ON THE INFINITE PRODUCT REPRESENTATION OF SOLUTION AND DUAL EQUATIONS OF
STURM-LIOUVILLE EQUATION WITH TURNING POINT OF ORDER 4M+1

H. KHEIRI AND A. JODAYREE AKBARFAM

Abstract. The purpose of this paper is studying the infinite product representation of solution of boundary value problem:

\[-y'' + q(x)y = \lambda R^2(x)y, \quad 0 \leq x \leq 1\]

\[y(0) = 0 = y(1), \tag{I}\]

where \(\lambda = \rho^2\) is the spectral parameter and \(q(x)\) is a integrable function. We also suppose that

\[R^2(x) = (x - x_1)^{4m+1}R_0(x)\]

where \(0 < x_1 < 1, m \in \mathbb{N}, R_0 > 0\) for \(x \in [0, 1]\), \(R_0\) is twice continuously differentiable on \([0, 1]\) and \(R^2(x)\) has one zero in \([0, 1]\), so called turning point.

The product representation satisfies in the original equation (I). As a result we substituted the infinite product form in the equation (I) and derive the associate dual equations.

MSC(2000): Primary 34E

Keywords: Asymptotic distribution, Turning point, Sturm-Liouville problem, infinite product, dual equations

Received: 02 May 2004 , Revised: 9 July 2004

corresponding author: H. Kheiri

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1. Introduction

We consider boundary value problems $L$ of the form

$$Ly = -y'' + q(x) y = \lambda R^2(x)y, \quad x \in [0, 1] \quad (1)$$

$$y(0) = 0 = y(1) \quad (2)$$

where $\lambda = \rho^2$ is the spectral parameter and the functions $q(x)$ and $R^2(x)$ satisfy:

(i) $R^2(x) = R_0(x)(x - x_1)^l$ is real and has one zero $x_1$ of order $l = 4m + 1, m \in N$ in $[0, 1]$ and also $R_0$ is positive and twice continuously differentiable.

(ii) $q(x)$ is bounded and integrable on $I = [0, 1]$.

To study the infinite representation of the solution in this paper we use the asymptotic form of fundamental system of solutions (FSS) for equation (1) constructed in [7], [8] and we also use the asymptotic form of eigenvalues of equation (1) constructed in [9]. Note that the infinite product representation of solution with simple turning point has been studied in [10].

2. Notations and preliminary results

Let $\varepsilon > 0$ be fixed and sufficiently small, and let $D_\varepsilon = [0, x_1 - \varepsilon] \cup [x_1 + \varepsilon, 1]$. Further, we set $\mu = \frac{1}{2 \varepsilon t}$ and $\theta = 4 \mu$. we also denote

$$I_+ = \{x : R^2(x) > 0\} \quad I_- = \{x : R^2(x) < 0\}$$

$$\xi(x) = \begin{cases} 0 & \text{for } x \in I_+(x) \\ 1 & \text{for } x \in I_-(x) \end{cases}$$

$$R^2_+ = \max(0, R^2(x)) \quad R^2_- = \max(0, -R^2(x))$$

$$\gamma = 2 \sin \left( \frac{\pi \mu}{2} \right)$$

$$K_\pm(x) = \begin{cases} 1 & \text{for } x \in I_-(x) \\ \frac{1}{2} \csc \left( \frac{\pi \mu}{2} \right) \exp \left( \mp i \frac{\pi}{4} \right) & \text{for } x \in I_+(x) \end{cases}$$

$$K_{\pm}^*(x) = \begin{cases} \pm i & \text{for } x \in I_-(x) \\ 2 \sin \left( \frac{\pi \mu}{2} \right) \exp \left( \pm i \frac{\pi}{4} \right) & \text{for } x \in I_+(x) \end{cases}$$
Let

\[ S_k = \{ \rho \mid \arg \rho \in \left[ \frac{k\pi}{4}, \frac{(k+1)\pi}{4} \right] \}, \quad k = \{0, 1\}. \]

In [?2] it is shown that for each fixed sector \( S_k(k = 0, 1) \) there exists a FSS of (??) \( \{z_1(x, \rho), z_2(x, \rho)\}, \ x \in I, \ \rho \in S_k \) such that the functions \( (x, \rho) \rightarrow z^{(j)}_1(x, \rho)(s = 1, 2; j = 0, 1) \) are continuous for \( x \in I, \ \rho \in S_k \) and holomorphic for each fixed \( x \in I \) with respect to \( \rho \in S_k \); moreover for \( |\rho| \rightarrow \infty, \ \rho \in S_k, \ x \in D_e, \quad j = 1, 2 \)

\[
z^{(j)}_1(x, \rho) = (\pm i\rho)^{\frac{1}{2}} |R(x)|^{\frac{1}{2}} \left[ \frac{1}{2} \exp(\mp i\frac{\pi}{2} \xi(x)) \right]^{\frac{1}{2}} \exp(\pm i\rho \int_0^x |R_-(t)| dt) \times \exp(\pm i\rho \int_0^x |R_+(t)| dt) K_\pm(x) \kappa(x, \rho), \quad (3)
\]

\[
z^{(j)}_2(x, \rho) = (\mp i\rho)^{\frac{1}{2}} |R(x)|^{\frac{1}{2}} \left[ \frac{1}{2} \exp(\mp i\frac{\pi}{2} \xi(x)) \right]^{\frac{1}{2}} \exp(-\rho \int_0^x |R_-| dt) \times \exp(-\rho \int_0^x |R_+| dt) K_\pm(x) \kappa(x, \rho), \quad (4)
\]

\[
\left| \begin{array}{cc}
z_1(x, \rho) & z_2(x, \rho) \\
z^{(1)}_1(x, \rho) & z^{(2)}_1(x, \rho)
\end{array} \right| = \mp(2i\rho)[1].
\]

Here and in the following:

(i) The upper or lower signs in formulae correspond to the sectors \( S_0, S_1 \) respectively.

(ii) \( [1] = 1 + O(\frac{1}{|\rho|}) \) uniformly in \( x \in D_e. \)

(iii) \( \kappa(x, \rho) = O(1) \) as \( |\rho| \rightarrow \infty, \ \rho \in S_k. \)

3. Representation of the solution in the form of infinite product

Let \( \varphi(x, \lambda) \) be solution of equation (??) with initial conditions

\[
\varphi(0, \lambda) = 0, \quad \frac{\partial \varphi}{\partial x}(0, \lambda) = 1. \quad (5)
\]

Using the FSS \( \{z_1(x, \rho), z_2(x, \rho)\} \) we obtain:

\[
\varphi(x, \lambda) = \frac{1}{\omega(\lambda)}(z_1(0, \rho)z_2(x, \rho) - z_2(0, \rho)z_1(x, \rho))
\]
where \( \omega(\lambda) = \mp(2i\rho)[1] \). By virtue of (6)—(7), we infer that for \( \rho \in S_k, \ x \in D, \ j = 0, 1 \)

\[
\varphi^j(x, \lambda) = \frac{1}{2}(\pm i\rho)^j-1 |R(0)|^{-\frac{1}{2}} |R(x)|^{\frac{1}{2}} (\exp(\mp i\frac{\pi}{2} \xi(x))^j \exp(\pm i\frac{\pi}{2}) \times \exp(\rho \int_0^x \frac{|R_+|}{|R_-|} dt) \exp(\pm i\rho \int_0^x |R_-| dt) K_\pm(x, \rho) \quad (6)
\]

and

\[
|\varphi^j(x, \lambda)| \leq C|\rho|^{j-1} \exp(\rho \int_0^x |R_-| dt) \exp(\pm i\rho \int_0^x |R_+| dt). \quad (7)
\]

It follows from (6) that the function \( \varphi^j(x, \lambda) \) are entire of order \( \frac{1}{2} \).

The function \( \varphi(x, \lambda) \) has a zero set for each \( x \), say \( \{ \lambda_n \} \), so that \( \varphi(x, \lambda_n(x)) = 0 \) which corresponds to eigenvalues of the Dirichlet problem for equation (7) on the closed interval \([0, x]\). Note that \( \lambda_n \neq 0 \) for any \( x \) by Sturm’s comparison theorem since we assume that \( q(x) \geq 0 \). The eigenvalues of the Dirichlet problem on \([0, x]\) for (7) are real and simple (see [2], §10.61), so we have

\[
\frac{\partial \varphi}{\partial \lambda}(x, \lambda_n(x)) \neq 0.
\]

We consider the Dirichlet problem corresponding to equation (7) on \([0, x]\) for fixed \( x, x < x_1 \). By result of [2] this problem has an infinite number of negative eigenvalues, which we denote by \( \{ \lambda_n \} \). By the Hadamard’s theorem, the product formula is of the form

\[
\varphi(x, \lambda) = C(x) \prod_{n=1}^\infty \left( 1 - \frac{\lambda}{\lambda_n} \right) \quad (8)
\]

where \( \varphi \) satisfies in initial condition (7) and \( C(x) \) is a function of \( x \) and independent of \( \lambda \), by [2], each function \( \lambda_n \) is of the form

\[
\sqrt{\lambda_n} = \frac{n\pi}{p(x)} \cdot i + O\left(\frac{1}{n}\right), \quad x < x_1
\]

where

\[
\lim_{x \to 0} \lambda_n(x) = -\infty, \quad \lambda_1 > \lambda_2 > ...
\]

and

\[
p(x) = \int_0^x |R(t)| dt. \quad (9)
\]
In order to estimate $C(x)$ we rewrite the infinite product as

$$
\varphi(x, \lambda) = C(x) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \\
= C(x) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n} \\
= C_1(x) \prod_{n=1}^{\infty} \frac{\lambda - \lambda_n}{z_n^2}
$$

(10)

with

$$
C_1(x) = C(x) \prod_{n=1}^{\infty} \frac{z_n^2}{\lambda_n(x)}
$$

(11)

where $z_n = \frac{n\pi}{p(x)}$.

It follows from the asymptotic form of eigenvalues, $-\frac{z_n^2}{\lambda_n} = 1 + O\left(\frac{1}{n^2}\right)$, then the infinite product $\prod_{n=1}^{\infty} \frac{z_n^2}{\lambda_n(x)}$ is absolutely convergent on any compact subinterval of $(0, x_1)$.

For $x \in (x_1, 1]$, fixed, the Dirichlet problem for (10) on $[0, x]$ has an infinite number of positive and negative eigenvalues, which we denote by $\{u_n\}, \{r_n\}$ respectively, it follows from results of [7]

$$
\sqrt{u_n(x)} = \frac{n\pi - \frac{\pi}{2}}{f(x)} + \left(\frac{1}{n}\right), \quad x_1 < x
$$

where

$$
f(x) = \int_{x_1}^{x} |R(t)|dt
$$

(12)

and $r_n(x)$ is of the form

$$
\sqrt{r_n(x)} = \frac{n\pi - \frac{\pi}{2}}{p(x_1)} + \left(\frac{1}{n}\right), \quad x_1 < x
$$

where $p(x)$ is defined in (12). By Hadamard’s theorem, the solution on $[0, x]$ for $x > x_1$ is of the form:

$$
\varphi(x, \lambda) = C(x) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{r_n}\right)^{\frac{1}{r_n}} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{u_n}\right).
$$

(13)
Let \( \tilde{j}_m \) be the positive zeros of \( J'_1(z) \), then
\[
\tilde{j}_m = m^2 \pi^2 - \frac{m \pi^2}{2} + O(1)
\]
for more details (See [?], §9.5.11). Consequently we have
\[
\frac{\tilde{j}_m^2}{f^2(x)u_n(x)} = 1 + O\left(\frac{1}{n^2}\right),
\]
\[
\frac{-\tilde{j}_m^2}{p^2(x) r_n(x)} = 1 + O\left(\frac{1}{n^2}\right)
\]
where \( p(x) \) and \( f(x) \) are defined in (??) and (??) respectively. Therefore the infinite products \( \prod_{n=1}^{\infty} \frac{\tilde{j}_m^2}{f^2(x)u_n(x)} \) and \( \prod_{n=1}^{\infty} \frac{-\tilde{j}_m^2}{p^2(x) r_n(x)} \) are absolutely convergent for each \( x > x_1 \). Then we may write
\[
\varphi(x, \lambda) = C_2(x) \prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_m^2} \prod_{n=1}^{\infty} \frac{u_n(x) - \lambda f^2(x)}{\tilde{j}_m^2}
\]
where
\[
C_2(x) = C(x) \prod_{n=1}^{\infty} \frac{\tilde{j}_m^2}{f^2(x)u_n(x)} \prod_{n=1}^{\infty} \frac{-\tilde{j}_m^2}{p^2(x) r_n(x)}.
\]

Now we will first approximate infinite products, then by using the asymptotic form of \( \varphi(x, \lambda) \), we will determine \( C_i, i = 1,2 \).

**Lemma 3.1.** Let \( z_m = \frac{m \pi}{p(x)} \) and \( \lambda_m(x), 1 \leq m \) be a sequence of continuous functions such that for each \( x \)
\[
\lambda_m(x) = -\frac{m^2 \pi^2}{p^2(x)} + O(1) \quad 0 < x < x_1.
\]
Then the infinite product
\[
\prod_{m=1}^{\infty} \left(\frac{\lambda - \lambda_m}{z_m^2}\right)
\]
is an entire function of \( \lambda \) for fixed \( x \) in \((0, x_1)\) whose roots are precisely \( \lambda_m(x), 1 \leq m \). Moreover
\[
\prod_{m=1}^{\infty} \left(\frac{\lambda - \lambda_m}{z_m^2}\right) = \frac{\sinh(\rho p(x))}{\rho p(x)} (1 + O\left(\frac{\log m}{m}\right)),
\]
uniformly on the circles $|\lambda| = \frac{(n + \frac{1}{2})^2 \pi^2}{\rho^2(x)}$, where $\rho = \sqrt{\lambda}$ and $p(x)$ is a function of $x$ as defined in (??)

**Proof.** See [?].

\[\square\]

**Lemma 3.2.** Let \( \tilde{\gamma}_n \) be the positive zeros of \( J'_1(z) \) and for fixed \( x \) in \((x_1, 1)\)

\[ u_n(x) = \frac{n^2 \pi^2}{f'^2(x)} - \frac{n \pi^2}{2 f'^2(x)} + O(1) \quad 1 \leq n \]

be a positive sequence of continuous functions. Then the infinite product

\[
\prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f'^2(x)}{\tilde{\gamma}^2_n} = 2J'_1(f(x)\rho)[1 + O(\frac{\log n}{n})]
\]

is an entire function of \( \lambda \) for fixed \( x \), whose roots are precisely \( u_n(x), 1 \leq n \). Moreover

\[
\prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f'^2(x)}{\tilde{\gamma}^2_n} = 2J'_1(f(x)\rho)[1 + O(\frac{\log n}{n})]
\]

uniformly on the circles $|\lambda| = \frac{n^2 \pi^2}{f'^2(x)}$.

**Proof.** See [?] and [?].

\[\square\]

**Lemma 3.3.** Let \( \tilde{\gamma}_n \) be the positive zeros of \( J'_1(z) \) and for fixed \( x \) in \((x_1, 1)\)

\[ r_n(x) = -\frac{n^2 \pi^2}{p^2(x_1)} + \frac{n \pi^2}{2 p^2(x_1)} + O(1) \quad 1 \leq n \]

be a negative sequence of continuous functions. Then the infinite product

\[
\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{\gamma}^2_n}
\]
is an entire function of $\lambda$ for fixed $x$, whose roots are precisely $r_n(x), 1 \leq n$. Moreover
\[
\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))f^2(x)}{J_n^2} = 2J_1(ip(x_1)\rho) \{1 + O\left(\frac{\log n}{n}\right)\}
\]
uniformly on the circles $|\lambda| = \frac{n\pi^2}{p\rho^2(x_1)}$.

**Proof.** See [?] and [?]. \(\square\)

**Theorem 3.4.** Let $\varphi(x, \lambda)$ be the solution of (3.1) with the initial conditions (3.2), then for $0 \leq x < x_1$,
\[
\varphi(x, \lambda) = |R(0)R(x)|^{-\frac{1}{2}} p(x) \prod_{k=1}^{\infty} \frac{\lambda - \lambda_k(x)}{z_k^2}
\]
where $p(x)$ is defined (3.3), $z_k = \frac{k\pi}{p(x)}$ and $\{\lambda_k(x)\}$ is the sequence of eigenvalues for the Dirichlet problem associated with (3.1) on $[0, x]$.

**Proof.** For $0 < x < x_1$, $\rho \in S_0$, $|\rho| \to \infty$ by virtue of (3.1) we calculate
\[
\varphi(x, \rho) = \frac{1}{2}(\rho)\frac{1}{\rho} |R(0)R(x)|^{-\frac{1}{2}} \exp\left(\frac{\pi}{2}\right) \times \exp(\rho \int_0^x |R(t)| dt) k_+(x) \kappa(x, \rho)
\]
(14)

Now from (3.2) and (3.3) we have
\[
\varphi(x, \rho) = C_1 \prod_{n=1}^{\infty} \frac{\lambda - \lambda_k}{z_k^2}
= \frac{1}{2}(\rho)\frac{1}{\rho} |R(0)R(x)|^{-\frac{1}{2}} \exp\left(\frac{\pi}{2}\right) \times \exp(\rho \int_0^x |R(t)| dt) k_+(x) \kappa(x, \rho).
\]

From Lemma 3.2, uniformly on the circles $|\lambda| = \frac{(n+1/2)^2\pi^2}{p^2(x)}$, we have
\[
\prod_{n=1}^{\infty} \frac{\lambda - \lambda_k}{z_k^2} = \frac{\sinh\left(\rho p(x)\right)}{\rho p(x)} (1 + O\left(\frac{\log k}{k}\right))
\]
whence on the circles $|\lambda| = \frac{(n+1/2)^2 \pi^2}{p^2(x)}$ we obtain

$$C_1(x) = \frac{\varphi(x, \rho)}{\prod_{k=1}^{\infty} \lambda - \lambda_k \over z_k^i} = |R(0)R(x)|^{-\frac{1}{p}}p(x).$$

as $|\rho| \rightarrow \infty$  

\[\square\]

**Theorem 3.5.** Let $\varphi(t, \lambda)$ be the solution of the initial value problem (??), (??). Then for $x_1 < x$,

$$\varphi(x, \lambda) = \frac{|R(0)R(x)|^{-\frac{1}{2}}}{2} \pi p^\frac{1}{2}(x_1) f^\frac{1}{2}(x) \csc(\frac{\pi \mu}{2})$$

$$\times \prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{r_n^2} \prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f^2(x)}{r_n^2} \tag{15}$$

\[\text{Proof.}\] For $x_1 < x < 1$, $|\rho| \rightarrow \infty$ by use of (??) it is obtained

$$\varphi(x, \lambda) = \frac{1}{4}(-i \rho)^{-1} |R(0)R(x)|^{-\frac{1}{2}}(-i) \exp(\rho p(x_1))$$

$$\times \exp(-i \rho f(x)) \csc(\frac{\pi \mu}{2}) \exp(i \frac{\pi}{4} \kappa(x, \rho))$$

$$= \frac{1}{4 \rho} |R(0)R(x)|^{-\frac{1}{2}} \exp(\rho p(x_1))$$

$$\times \cos(\rho f(x) - \frac{\pi}{4}) \csc(\frac{\pi \mu}{2})(1 + O(\frac{1}{\rho}))$$

By Lemma 3.2 and 3.3, on the circles $|\lambda| = \min\{\frac{n^2 \pi^2}{p^2(x_1)}, \frac{n^2 \pi^2}{f^2(x)}\}$ we have

$$\varphi(x, \lambda) = C_2(x) \prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{r_n^2} \prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f^2(x)}{r_n^2}$$

$$= 4 J_1'(f(x)\rho)J_1'(i p(x_1)\rho)\{1 + O(\frac{\log n}{n})\} \tag{17}$$

$$= \frac{4 \exp(\rho p(x_1))}{\pi p^\frac{1}{2}(x_1)f^\frac{1}{2}(x)\rho} \left(\cos(f(x)\rho - \frac{\pi}{4}) + O(\frac{1}{\rho})\right).$$
Now by virtue of (??), (??) and let of $|\rho| \to \infty$ it is calculated

$$C_2(x) = \frac{\varphi(x, \rho)}{\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x)) p^2(x_1)}{f_n^2} \prod_{n=1}^{\infty} \frac{|u_n(x) - \lambda| f^2(x)}{f_n^2}}$$

$$= \frac{1}{16} \pi R(0) R(x) \csc \left( \frac{\pi \mu}{2} \right) p(x_1) f^2(x)$$

$$\Box$$

4. Dual equations

By the implicit function theorem $\lambda_n(x), u_n(x)$ and $r_n(x)$ are twice continuously differentiable functions. For $x < x_1$, the condition

$$\varphi(x, \lambda_n(x)) = 0$$

gives, as usual,

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial \lambda} \lambda'_n = 0$$

(18)

and differentiating again

$$\frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \varphi}{\partial x \partial \lambda} \lambda'_n + \frac{\partial^2 \varphi}{\partial \lambda^2} (\lambda'_n)^2 + \frac{\partial \varphi}{\partial \lambda} \lambda''_n = 0$$

(19)

The first term in (??) is zero at $(x, \lambda_n(x))$ by virtue of (??) . Thus

$$2 \frac{\partial^2 \varphi}{\partial x \partial \lambda} \lambda'_n + \frac{\partial^2 \varphi}{\partial \lambda^2} (\lambda'_n)^2 + \frac{\partial \varphi}{\partial \lambda} \lambda''_n = 0.$$  

(20)

Similarly for $x_1 < x$, the conditions

$$\varphi(x, u_n(x)) = 0$$

$$\varphi(x, r_n(x)) = 0$$

give the equations

$$2 \frac{\partial^2 \varphi}{\partial x \partial \lambda} u'_n + \frac{\partial^2 \varphi}{\partial \lambda^2} (u'_n)^2 + \frac{\partial \varphi}{\partial \lambda} u''_n = 0$$

$$2 \frac{\partial^2 \varphi}{\partial x \partial \lambda} r'_n + \frac{\partial^2 \varphi}{\partial \lambda^2} (r'_n)^2 + \frac{\partial \varphi}{\partial \lambda} r''_n = 0$$

(21)

If we make use of the infinite product form of $\varphi(x, \lambda)$, substitute this in (??), in the case $x < x_1$ and in (??) for $x > x_1$ it will be
obtained the dual of the equation (22). Indeed we need the various derivatives of \( \varphi(x, \lambda) \) at the points \((x, \lambda_n(x))\) for \(x < x_1\) and at the points \((x, u_n(x))\) and \((x, r_n(x))\) for \(x > x_1\). Now, we first calculate the various derivatives of \( \varphi(x, \lambda) \) for \(x < x_1\). In this case, from (22), it can be written
\[
\varphi(x, \lambda) = C(x) \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k(x)}\right)
\]
(22)
where \(C\) is a function independent of \(\lambda\); by using (22) it is obtained
\[
C_1 = |R(0)R(x)|^{-\frac{1}{2}} p(x) = C' \prod_{k=1}^{\infty} \frac{-z_k^2}{\lambda_k}
\]
where \(z_k = \frac{k\pi}{p(x)}\) and \(p(x)\) is determined in (22). Therefore
\[
C(x) = |R(0)R(x)|^{-\frac{1}{2}} p(x) \prod_{k=1}^{\infty} \frac{-\lambda_k}{z_k^2}
\]
(23)
We calculate \(\frac{\partial \varphi}{\partial x}\), \(\frac{\partial^2 \varphi}{\partial x^2}\) and \(\frac{\partial^2 \varphi}{\partial x \partial \lambda}\) at the points \((x, \lambda_n(x))\) by using (22). In determining of \(\frac{\partial^2 \varphi}{\partial x \partial \lambda}\), the interchange of summation and differentiation in
\[
\frac{d}{dx} \sum_{k=1}^{\infty} \log(1 - \frac{\lambda}{\lambda_k(x)})
\]
is valid, because by results of [????], the differentiated series
\[
\sum_{k \neq n} \frac{-\lambda_n \lambda_k(x)}{(\lambda_k(x) - \lambda_n)\lambda_k(x)}
\]
is uniformly convergent. We define \(F_n\) by
\[
F_n = F_n(x, \lambda_n(x)) = \prod_{k \neq n, 1 \leq k} \left(1 - \frac{\lambda_n(x)}{\lambda_k(x)}\right).
\]
(24)
Since
\[
\frac{\partial \varphi}{\partial \lambda} = C \sum_{i=1}^{\infty} \frac{-1}{\lambda_i(x)} \prod_{k \neq i, 1 \leq k} \left(1 - \frac{\lambda}{\lambda_k(x)}\right),
\]
we have
\[
\frac{\partial \varphi}{\partial \lambda}(x, \lambda_n) = -\frac{C F_n}{\lambda_n(x)}.
\]
\[ \frac{\partial^2 \varphi}{\partial \lambda^2}(x, \lambda_n(x)) = \frac{2C_n}{\lambda_n(x)} \sum_{i \neq n, 1 \leq i} \frac{1}{\lambda_i} \left( 1 - \frac{\lambda_n(x)}{\lambda_i(x)} \right)^{-1}, \]

\[ \frac{\partial^2 \varphi}{\partial \lambda \partial x} = -\frac{C(x)F_n}{\lambda_n(x)} + \frac{C(x)\lambda_n(x)}{\lambda_n^2} \frac{\partial}{\partial x} + C(x)F_n \sum_{i \neq n, 1 \leq i} \frac{\lambda_i}{\lambda^2_i} \left( 1 - \frac{\lambda_n(x)}{\lambda_i(x)} \right)^{-1} \]

\[ \frac{C(x)F_n\lambda_i}{\lambda_n} \sum_{i \neq n, 1 \leq i} \frac{1}{\lambda_i^2} \left( 1 - \frac{\lambda_n(x)}{\lambda_i(x)} \right)^{-1}. \]

Placing these terms into (23) we obtain

\[ \lambda_n'' + \frac{2C(x)\lambda_n'}{\lambda_n} + 2\lambda_n \lambda_n' \sum_{i \neq n, 1 \leq i} \frac{\lambda_i}{\lambda^2_i} \left( 1 - \frac{\lambda_n(x)}{\lambda_i(x)} \right)^{-1} - 2 \frac{\lambda_n'^2}{\lambda_n} = 0. \]

(25)

Dividing the above equation by \( \lambda_n \) and integrating from a fixed number \( \alpha \neq 0 \) up to \( x \), we obtain

\[ \lambda_n(x) = \frac{\lambda_n^2(x)}{\lambda_n^2(\alpha)} \frac{\lambda_n'(\alpha)C^2(\alpha)}{C^2(x)} e^{-2S_n(x, \lambda_n)} \]

(26)

where

\[ S_n(x, \lambda_n) = \sum_{i \neq n} \int_{\alpha}^{x} \frac{\lambda_i}{\lambda_i^2} (\lambda_i - \lambda_n)^{-1} \]

(27)

and \( C(x) \) is determined in (22).

Similarly, for the case \( x > x_1 \) from (22) and theorem 2, we have

\[ \varphi(x, \lambda) = a(x) \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{r_k(x)} \right) \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{u_k(x)} \right) \]

(28)

with

\[ a(x) = \frac{1}{16} \pi |R(0)R(x)|^{-\frac{1}{2}} \csc\left( \frac{\pi \mu}{2} \right) p(x_1)^{\frac{1}{2}} f^{\frac{1}{2}}(x) \]

\[ \times \prod_{k=1}^{\infty} \frac{f^2(x)u_k(x)}{\tilde{J}_k} \prod_{k=1}^{\infty} \frac{f^2(x_1)r_k(x)}{\tilde{J}_k} \]

(29)

where \( f(x), p^2(x) \) are defined in (22), (25) and \( \tilde{J}_k, k = 1, 2, \ldots \) are the positive zeros of \( J_1^2(z) \).
as before, we calculate the various derivatives of $\varphi(x, \lambda)$ and evaluate these at the fixed points $(x, u_n(x)), (x, r_n(x))$. Since, by results of [?], the series
\[
\sum_{k \neq n} \frac{-u_n u'_k(x)}{(u_k(x) - u_n(x)) u_k(x)}
\]
is uniformly convergent, we obtain $\frac{\partial^2 \varphi}{\partial \lambda^2}$ from (??) in terms of $u_n$ and $r_n$.
Suppose
\[
G_n(x, \lambda) = \prod_{k \neq n, 1 \leq k} \left(1 - \frac{\lambda}{u_k(x)} \right)
\]
\[
H_n(x, \lambda) = \prod_{1 \leq k} \left(1 - \frac{\lambda}{r_k(x)} \right).
\]
Then,
\[
G_n = G_n(x, u_n(x)) = \prod_{k \neq n, 1 \leq k} \left(1 - \frac{u_n(x)}{u_k(x)} \right) \tag{30}
\]
\[
H_n = H_n(x, u_n(x)) = \prod_{1 \leq k} \left(1 - \frac{u_n(x)}{r_k(x)} \right) \tag{31}
\]
so that
\[
\prod_{k \neq i, 1 \leq k} \left(1 - \frac{u_n(x)}{r_k(x)} \right) = H_n(1 - \frac{u_n}{r_i})^{-1} \tag{32}
\]
We have
\[
\frac{\partial \varphi}{\partial \lambda}(x, u_n) = \frac{-a H_n G_n}{u_n(x)}
\]
\[
\frac{\partial^2 \varphi}{\partial \lambda^2}(x, u_n(x)) = \frac{2a H_n G_n}{u_n(x)} \sum_{1 \leq i} \frac{1}{r_i(x) - u_n(x)}
\]
\[
+ \frac{2a H_n G_n}{u_n(x)} \sum_{1 \leq i, i \neq n} \frac{1}{u_i(x) - u_n(x)}
\]
\[
\frac{\partial^2 \varphi}{\partial \lambda \partial x}(x, u_n(x)) = -a'(x) H_n G_n \frac{u_n'}{u_n(x)} + a(x) H_n G_n \frac{u_n'}{u_n^2(x)}
\]
\[- \frac{a(x)H_n G_n u_n'}{u_n} \sum_{1 \leq i \leq r} \frac{1}{r_i(x) - u_n(x)} \]
\[- a(x)H_n G_n \sum_{1 \leq i \leq r} \frac{u_i'}{u_i} (u_i(x) - u_n(x))^{-1} \]
\[- a(x)H_n G_n \sum_{1 \leq i \neq n} \frac{u_i'}{u_i} (u_i(x) - u_n(x))^{-1} \]
\[- a(x)H_n G_n u_n' \sum_{1 \leq i \neq n} \frac{1}{u_i(x) - u_n(x)} \]

Placing these terms into (??) we obtain
\[
\frac{2 a' u_n'}{a} + 2 a u_n u_n' \sum_{i \neq n, 1 \leq i} \frac{u_i'}{u_i} (u_i(x) - u_n(x))^{-1} + \sum_{1 \leq i \neq n} \frac{r_i'}{r_i} (r_i(x) - u_n(x))^{-1} - 2 \frac{(u_n')^2}{u_n} = 0. \tag{33}
\]

Similarly for negative eigenvalue \( r_n(x) \) we get
\[
\frac{2 a' r_n'}{a} + 2 r_n r_n' \sum_{i \neq n, 1 \leq i} \frac{r_i'}{r_i} (r_i(x) - r_n(x))^{-1} + \sum_{1 \leq i \neq n} \frac{r_i'}{r_i} (u_i(x) - r_n(x))^{-1} - 2 \frac{(r_n')^2}{r_n} = 0. \tag{34}
\]

Dividing the equation (??) by \( u_n' \), the equation (??) by \( r_n' \) and integrating from \( x \) up to 1, we obtain
\[
\frac{u_n'(x)}{u_n'(1)} = \frac{a^2(1)}{a^2(1)} e^{2T_n(x, u_n, r_n)} \tag{35}
\]
\[
\frac{r_n'(x)}{r_n'(1)} = \frac{a^2(1)}{a^2(1)} e^{2T_n(x, r_n, u_n)} \tag{36}
\]

where
\[
T_n(x, u_n, r_n) = \sum_{i \neq n} \int_x^1 \frac{u_i u_n}{u_i} (u_i - u_n)^{-1} dv + \sum_i \int_x^1 \frac{r_i' u_n}{r_i} (r_i - u_n)^{-1} dv, \tag{37}
\]
and $a(x)$ is determined in (??).

The system of equations (??), (??) and (??) are dual to the original equation (??) and involves only the functions $\lambda_n(x), \, u_n(x)$ and $r_n(x)$.

Note that the proof of existence and uniqueness solution for dual equation (??), (??) and (??) in one simple turning point case is given by authors in submitted paper [?].

References


H. Kheiri
Faculty of Mathematical Sciences
Tabriz University
Tabriz
Iran
e-mail:h-kheiri@tabrizu.ac.ir

A. Jodayree Akbarfam
Faculty of Mathematical Sciences
Tabriz University
Tabriz
Iran
e-mail:akbarfam@yahoo.com