, the authors of [5] introduced a new type of this notion, called *modified category of interest,* that satisfies all axioms of the former notion except one, which is replaced by a new and modified axiom. The main examples are those which are equivalent to the categories of crossed modules in the categories of groups, (commutative) algebras, dialgebras, Lie and Leibniz algebras, etc. See [4, 7, 11, 18, 20] for more examples.

Crossed modules were introduced by Whitehead in [23] as a model of homotopy 2-types and they were used to classify higher dimensional cohomology groups. The notion of crossed module is also defined for various algebraic structures. However, the definition of crossed modules in modified categories of interest unifies all of these definitions. As an equivalent model of homotopy 2-types, cat^1-groups were introduced by Loday in [14]. This notion and the

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corresponding equivalence were also adapted to several algebraic structures, as well as to modified category of interest [22].

In this paper, we first prove that the category of crossed modules in a modified category of interest $\mathbb{C}$ is finitely complete. This unifies a number of constructions given in [21]. Then, we define pullback crossed modules and pullback cat^1-objects in $\mathbb{C}$ that are both obtained by pullback diagrams with extra categorical structures on certain arrows. These definitions unify the constructions and results given in [2,3,6]. Moreover, one can adapt them to other different algebraic structures such as Lie algebras, Leibniz algebras, dialgebras, etc.

2. **Preliminaries**

In this section, we recall some notions from [5,14,22].

2.1. **Modified Category of Interest.**

Definition 2.1. Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$ such that $\mathbb{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, $i = 0, 1, 2$, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(b) the group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$. Assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$. Furthermore, assume $\Omega_0 = \{0\}$;
(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;
(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and *either* the identity $\omega(x * y) = \omega(x) * \omega(y)$ *or* the identity $\omega(x * y) = \omega(x) * y$.

Denote by Ω_{1S}' the subset of those elements in Ω_1' which satisfy the identity $\omega(x * y) = \omega(x) * y$, and by Ω_1'' all other unary operations, i.e., those which satisfy the first identity in (d).

Let C be an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

(e) $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$,

(f) for each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that:

$$(x_1 * x_2) \mathbin{\overline{*}} x_3 = W\big(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,\\ x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2\big),$$

where each juxtaposition represents an operation in Ω_2'.

A category of groups with operations $\mathbb{C}$ satisfying conditions (a)-(f) is called a **modified category of interest**, or **MCI** for short.

As indicated in [5], the difference between this definition and that of the original **category of interest** is the modification of the second identity in (d). According to this definition every category of interest is also a modified category of interest.

Definition 2.2. Let A, B be two objects of $\mathbb{C}$. A map $f\colon A \to B$ is called a morphism of $\mathbb{C}$ if it satisfies

$$f(a + a') = f(a) + f(a'),$$
$$f(a * a') = f(a) * f(a'),$$

for all $a, a' \in A$, $* \in \Omega_2'$ and commutes with all $w \in \Omega_1'$.

Example 2.3. The categories of groups, algebras, commutative algebras, Lie algebras, Leibniz algebras, dialgebras are all (modified) categories of interest.

Example 2.4. The categories $\mathbf{Cat^1Ass}$, $\mathbf{Cat^1Lie}$, $\mathbf{Cat^1Leibniz}$, i.e., the categories of cat^1-associative algebras, cat^1-Lie algebras and cat^1-Leibniz algebras are the examples of modified categories of interest, which are not categories of interest (see [5] for details).

Notation. From now on, $\mathbb{C}$ will denote an arbitrary but a fixed modified category of interest.

Definition 2.5. Let $B \in \mathbb{C}$. A subobject of B is called an ideal if it is the kernel of some morphism.

In other words, A is an ideal of B if and only if A is a normal subgroup of B and $a * b \in A$, for all $a \in A$, $b \in B$ and $* \in \Omega_2'$.

Definition 2.6. Let $A, B \in \mathbb{C}$. An extension of B by A is a sequence:

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0 \tag{2.1}$$

where p is surjective and i is the kernel of p. We say that an extension is split if there exists a morphism $s\colon B \to E$ such that $ps = 1_B$.

Definition 2.7. The split extension (2.1) induces an action of B on A corresponding to the operations of $\mathbb{C}$ with

$$b \cdot a = s(b) + a - s(b),$$
$$b * a = s(b) * a,$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2'$.

The actions defined by the previous equations are called *derived actions* of B on A. Note that we use the notation "$*$" to denote both the star operation and the star action.

Given an action of B on A, a semi-direct product $A \rtimes B$ is a universal algebra, whose underlying set is $A \times B$ and the operations are defined by

$$\begin{aligned}
\omega(a,b) &= (\omega(a), \omega(b)),\\
(a',b') + (a,b) &= (a' + b' \cdot a, b' + b),\\
(a',b') * (a,b) &= (a' * a + a' * b + b' * a, b' * b)
\end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$, $* \in \Omega_2'$. An action of B on A is a derived action if and only if $A \rtimes B$ is an object of $\mathbb{C}$.

Denote a general category of groups with operations of a modified category of interest $\mathbb{C}$ by $\mathbb{C}_G$. A set of actions of B on A in $\mathbb{C}_G$ is a set of derived actions if and only if it satisfies the following conditions

(1) $0 \cdot a = a$,
(2) $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
(3) $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
(4) $b * (a_1 + a_2) = b * a_1 + b * a_2$,
(5) $(b_1 + b_2) * a = b_1 * a + b_2 * a$,
(6) $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$,
(7) $(b_1 * b_2) \cdot (a * b) = a * b$,
(8) $a_1 * (b \cdot a_2) = a_1 * a_2$,
(9) $b * (b_1 \cdot a) = b * a$,
(10) $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
(11) $\omega(a * b) = \omega(a) * b = a * \omega(b)$ for any $\omega \in \Omega_{1S}'$, and $\omega(a * b) = \omega(a) * \omega(b)$ for any $\omega \in \Omega_1''$,
(12) $x * y + z * t = z * t + x * y$,

for each $\omega \in \Omega_1'$, $* \in \Omega_2'$, b, b_1, $b_2 \in B$, $a, a_1, a_2 \in A$; and for $x, y, z, t \in A \cup B$ whenever both sides of the last condition are defined.

2.2. **Crossed Modules.**

Definition 2.8. A crossed module (C_1, C_0, ∂) in $\mathbb{C}$ is given by a morphism $\partial\colon C_1 \to C_0$ with a derived action of C_0 on C_1 such that

XM1) $\begin{array}{l}\partial(c_0 \cdot c_1) = c_0 + \partial(c_1) - c_0\\ \partial(c_0 * c_1) = c_0 * \partial(c_1)\end{array}$

XM2) $\begin{array}{l}\partial(c_1) \cdot c_1' = c_1 + c_1' - c_1\\ \partial(c_1) * c_1' = c_1 * c_1'\end{array}$

for all $c_0 \in C_0$, $c_1, c_1' \in C_1$, $* \in \Omega_2'$.

A morphism between two crossed modules $(C_1, C_0, \partial) \to (C_1', C_0', \partial')$ is a pair (μ_1, μ_0) of morphisms $\mu_0 \colon C_0 \to C_0'$, $\mu_1 \colon C_1 \to C_1'$, such that the diagram

$$\begin{tikzcd} C_1 \arrow[r, "\partial"] \arrow[d, "\mu_1"'] & C_0 \arrow[d, "\mu_0"] \\ C_1' \arrow[r, "\partial'"'] & C_0' \end{tikzcd}$$

commutes and

$$\mu_1(c_0 \cdot c_1) = \mu_0(c_0) \cdot \mu_1(c_1)\,,$$
$$\mu_1(c_0 * c_1) = \mu_0(c_0) * \mu_1(c_1)$$

for all $c_0 \in C_0$, $c_1 \in C_1$ and $* \in \Omega_2'$.

Crossed modules and their morphisms form the category of crossed modules in $\mathbb{C}$ that will be denoted by **XMod**.

Example 2.9 ([12])**.** A crossed module of groups is given by a group homomorphism $\partial \colon E \to G$ together with an action $\triangleright$ of G on E such that (for all $e, f \in E$ and $g \in G$)

- $\partial(g \triangleright e) = g\,\partial(e)\,g^{-1}$,
- $\partial(e) \triangleright f = e\,f\,e^{-1}$.

Example 2.10 ([12])**.** A crossed module of Lie algebras is given by a Lie algebra homomorphism $\partial \colon \mathfrak{e} \to \mathfrak{g}$ together with an action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{e}$ such that (for all $e, f \in \mathfrak{e}$ and $g \in \mathfrak{g}$)

- $\partial(g \triangleright e) = [g, \partial(e)]$,
- $\partial(e) \triangleright f = [e, f]$.

Note that in the previous examples $\triangleright$ denotes both the group action and the Lie algebra action, respectively.

2.3. Cat^1 Objects.

Definition 2.11. Let S be a subobject of R. A cat^1-object $(e; s, t, R \to S)$ in $\mathbb{C}$ is an object C together with morphisms $s, t \colon R \to S$ and $e \colon S \to R$ such that the following conditions are satisfied

- $se = \mathrm{id}_S$ and $te = \mathrm{id}_S$,
- $x * y = 0$, $x + y - x - y = 0$

for all $* \in \Omega_2'$ and $x \in \ker s$, $y \in \ker t$.

Let $C = (e; s, t : R \to S)$ and $C' = (e'; s', t' : R' \to S')$ be two cat^1-objects. A cat^1-morphism $(\phi, \varphi) : C \to C'$ is a tuple which consists of morphisms $\phi : R \to R'$ and $\varphi : S \to S'$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & R' \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\ S \arrow[r, "\varphi"'] \arrow[u, bend left=60, "e"] & S' \arrow[u, bend right=60, "e'"'] \end{tikzcd} .$$

Cat^1-objects and their morphisms form the category of cat^1-objects in $\mathbb{C}$ that will be denoted by $\mathbf{Cat^1}$.

Notation. We denote any cat^1-object in $\mathbb{C}$ by (R, S) for short.

Example 2.12 ([10])**.** A cat^1-Leibniz algebra consists of a Leibniz algebra L, a sub Leibniz algebra M and Leibniz algebra homomorphisms $s, t\colon L \to M$ and $e\colon M \to L$ such that

- $se = \mathrm{id}_M$ and $te = \mathrm{id}_M$,
- $[x, y] = 0 = [y, x]$,

for all $x \in \ker s$, $y \in \ker t$.

Example 2.13. A cat^1-dialgebra consists of a dialgebra [16] D, a sub dialgebra and dialgebra homomorphisms $s, t\colon D \to F$ and $e\colon F \to D$ such that

- $se = \mathrm{id}_F$ and $te = \mathrm{id}_F$,
- $x \dashv y = 0 = y \dashv x$, $x \vdash y = 0 = y \vdash x$

for all $x \in \ker s$, $y \in \ker t$.

Proposition 2.14. *The categories* $\mathbf{XMod}$ *and* $\mathbf{Cat^1}$ *are naturally equivalent.*

Proof. Let (C_1, C_0, ∂) be a crossed module in $\mathbb{C}$. Consider the corresponding semi-direct product $C_1 \rtimes C_0$ induced from the action of C_0 on C_1. By using the morphisms $s, t\colon C_1 \rtimes C_0 \to C_0$ and $e\colon C_0 \to C_1 \rtimes C_0$ defined by $s(c_1, c_0) = c_0$, $t(c_1, c_0) = \partial(c_1) + c_0$ and $e(c_0) = (0, c_0)$, we obtain a cat^1-object. This yields to the functor $\mathbf{C^1}\colon \mathbf{XMod} \to \mathbf{Cat^1}$. See [22] for the converse. $\square$

3. Limits in MCI

The cartesian product $P \times R$ is the product object of P and R in $\mathbb{C}$ with the projection morphisms satisfying the universal property.

Suppose that $\alpha : P \to S$ and $\beta : R \to S$ are two morphisms in $\mathbb{C}$. Then the subobject of the cartesian product

$$P \times_S R = \{(p, r) \mid \alpha(p) = \beta(r)\},$$

i.e., the *fiber product*, defines the pullback of α, β.

Therefore, a modified category of interest $\mathbb{C}$ has products and pullbacks which guarantees the existence of equalizer objects. Briefly, suppose that we have two parallel morphisms $f, g\colon P \to R$. Their equalizer is defined as $\mathrm{Eq}(f,g) = \{x \in P \mid f(x) = g(x)\}$.

Consequently, we can say that $\mathbb{C}$ has all finite limits since it has both products and equalizers. Thus, $\mathbb{C}$ is finitely complete.

3.1. **Limits in the Category of Crossed Modules in MCI.**

Definition 3.1. The category of crossed modules in $\mathbb{C}$ with fixed codomain X forms a full subcategory of **XMod** that is denoted by **XMod/X**. These kind of crossed modules will be called *crossed X-modules*.

Lemma 3.2. *Given two crossed modules (P, S, α) and (R, S, β) there is a crossed module*

$$\partial : P \times_S R \to S,$$

where $\partial\,(p, r) = \alpha\,(p) = \beta\,(r)$ and the action of S on $P \times_S R$ is defined by

$$s \cdot (p, r) = (s \cdot p, s \cdot r)\,, \qquad s * (p, r) = (s * p, s * r)\,.$$

Proof. The action given above is well-defined and the action conditions are already satisfied. Moreover, $\partial : P \times_S R \to S$ is a morphism of $\mathbb{C}$ since

$$\begin{aligned}\partial\,((p, r) + (p', r')) &= \partial\,(p + p', r + r')\\ &= \alpha\,(p + p')\\ &= \alpha\,(p) + \alpha\,(p')\\ &= \partial\,(p, r) + \partial\,(p', r')\,.\end{aligned}$$

Similarly, we have

$$\partial\,((p, r) * (p', r')) = \partial\,(p, r) * \partial\,(p', r')\,,$$

for all $(p, r)\,, (p', r') \in P \times_S R$. Also ∂ commutes with all $w \in \Omega_1'$ since

$$\partial\big(w(p, r)\big) = \partial\big(w(p), w(r)\big) = \alpha\big(w(p)\big) = w\big(\alpha(p)\big) = w\big(\partial(p, r)\big).$$

Finally, ∂ satisfies the crossed module conditions

XM1)

$$\begin{aligned}&\partial\,(s \cdot (p, r)) = \partial\,(s \cdot p, s \cdot r) = \alpha(s \cdot p) = s + \alpha\,(p) - s = s + \partial\,(p, r) - s,\\ &\partial\,(s * (p, r)) = \partial\,(s * p, s * r) = \alpha\,(s * p) = s * \alpha\,(p) = s * \partial\,(p, r)\,,\end{aligned}$$

XM2)

$$\begin{aligned}
\partial\,(p',r')\cdot(p,r) &= \alpha\,(p')\cdot(p,r)\\
&= (\alpha\,(p')\cdot p, \alpha\,(p')\cdot r)\\
&= (\alpha\,(p')\cdot p, \beta\,(r')\cdot r)\\
&= (p'+p-p', r'+r-r')\\
&= (p',r')+(p,r)-(p',r'),
\end{aligned}$$

$$\begin{aligned}
\partial\,(p',r')*(p,r) &= \alpha\,(p')*(p,r)\\
&= (\alpha\,(p')*p, \alpha\,(p')*r)\\
&= (\alpha\,(p')*p, \beta\,(r')*r)\\
&= (p'*p, r'*r)\\
&= (p',r')*(p,r)\,,
\end{aligned}$$

for all $(p,r)\,,(p',r')\in P\times_S R$ and $s\in S$. □

Lemma 3.3. *Let $(\alpha,\mathrm{id})\colon (P,X,\gamma)\to(S,X,\partial')$ be a crossed module morphism. Then there exists a crossed module (P,S,α) where the action of S on P are defined along ∂', namely,*

$$s\cdot p=\partial'(s)\cdot p,\quad s*p=\partial'(s)*p.$$

Proof. Since (α,id) is a crossed module morphism, the diagram

$$\begin{tikzcd}
P \arrow[dd, "\gamma"'] \arrow[dr, "\alpha"] & \\
& S \arrow[dl, "\partial'"] \\
X &
\end{tikzcd}$$

commutes; namely, $\alpha(x\cdot p)=x\cdot\alpha(p)$ and $\alpha(x*p)=x*\alpha(p)$ for all $x\in X$ and $p\in P$. Thus,

XM1)

$$\begin{aligned}
\alpha(s\cdot p)&=\alpha\big(\partial'(s)\cdot p\big)=\partial'(s)\cdot\alpha(p)=s+\alpha(p)-s\,,\\
\alpha(s*p)&=\alpha\big(\partial'(s)*p\big)=\partial'(s)*\alpha(p)=s*\alpha(p)\,,
\end{aligned}$$

XM2)

$$\begin{aligned}
\alpha(p)\cdot p'&=\partial'(\alpha(p))\cdot p'=\gamma(p)\cdot p'=p+p'-p\,,\\
\alpha(p)*p'&=\partial'(\alpha(p))*p'=\gamma(p)*p'=p*p'\,,
\end{aligned}$$

for all $s\in S$ and $p,p'\in P$. □

Remark 3.4. If (A, B, ∂) and (B, C, ∂') are crossed modules such that C acts on A in a compatible way with B (i.e., $(\partial' b \cdot a) = b \cdot a)$, then $(A, C, \partial'\partial)$ becomes a crossed module as well, see [21] for details.

Lemma 3.5. *Suppose that we have crossed module morphisms*

$$(\alpha, \mathrm{id})\colon (P, X, \gamma) \to (S, X, \partial') \text{ and } (\beta, \mathrm{id})\colon (R, X, \delta) \to (S, X, \partial').$$

Then there exists a crossed module

$$P \times_S R \to X\,,$$

which leads to the pullback object in **XMod/X**.

Proof. By using crossed module morphisms (α, id) and (β, id), we get the following morphisms of $\mathbb{C}$

$$\alpha\colon P \to S \text{ and } \beta\colon R \to S.$$

We already know that the pullback of these morphisms in $\mathbb{C}$ are defined by the fiber product $P \times_S R$ that makes the following diagram commutative and satisfies the universal property

$$\begin{tikzcd}
 & P \times_S R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
P \arrow[dr, "\alpha"'] & & R. \arrow[dl, "\beta"] \\
 & S &
\end{tikzcd}$$

By using Lemma 3.3, α and β turn into crossed modules. Thus, we get a crossed module $\partial\colon P \times_S R \to S$ in the sense of Lemma 3.2. Moreover, $\partial'\colon S \to X$ is already a crossed module and X acts on $P \times_S R$ in a natural way. Therefore, by using Remark 3.4, we get the crossed module

$$\partial'\partial\colon P \times_S R \to X,$$

which leads to the pullback object in the category of crossed X-modules. All fitting into the diagram

$$\begin{tikzcd}
 & P \times_S R \arrow[dl, "\pi_1"'] \arrow[d, "\partial"] \arrow[dr, "\pi_2"] & \\
P \arrow[r, "\alpha"] \arrow[dr, bend right, "\gamma"'] & S \arrow[d, "\partial'"] & R \arrow[l, "\beta"'] \arrow[dl, bend left, "\delta"] \\
 & X &
\end{tikzcd}$$

□

Proposition 3.6. *The category of crossed X-modules has an initial object $0 \to X$ and a terminal object $\mathrm{id}\colon X \to X$. Consequently, one can construct the product object as a pullback of the morphisms*

$$\begin{tikzcd}
\mathcal{X} \arrow[dr] & & \mathcal{X}' \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

where $\mathcal{X}, \mathcal{X}'$ are two crossed X-modules and 1 is the terminal object.

This yields the following:

Proposition 3.7. *Given two crossed modules $\alpha\colon P \to S$ and $\beta\colon R \to S$ in a modified category of interest $\mathbb{C}$, their product is the crossed module $\partial : P \times_S R \to S$.*

Thus, we have proved the following theorem:

Theorem 3.8. *The category* **XMod/X** *is finitely complete.*

Remark 3.9. As a consequence of this section, one can obtain the completeness of the categories of crossed X-modules of groups, (commutative) algebras, Lie and Leibniz algebras, dialgebras, etc.

4. **Pullback Crossed Modules**

Definition 4.1. For a given crossed module (P, R, ∂) and a morphism $\phi\colon S \to R$ in $\mathbb{C}$, the pullback crossed module is defined as a crossed module morphism

$$(\phi', \phi)\colon \phi^{\star}(P, R, \partial) \to (P, R, \partial)\,,$$

where the crossed module:

$$\phi^\star(P,R,\partial) = (\phi^\star(P), S, \partial^\star)$$

satisfies the following universal property:

For any crossed module morphism

$$(f,\phi):\ (X,S,\mu) \to (P,R,\partial),$$

there exists a unique crossed module morphism

$$(f^\star, \mathrm{id}_S) : (X,S,\mu) \to (\phi^\star(P), S, \partial^\star),$$

such that the following diagram commutes

(4.1)
$$\begin{tikzcd} & (X,S,\mu) \arrow[dl, dashed, "(f^\star,\mathrm{id}_S)"'] \arrow[d, "(f,\phi)"] \\ (\phi^\star(P),S,\partial^\star) \arrow[r, "(\phi',\phi)"'] & (P,R,\partial). \end{tikzcd}$$

In other words, it can be seen as a pullback diagram [1],

(4.2)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dotted, "f^\star"] \arrow[dd, "\mu"'] & & P \arrow[dd, "\partial"] \\ & \phi^\star(P) \arrow[ur, "\phi'"'] \arrow[dl, "\partial^\star"] & \\ S \arrow[rr, "\phi"'] & & R. \end{tikzcd}$$

In order to give a particular construction for the pullback crossed module, let (P,R,∂) be a crossed module and let $\phi : S \to R$ be a morphism in $\mathbb{C}$. Define

$$\phi^\star(P) = P \times_R S = \{(p,s) \mid \partial(p) = \phi(s)\},$$

and define the morphism $\partial^\star : \phi^\star(P) \to S$ by

$$\partial^\star(p,s) = s.$$

There exists an action of S on $\phi^\star(P)$ defined by

$$\begin{array}{ccc} S \times \phi^\star(P) & \to & \phi^\star(P) \\ (t,(p,s)) & \mapsto & t \cdot (p,s) = (\phi(t)\cdot p, t+s-t), \end{array}$$

and

$$\begin{array}{ccc} S \times \phi^\star(P) & \to & \phi^\star(P) \\ (t,(p,s)) & \mapsto & t * (p,s) = (\phi(t) * p, t * s). \end{array}$$

Then, $(\phi^\star(P), S, \partial^\star)$ defines a crossed module since

XM1)

$$\partial^{\star}\left(t\cdot(p,s)\right)=\partial^{\star}\left(\phi\left(t\right)\cdot p,t+s-t\right)=t+s-t=t+\partial^{\star}\left(p,s\right)-t\,,$$
$$\partial^{\star}\left(t\star(p,s)\right)=\partial^{*}\left(\phi\left(t\right)*p,t*s\right)=t*s=t*\partial^{\star}\left(p,s\right)\,,$$

XM2)

$$\begin{aligned}\partial^{\star}\left(p',s'\right)\cdot(p,s)&=s'\cdot(p,s)\\&=\left(\phi\left(s'\right)\cdot p,s'+s-s'\right)\\&=\left(\partial\left(p'\right)\cdot p,s'+s-s'\right)\\&=\left(p'+p-p',s'+s-s'\right)\\&=\left(p',s'\right)+(p,s)-\left(p',s'\right),\end{aligned}$$

$$\begin{aligned}\partial^{\star}\left(p',s'\right)*(p,s)&=s'*(p,s)\\&=\left(\phi\left(s'\right)*p,s'*s\right)\\&=\left(\partial\left(p'\right)*p,s'*s\right)\\&=\left(p'*p,s'*s\right)\\&=\left(p',s'\right)*(p,s)\,,\end{aligned}$$

for all $(p,s)\,(p',s')\in\phi^{\star}(P)$ and $t\in S$.
This construction satisfies the universal property. Consider the crossed module morphism

$$(\phi',\phi):(\phi^{\star}(P),S,\partial^{\star})\to(P,R,\partial)\,,$$

where $\phi':\phi^{\star}(P)\to P$ is defined by $\phi'\left(p,s\right)=p$.
Suppose that (X,S,μ) is a crossed module and the tuple

$$(f,\phi):(X,S,\mu)\to(P,R,\partial) \tag{4.3}$$

is a crossed module morphism.
Define: $f^{\star}:X\to\phi^{\star}(P)$ by $f^{\star}(x)=\left(f\left(x\right),\mu\left(x\right)\right)$. Then,

$$(f^{\star},\mathrm{id}_S):(X,S,\mu)\to(\phi^{\star}(P),S,\partial^{\star})$$

becomes a crossed module morphism. In fact the diagram

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "f^{\star}"'] & S \arrow[d, "\mathrm{id}_S"] \\ \phi^{\star}(P) \arrow[r, "\partial^{\star}"'] & S \end{tikzcd}$$

is commutative since

$$\partial^{\star}f^{\star}(x)=\partial^{\star}\left(f\left(x\right),\mu\left(x\right)\right)=\mu\left(x\right)=\mathrm{id}_S\mu\left(x\right), \tag{4.4}$$

and

$$\begin{aligned} f^{\star}(s \cdot x) &= (f(s \cdot x), \mu(s \cdot x)) \\ &= (\phi(s) \cdot f(x), s \cdot \mu(x)) \\ &= s \cdot (f(x), \mu(x)) \\ &= \mathrm{id}_S(s) \cdot f^{\star}(x), \end{aligned}$$

$$\begin{aligned} f^{\star}(s * x) &= (f(s * x), \mu(s * x)) \\ &= (\phi(s) * f(x), s * \mu(x)) \\ &= s * (f(x), \mu(x)) \\ &= \mathrm{id}_S(s) * f^{\star}(x), \end{aligned}$$

for all $s \in S$ and $x \in X$. Moreover, we have

$$\phi' f^{\star}(x) = \phi'(f(x), \mu(x)) = f(x), \tag{4.5}$$

that makes diagram (4.1) commutative. In other words, pullback diagram (4.2) commutes since

- $\partial^{\star} f^{\star}(x) = \mu(x)$ from (4.4),
- $\phi\mu = \partial f$ since (4.3) is a crossed module morphism,
- $\phi' f^{\star} = f$ from (4.5).

Finally, we need to prove that $(f^{\star}, \mathrm{id})$ is unique in (4.1). Suppose that

$$(f^{\star\star}, \mathrm{id}_S) : (X, S, \mu) \to (\phi^{\star}(P), S, \partial^{\star})$$

is a crossed module morphism with the same property as $(f^{\star}, \mathrm{id})$. We get

$$\partial^{\star} f^{\star\star}(x) = f(x), \qquad \partial^{\star} f^{\star\star}(x) = \mu(x),$$

for all $x \in X$ which implies

$$f^{\star\star}(x) = (p, s) = (f(x), \mu(x)) = f^{\star}(x),$$

and proves that $(f^{\star}, \mathrm{id})$ is unique.

Therefore, we have the following:

Corollary 4.2. *We get a functor in* $\mathbb{C}$

$$\phi^{\star} \colon \mathbf{XMod/R} \to \mathbf{XMod/S}.$$

Moreover, let (P, R, ∂) *be a crossed module and let* $\phi \colon S \to R$ *be a morphism in* $\mathbb{C}$*. We have the pullback diagram*

$$\begin{tikzcd} \phi^\star(P) \arrow[r, "\phi'"] \arrow[d, "\partial^\star"'] & P \arrow[d, "\partial"] \\ S \arrow[r, "\phi"'] & R. \end{tikzcd}$$

Example 4.3. Given an object R and a normal subobject N of R, then (N, R, ∂) is a crossed module where ∂ is the inclusion map. Suppose that $\phi : S \to R$ is a morphism. Then the pullback crossed module is defined by

$$\begin{aligned} \phi^\star(N) &= \{(n, s) \mid \partial(n) = \phi(s),\, n \in N,\, s \in S\} \\ &\cong \{s \in S \mid \phi(s) = n,\ n \in N\} \\ &= \phi^{-1}(N), \end{aligned}$$

and the pullback diagram is

$$\begin{tikzcd} \phi^{-1}(N) \arrow[r, "\phi'"] \arrow[d, "\partial^\star"'] & N \arrow[d, "\partial"] \\ S \arrow[r, "\phi"'] & R. \end{tikzcd}$$

where the preimage $\phi^{-1}(N)$ is a normal subobject of S.
In particular, if $N = \{0\}$, then

$$\phi^\star(\{0\}) \cong \{s \in S \mid \phi(s) = 0\} = \ker \phi.$$

So kernels are particular cases of pullback crossed modules.

5. Pullback Cat^1-Objects

Definition 5.1. The definition of pullback cat^1-object along a morphism is similar to the one for crossed modules given in Definition 4.1. For a given cat^1-object (R, S) and a morphism $\phi\colon Q \to S$ in $\mathbb{C}$, we require a cat^1-object $\phi^\star(R, S) = (\phi^\star(R), Q)$ to fill the pullback diagrams

(5.1)
$$\begin{tikzcd} & (P, Q) \arrow[dl, dashed, "(\psi, \mathrm{id}_S)"'] \arrow[d, "(\varphi, \phi)"] \\ (\phi^\star(R), Q) \arrow[r, "(\pi, \phi)"'] & (R, S) \end{tikzcd}$$

and

(5.2)
$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, dotted, "\psi"] \arrow[dd, shift right, "t'"'] \arrow[dd, shift left, "s'"] & & R \arrow[dd, shift right, "t"'] \arrow[dd, shift left, "s"] \\ & \phi^\star(R) \arrow[ur, "\pi"'] \arrow[dl, shift right, "t^\star"'] \arrow[dl, shift left, "s^\star"] & \\ Q \arrow[rr, "\phi"'] & & S. \end{tikzcd}$$

Note that we do not include the embedding morphisms in the above diagrams for the sake of simplicity.

In order to give a particular construction for the pullback cat^1-object, let $(e; s, t : R \to S)$ be a cat^1-object and let $\phi : Q \to S$ be a morphism. Define

$$\phi^\star (e; s, t : R \to S) = (e^\star; s^\star, t^\star : \phi^\star(R) \to Q),$$

where

$$\phi^\star(R) = \{(q_1, r, q_2) \in Q \times R \times Q \mid \phi(q_1) = s(r),\ \phi(q_2) = t(r)\}$$

is a subobject of $Q \times R \times Q$. Define the morphisms:

$$s^\star (q_1, r, q_2) = q_1, \quad t^\star (q_1, r, q_2) = q_2, \quad e^\star (q) = (q, e\phi(q), q).$$

It is easily verified that $s^\star e^\star = t^\star e^\star = \mathrm{id}_Q$. Moreover, let $(q_1', r_1, q_1) \in \ker s^\star$ and $(q_2, r_2, q_2') \in \ker t^\star$. Then

$$s^\star (q_1', r_1, q_1) = 0_Q \quad \text{and} \quad t^\star (q_2, r_2, q_2') = 0_Q,$$

which implies $q_1' = q_2' = 0_Q$, hence we get $r_1 \in \ker s$ and $r_2 \in \ker t$. Therefore,

$$\begin{aligned} (q_1', r_1, q_1) * (q_2, r_2, q_2') &= (0_Q * q_2, r_1 * r_2, q_1 * 0_Q) \\ &= (0_Q, r_1 * r_2, 0_Q) \\ &= (0_Q, 0_R, 0_Q), \end{aligned}$$

and

$$\begin{aligned} (q_1', r_1, q_1) + (q_2, r_2, q_2') &= (0_Q + q_2, r_1 + r_2, q_1 + 0_Q) \\ &= (q_2, r_2 + r_1, q_1) \\ &= (q_2, r_2, q_2') + (q_1', r_1, q_1), \end{aligned}$$

which implies

$$(q_1', r_1, q_1) + (q_2, r_2, q_2') - (q_1', r_1, q_1) - (q_2, r_2, q_2') = 0_{\phi^*(R)}.$$

Consequently, we get the cat^1-object structure

$$(e^\star; s^\star, t^\star : \phi^\star(R) \to Q).$$

Define the morphism

$$\begin{array}{rccl} \pi: & \phi^\star(R) & \to & R \\ & (q_1, r, q_2) & \mapsto & \pi\left(q_1, r, q_2\right) = r. \end{array}$$

Since

$$\begin{aligned} \phi s^\star\left(\left(q_1, r, q_2\right)\right) &= \phi\left(q_1\right) = s\left(r\right) = s\pi\left(q_1, r, q_2\right), \\ \phi t^\star\left(\left(q_1, r, q_2\right)\right) &= \phi\left(q_2\right) = t\left(r\right) = t\pi\left(q_1, r, q_2\right), \\ \pi e^\star\left(q\right) &= \pi\left(q, e\phi\left(q\right), q\right) = e\phi\left(q\right), \end{aligned}$$

for all $(q_1, r, q_2) \in \phi^\star(R)$, $q \in Q$, the following diagram commutes

$$\begin{tikzcd} \phi^\star(R) \arrow[r, "\pi"] \arrow[d, shift right, "t^\star"'] \arrow[d, shift left, "s^\star"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ Q \arrow[r, "\phi"'] \arrow[u, bend left=50, "e^\star"] & S \arrow[u, bend right=50, "e"'] \end{tikzcd} .$$

Hence (π, ϕ) becomes a cat^1-object morphism.

Now we need to prove the universal property. Let

$$(\varphi, \phi): \ (e'; s', t' : P \to Q) \to (e; s, t : R \to S)$$

be any cat^1-morphism such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ Q \arrow[r, "\phi"'] \arrow[u, bend left=60, "e'"] & S \arrow[u, bend right=60, "e"'] \end{tikzcd} .$$

Define $\psi : P \to \phi^\star(R)$ by $\psi\left(p\right) = \left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right)$. Then

$$(\psi, \mathrm{id}_Q): \ (e'; s', t' : P \to Q) \to (e^\star; s^\star, t^\star : \phi^\star(R) \to Q)$$

becomes a cat^1-object since

$$\begin{aligned} s^\star\psi\left(p\right) &= s^\star\left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right) = s'\left(p\right) = \mathrm{id}_Q s'\left(p\right), \\ t^\star\psi\left(p\right) &= t^\star\left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right) = t'\left(p\right) = \mathrm{id}_Q t'\left(p\right), \end{aligned}$$

and

$$\psi e'\left(q\right) = \left(s'e'\left(q\right), \varphi e'\left(q\right), t'e'\left(q\right)\right) = \left(q, e\phi\left(q\right), q\right) = e^\star \mathrm{id}_Q\left(q\right),$$

for all $p \in P$ and $q \in Q$. Moreover we have

$$\pi\psi(p) = \pi(s'(p), \varphi(p), t'(p)) = \varphi(p),$$

which makes (5.1) commutative and leads to the pullback diagram (5.2). The uniqueness of (π, ϕ) can be proven analogous to the crossed module case given in the previous section.

6. Conclusion

We have provided the commutativity of the following diagram (up to isomorphism) for a fixed morphism ϕ of $\mathbb{C}$.

$$\begin{tikzcd}
\mathbf{XMod/R} \arrow[r, "\phi^{\star}"] \arrow[d, "\mathbf{C^1}"'] & \mathbf{XMod/S} \arrow[d, "\mathbf{C^1}"] \\
\mathbf{Cat^1} \arrow[r, "\phi^{\star}"'] & \mathbf{Cat^1}.
\end{tikzcd}$$

Another main outcome of the paper is the following:

One can obtain pullback crossed modules and pullback cat^1-objects in many well-known algebraic categories listed in Example 2.3 such as category of groups, (commutative) algebras, dialgebras, Lie algebras and Leibniz algebras, etc. For instance, if we consider the cases of category of groups and commutative algebras, we are lead to the constructions given in [2,3,6].

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