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A WEAK APPROXIMATION FOR THE EXTREMA'S DISTRIBUTIONS OF LÉVY PROCESSES

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Abstract. Suppose that X_t is a one-dimensional and real-valued Lévy process started from $X_0 = 0$, which (**1**) its nonnegative jumps measure ν satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$ and (**2**) its stopping time $\tau(q)$ is *either* a geometric *or* an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$. This article employs the Wiener-Hopf Factorization (WHF) to find, an $L^{p^*}(\mathbb{R})$ (where $1/p^* + 1/p = 1$ and $1 < p \leq 2$), approximation for the extrema's distributions of X_t. Approximating the finite (infinite)-time ruin probability as a direct application of our findings has been given. Estimation bounds, for such approximation method, along with two approximation procedures and several examples are explored.

Keywords: Lévy processes, positive-definite function, extrema's distributions, the Fourier transform, the Hilbert transform.
MSC(2010): Primary: 60G51; Secondary: 11A55, 42A38, 60J50, 60E10.

1. Introduction

Suppose that X_t is a one-dimensional and real-valued Lévy process started from $X_0 = 0$ and defined by a triple (μ, σ, ν): the drift $\mu \in \mathbb{R}$, the volatility $\sigma \geq 0$, and the jumps measure ν which is given by a nonnegative function defined on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$. Moreover, suppose that the stopping time $\tau(q)$ is *either* a geometric *or* an exponential distribution with parameter q independent of the Lévy process X_t and $\tau(0) = \infty$. The Lévy-Khintchine formula states that the characteristic exponent ψ (i.e., $\psi(\omega) = \ln(E(\exp(i\omega X_1)))$, $\omega \in \mathbb{R}$) can be represented by

$$\psi(\omega) = i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x I_{[-1,1]}(x))\nu(dx), \ \omega \in \mathbb{R}. \tag{1.1}$$

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The extrema of the Lévy process X_t are given by

$$M_q = \sup\{X_s : \ s \leq \tau(q)\} \quad \& \quad I_q = \inf\{X_s : \ s \leq \tau(q)\}.$$

The Wiener-Hopf Factorization (WHF) is a well known technique to study the characteristic functions of the extrema random variables (see [2]). Namely, the WHF states that: (**i**) product of their characteristic functions equal to the characteristic function of Lévy process X_t at its stopping time $\tau(q)$, say $X_{\tau(q)}$ and (**ii**) random variable M_q (I_q) is infinitely divisible, positive (negative), and has zero drift.

In the cases that the characteristic function of Lévy process X_t is either a rational function or can be decomposed as a product of two sectionally analytic functions in the closed upper, i.e., $\mathbb{C}^+ := \{\lambda : \ \lambda \in \mathbb{C} \text{ and } \Im(\lambda) \geq 0\}$, and lower half complex planes, i.e., $\mathbb{C}^- := \{\lambda : \ \lambda \in \mathbb{C} \text{ and } \Im(\lambda) \leq 0\}$. Then the characteristic functions of random variables M_q and I_q can be determined explicitly (see [25]). In [17] the authors considered a Lévy process X_t which its negative jumps is distributed according to a mixture-gamma family of distributions and its positive jumps measure has an arbitrary distribution. They established that the characteristic function of such a Lévy process can be decomposed as a product of a rational function in an arbitrary function, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. They also provided an analog result for a Lévy process whose its corresponding positive jumps measure follows from a mixture-gamma family of distributions while its negative jumps measure is an arbitrary one, more details can be found in [18].

Unfortunately, in the most situations, the characteristic function of the process *neither* is a rational function *nor* can be decomposed as a product of two sectionally analytic functions in $\mathbb{C}^+$ and $\mathbb{C}^-$. Therefore, the characteristic functions of M_q and I_q should be expressed in terms of a Sokhotskyi-Plemelj integral (see Equation, 2.1). But, this form, also, presents some difficulties in numerical work due to slow evaluation and numerical problems caused by singularities near the integral contour (see [12]). To overcome these difficulties, an appropriate (in some sense) approximation method has to be considered. It is well known that a Lévy process X_t which its jumps distribution follows from the phase-type distribution has a rational characteristic function (see [8]). In [14] the authors utilized this fact and approximated a jumps measure ν of a ten-parameter Lévy processes (named $\beta-$family of Lévy process) by a sequence of the phase-type measures. Then he determined the characteristic functions of random variables M_q and I_q, approximately. [15] extended [14]'s findings to class of Meromorophic Lévy processes. Moreover, [16] provided a uniform approximation for the cumulative distribution function of $M_{\tau(q)}$ whenever X_t is a symmetric Lévy process. [13] employed the Shannon sampling method to find the distributions of the extrema for a wide class of Lévy processes.

This article begins with an extension of [12]'s results for the multiplicative WHF

$$\Phi^{+}(\omega)\Phi^{-}(\omega) = g(\omega)\ \omega \in \mathbb{R}, \tag{1.2}$$

where $g(\cdot)$ is a given function with some certain conditions (see below) and $\Phi^{\pm}(\cdot)$ are to be determined. Then it utilizes such results to approximate the extrema's distributions of a class of Lévy processes. Estimation bounds, for such approximate method, along with two approximation procedures are given.

Section 2 collects some useful elements for other sections. Moreover, it provides an $L^{p}(\mathbb{R})$, $1 < p \leq 2$ approximation technique for solving a multiplicative WHF (1.2). Section 3 considers the problem of approximating the extrema's density functions for a class of Lévy processes. Then it develops two approximate techniques for situations where those density functions cannot be determined, explicitly. Error bounds for such techniques are given. Several examples are given in Sections 4. Section 5 provides concluding remarks along with some suggestions for other application of our techniques.

2. Preliminaries

The *Sokhotskyi-Plemelj integral* for $s(\cdot)$, which satisfies the Hölder condition, is defined by a principal value integral, as follows

$$\phi_s(\lambda) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{s(x)}{x-\lambda}dx, \text{ for } \lambda \in \mathbb{C}. \tag{2.1}$$

It is worth mentioning that, the Sokhotskyi-Plemelj integral can be existed for non-integrable function. Therefore, the Sokhotskyi-Plemelj integral $\phi_s(\cdot)$ should be viewed as a principal value integral over $\mathbb{R}$.

The radial limits of the Sokhotskyi-Plemelj integral of $s(\cdot)$, are given by $\phi_s^{\pm}(\omega) = \lim_{\lambda\to\omega+i0^{\pm}} \phi_s(\lambda)$ and satisfy the following *jump formulas:* (**1**) $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + \phi_s(\omega)$, for $\omega \in \mathbb{R}$ and (**2**) $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + H_s(\omega)/(2i)$, where $H_s(\omega)$ stands for the *Hilbert transform* of $s(\cdot)$ and $\omega \in \mathbb{R}$.

The multiplicative WHF is the problem of finding an analytic and bounded, except on the real line, function $\Phi(\cdot)$ where its upper and lower radial limits $\Phi^{\pm}(\cdot)$ satisfy Equation (1.2). Given function $g(\cdot)$ is a bounded above by 1, zero index[1], continuous, and positive function which satisfies the Hölder condition on $\mathbb{R}$, $g(0) = 1$, and $g(\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

The following extends [12]'s results to the multiplicative WHF (1.2). We begin with what we term the Resolvent Equation for Sokhotskyi-Plemelj integrals.

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[1] The index of a complex-valued function f on a smooth oriented curve Γ, such that $f(\Gamma)$ is closed and compact, is defined to be the winding number of $f(\Gamma)$ about the origin (see [24], §1), for more technical details.

Lemma 2.1. *The Sokhotskyi-Plemelj integral of a function* $f(\cdot)$ *satisfies* $\phi_f(\lambda) - \phi_f(\mu) = (\lambda - \mu)\phi_{\frac{f(x)}{x-\lambda}}(\mu)$, *where* λ *and* μ *are real or complex values.*

Proof. In general, $(x-\lambda)^{-1} - (x-\mu)^{-1} = (\lambda-\mu)(x-\mu)^{-1}(x-\lambda)^{-1}$. Then see [9], we have an equation of Cauchy integrals, where $\Gamma = \mathbb{R}$:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-\lambda}dx - \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-\mu}dx = \frac{\lambda-\mu}{2\pi i}\int_\Gamma \frac{f(x)}{(x-\mu)(x-\lambda)}dx.$$

The above is valid only for λ and μ not on the real line. However, by Equation (2.1) the values of $\phi_f(\cdot)$ on the real line are obtained by averaging the limit from above, $\phi_f^+(\cdot)$, and the limit from below, ϕ_f^-. We thus obtain the stated equation in all cases. ☐

Lemma 2.2. *Suppose* $\Phi^\pm(\cdot)$ *are sectionally analytic functions that satisfy the multiplicative WHF* (1.2). *Moveover, suppose that given function* $g(\cdot)$ *is a zero index function which satisfies the Hölder condition and* $g(0) = 1$.[2] *Then* $\Phi^\pm(\lambda) = \exp\{\pm(\phi_{\ln g}(\lambda) - \phi_{\ln g}(0))\}$, *where* $\phi_{\ln g}(\cdot)$ *stands for the Sokhotskyi-Plemelj integral of* $\ln g(\cdot)$.

Proof. Using the [11]'s suggestion for solving the homogeneous WHF (1.2) gives, see also [18]:

$$\Phi^\pm(\lambda) \quad = \quad \exp\{\pm\frac{\lambda}{2\pi i}\int_\mathbb{R} \frac{\ln g(x)/x}{x-\lambda}dx\}.$$

Lemma 2.1 with $f \equiv \ln g$ gives $\phi_{\ln g}(\lambda) - \phi_{\ln g}(\mu) = (\lambda-\mu)\phi_{\frac{\ln g(x)}{x-\lambda}}(\mu)$. Letting λ goes to zero from the above, in the complex plane, and using the fact that $\ln g(0) = 0$, Equation (2.1) lets us to conclude that $\phi_{\ln g}(0) - \phi_{\ln g}(\mu) = -\mu\phi_{\frac{\ln g(x)}{x}}(\mu)$. Substituting this into the above equation for $\Phi^\pm(\cdot)$ gives our claimed result. ☐

Using the jump formula one can conclude that

$$\Phi^\pm(\omega) \quad = \quad \sqrt{g(\omega)}\exp\{\pm\frac{i}{2}(H_{\ln g}(0) - H_{\ln g}(\omega))\}, \tag{2.2}$$

where $H_{\ln g}(\cdot)$ stands for the Hilbert transform of $\ln g(\cdot)$.

The Carlemann's method explores a situation which one may evaluate solutions of the multiplicative WHF (1.2) directly, rather than using the Sokhotskyi-Plemelj integrations. The Carlemann's method states that: if $g(\cdot)$ can be decomposed as a product of two sectionally analytic functions $g^+(\cdot)$ and $g^-(\cdot)$, respectively in $\mathbb{C}^+$ and $\mathbb{C}^-$, then solutions of the multiplicative WHF (1.2) are given by $\Phi^+ \equiv g^+$ and $\Phi^- \equiv g^-$.

---

[2] The condition $g(0) = 1$ does not always hold in the multiplicative WHF, but happen to arise in our application, and can lead to complications. Lemma 2.2 is used to simplifying this case.

In a situation that $g(\cdot)$ is a rational function $\frac{P(x)}{Q(x)}$ that has no poles or zeros on $\mathbb{R}$, using the Carlemann's method, we may conclude that the multiplicative WHF problem can be solved by factoring the polynomial P (Q), and then let P^+ (Q^+) be the product of those factors of P (Q) that have zeros in $\mathbb{C}^-$, and P^- (Q^-) be the product of those factors that have zeros in $\mathbb{C}^+$. Then setting $g^+(x) = \frac{P^+(x)}{Q^+(x)}$ and $g^-(x) = \frac{P^-(x)}{Q^-(x)}$ gives us (up to a scalar multiple) our desired factorization.

The *Hausdorff-Young* theorem (see [23]) states that: if $s(\cdot)$ is an $L^p(\mathbb{R})$ function, then $s(\cdot)$ and its corresponding the Fourier transform, say $\hat{s}(\cdot)$, satisfy $||\hat{s}||_{p^*} \leq (2\pi)^{-1/p}||s||_p$, where $1 \leq p \leq 2$ and $1/p + 1/p^* = 1$. From the Hausdorff-Young Theorem, one can observe that if $\{s_n(\cdot)\}$ is a sequence of functions converging in $L^p(\mathbb{R})$, $1 \leq p \leq 2$, to $s(\cdot)$. Then the Fourier transforms of the $s_n(\cdot)$ converges in $L^{p^*}(\mathbb{R})$, to the Fourier transform of $s(\cdot)$, where $1/p + 1/p^* = 1$. The converse is false.

A similar property for the Hilbert transform is well known as the Titmarsh-Riesz Lemma (see [23]). The Titmarsh-Riesz Lemma says that: if $s(\cdot)$ is an $L^p(\mathbb{R})$ function, where $1 < p \leq 2$. Then $||H_s||_p \leq tan(\pi/(2p))||s||_p$, where $H_s(\cdot)$ stands for the Hilbert transform of $s(\cdot)$. Using the Titmarsh-Riesz Lemma, one may conclude that if $\{f_n(\cdot)\}$, is a sequence of functions which converge, in $L^p(\mathbb{R})$, $1 < p \leq 2$, to $f(\cdot)$. Then the Hilbert transforms $H(f_n)$ converge, in $L^p(\mathbb{R})$, $1 < p \leq 2$, to the Hilbert transform of $f(\cdot)$.

The well known Paley-Wiener theorem states that: if $F(\cdot)$ is a function in $L^2(\mathbb{R})$. Then the real-valued function $F(\cdot)$ vanishes on $\mathbb{R}^-$ if and only if the Fourier transform $F(\cdot)$, say, $\hat{F}(\cdot)$ is holomorphic on $\mathbb{C}^+$ and the $L^2(\mathbb{R})$-norm of the functions $x \mapsto \hat{F}(x + iy_0)$ are uniformly bounded for all $y_0 \geq 0$.

The following lemma, from [12], recalls some further useful properties of functions in $L^p(\mathbb{R})$, for $1 < p \leq 2$, space.

Lemma 2.3. *Suppose $s(\cdot)$ and $r(\cdot)$ are two $L^p(\mathbb{R})$, $1 < p \leq 2$, functions. Then*

- **i):** $||\sqrt{s} - \sqrt{r}||_p \leq \frac{1}{2\sqrt{a}}||s - r||_p$, *whenever both $s(\cdot)$ and $r(\cdot)$ are bounded, above by a, functions;*
- **ii):** $||\ln s - \ln r||_p \leq a^{-1}||s - r||_p$, *whenever both $s(\cdot)$ and $r(\cdot)$ are positive-valued and bounded, above by a, functions ;*
- **iii):** $||e^{-is/2} - e^{-ir/2}||_p \leq \frac{1}{2}||s - r||_p$, *whenever $s(\cdot)$ and $r(\cdot)$ are real-valued functions.*

In many situations, WHF (1.2) cannot be solved explicitly and has to be solved approximately (see [12]). The following develops an approximation technique to solve a multiplicative WHF (1.2).

Theorem 2.4. *Suppose $\Phi^{\pm}(\cdot)$ are two sectionally analytic functions satisfying the multiplicative WHF (1.2) where*

A_1**):** *$g(\cdot)$ is real, positive, bounded above by a, index zero, satisfies the Hölder condition, and $g(0) = 1$;*

A_2**):** *There exist a sequence of functions $g_n(\cdot)$ where converge, in $L^p(\mathbb{R})$, $1 < p \le 2$, to $g(\cdot)$.*

Then $\Phi^{\pm}(\cdot)$ can be approximated by $\Phi^{\pm}_n(\cdot)$, where

$$||\Phi^{\pm}_n - \Phi^{\pm}||_p \quad \le \quad \frac{1}{2}tan(\frac{\pi}{2p})||g_n - g||_p^2 + (tan(\frac{\pi}{2p}) + \frac{1}{2})||g_n - g||_p.$$

Proof. Set $k(\omega) := -H_{\ln g}(\omega) + H_{\ln g}(0)$ and $k_n(\omega) := -H_{\ln g_n}(\omega) + H_{\ln g_n}(0)$. Now, from Equation (2.2) and Lemma 2.3 observe that $||\Phi^{\pm}_n - \Phi^{\pm}||_p$

$$\begin{aligned}
&= ||\sqrt{g_n}e^{\pm ik_n/2} - \sqrt{g}e^{\pm ik/2}||_p \\
&\le [||\sqrt{g_n} - \sqrt{g}||_p + ||\sqrt{g}||_p]\, ||e^{\pm ik_n/2} - e^{ik/2}||_p + |e^{\pm ik/2}|||\sqrt{g_n} - \sqrt{g}||_p \\
&\le \frac{1}{2}[||\sqrt{g_n} - \sqrt{g}||_p + ||\sqrt{g}||_p]\, || - H_{\ln g_n}(\omega) + H_{\ln g_n}(0) + H_{\ln g}(\omega) - H_{\ln g}(0)||_p \\
&\quad +|e^{\pm ik/2}|||\sqrt{g_n} - \sqrt{g}||_p \\
&\le [||\sqrt{g_n} - \sqrt{g}||_p + ||\sqrt{g}||_p]\, ||H_{\ln g_n} - H_{\ln g}||_p + ||\sqrt{g_n} - \sqrt{g}||_p \\
&\quad \text{since } k \text{ and } k_n \text{ are real-valued functions} \\
&\le [||\sqrt{g_n} - \sqrt{g}||_p + ||\sqrt{g}||_p]\, tan(\frac{\pi}{2p})||\ln(g_n) - \ln(g)||_p + ||\sqrt{g_n} - \sqrt{g}||_p \\
&\le tan(\frac{\pi}{2p})\left[\frac{1}{2}||g_n - g||_p + 1\right]||g_n - g||_p + \frac{1}{2}||g_n - g||_p \\
&= \frac{1}{2}tan(\frac{\pi}{2p})||g_n - g||_p^2 + (tan(\frac{\pi}{2p}) + \frac{1}{2})||g_n - g||_p.
\end{aligned}$$

□

Now, we recall definition of the positive-definite function which plays a vital role in the rest of this article.

Definition 2.5. A positive-definite function is a complex-valued function $f : \mathbb{R} \to \mathbb{C}$ such that for any real numbers $x_1, \ldots, x_n$, the $n \times n$ square matrix $A = (f(x_i - x_j))_{i,j=1}^n$ is a positive semi-definite matrix.

In the theory of Fourier transforms, it is well known that "$f(\cdot)$ is a continuous positive-definite function on $\mathbb{R}$ if and only if its corresponding Fourier transform is a (positive) measure", see [5] for more details.

Lemma 2.6. *Suppose $\phi : \mathbb{R} \to \mathbb{C}$ is a positive-definite function for which two equations $q_1 - \phi(\omega) = 0$ and $1 - q_2 \exp\{-\phi(\omega)\} = 0$ have not any solution on $\mathbb{R}$, where $q_1 > 0$ and $q_2 \in (0, 1)$. Then $h_1(\omega) = q_1/(q_1 - \phi(\omega))$ and $h_2(\omega) = (1 - q_2)/(1 - q_2 \exp\{-\phi(\omega)\})$ are positive-definite functions.*

Proof. Using Taylor expansion of $q_1/(q_1-x)$ and $(1-q_2)/(1-q_2\exp\{-x\})$, at zero, one can, respectively, obtain $h_1(\cdot)$ and $h_2(\cdot)$ as

$$\begin{aligned} h_1(\omega) &= \sum_{k=1}^{\infty}\frac{\phi^k(\omega)}{q_1^k} \\ h_2(\omega) &= 1+\frac{q_2}{q_2-1}\phi(\omega)+\frac{q_2(q_2+1)}{2(q_2-1)^2}\phi^2(\omega)+\frac{q_2(q_2^2+4q_2+1)}{6(q_2-1)^3}\phi^3(\omega)+\cdots \end{aligned}$$

Now, the desired proof arrives from the fact that the product of two positive-definite functions is again a positive-definite function (see [31]). ☐

Now, we provide two classes of positive-definite rational functions which play a vital role in numerical section of this article.

Lemma 2.7. *Consider the following two class of rational functions $\mathcal{D}$ and $\mathcal{D}^*$.*

$$\begin{aligned} \mathcal{D}: &= \{r(\omega);\ r(\omega)=A_0+\sum_{k=1}^{n}\sum_{j=1}^{m_k}\sum_{l=1}^{4}C_{kj}r_{lk}^{j}(\omega);\ A_0\ \&\ C_{kj}\geq 0\}; \\ \mathcal{D}^{\star}: &= \{r(\omega);\ r(\omega)=A_0+\sum_{k=1}^{n}\sum_{l=1}^{2}C_{kj}r_{lk}(\omega);\ A_0\ \&\ C_{kj}\geq 0\}, \end{aligned}$$

where

$$\begin{aligned} r_{1k}(\omega) &= \frac{1}{i\omega+\beta_k}\ (\textit{where } \beta_k>0); \\ r_{2k}(\omega) &= \frac{1}{-i\omega+\beta_k}\ (\textit{where } \beta_k>0); \\ r_{3k}(\omega) &= \frac{1}{(i\omega+\beta_k)(i\omega+\beta_k+\alpha_k i)(i\omega+\beta_k-\alpha_k i)}\ (\textit{where } \alpha_k,\ \beta_k>0); \\ r_{4k}(\omega) &= \frac{1}{(-i\omega+\beta_k)(-i\omega+\beta_k+\alpha_k i)(-i\omega+\beta_k-\alpha_k i)}\ (\textit{where } \alpha_k,\ \beta_k>0). \end{aligned}$$

Then

- **(i):** *the Fourier transform of functions in $\mathcal{D}$ are nonnegative and real-valued.*
- **(ii):** *The Fourier transform of functions in $\mathcal{D}^*$ are nonnegative, real-valued, and completely monotone functions.*

Proof. Nonnegativity of the Fourier transform of functions in $\mathcal{D}$ (or $\mathcal{D}^*$) arrives from the fact that r_{lk} (for $l=1,\ldots,4$) and their powers are positive-definite rational functions. Now, from Bernstein's theorem, we observe that a real-valued function defined on $\mathbb{R}^+$ is a completely monotone function, whenever it is a mixture of exponential functions, see [30] for more details. ☐

The following can be concluded from the properties of the WHF given in [2].

Lemma 2.8. *Suppose $g(\cdot)$ in the multiplicative WHF (1.2) is a positive-definite function. Then the solutions, of the multiplicative WHF (1.2), $\Phi^{\pm}(\cdot)$ are two positive-definite functions.*

Proof. First, using the multiplicative WHF (1.2), the characteristic function of Lévy process X_t at its stopping time $\tau(q)$, say $X_{\tau(q)}$, can be decomposed as a product of the characteristic functions of two random variables I_q and M_q, see [2]. Moreover, [4]'s theorem states that "$\phi : \mathbb{R}^n \to \mathbb{C}$ is the characteristic function of some random variable if and only if $\phi(\cdot)$ is positive-definite, continuous at the origin, with $\phi(0) = 1$". The desired proof follows from these observations. ☐

One may readily observe that the characteristic function of the mixed gamma family of distributions (given below) are belong to $\mathcal{D}$.

Definition 2.9 (Mixed gamma family of distributions)**.** A nonnegative random variable X is said to be distributed according to a mixed gamma distribution if its density function is given by

$$p(x) = \sum_{k=1}^{\nu}\sum_{j=1}^{n_\nu} c_{kj}\frac{\alpha_k^j x^{j-1}}{(j-1)!}e^{-\alpha_k x}I_{[0,\infty)}(x) + \sum_{k=1}^{\nu^*}\sum_{j=1}^{n_{\nu^*}} c_{kj}^*\frac{\beta_k^j(-x)^{j-1}}{(j-1)!}e^{\beta_k x}I_{(-\infty,0]}(x)$$

where c_{k_j} and α_k are positive value satisfying $\sum_{k=1}^{\nu}\sum_{j=1}^{n_\nu} c_{k_j} = 1$.

From [3] we recall some useful properties of a characteristic function, which plays an important role in the folowwing sections.

Lemma 2.10. *Suppose $\hat{p}(\cdot)$ stands for the characteristic function of a distribution. Then*

- **(i):** *$\hat{p}(\cdot)$ is a positive-definite function;*
- **(ii):** *$\hat{p}(\cdot)$ is a positive-definite rational function whenever its characteristic function belongs to $\mathcal{D}$;*
- **(iii):** *$\hat{p}(0) = 1$ and the norm of $\hat{p}(\cdot)$ is bounded by 1.*

The next section provides an application of Theorem 2.4 to the problem of finding distributions of the extrema of Lévy process X_t, approximately.

3. Main results

The following lemma restates the characteristic function of Lévy process X_t at its stopping time $\tau(q)$, say $X_{\tau(q)}$.

Lemma 3.1. *Suppose $X_{\tau(q)}$ represents Lévy process X_t at its stopping time $\tau(q)$. Then the characteristic function of $X_{\tau(q)}$ can be restated as:*

- (i) $q/(q-\psi(\omega))$, *for an exponential stopping time $\tau(q)$ with parameter $q > 0$;*
- (ii) $(1-q)/(1-q\exp\{-\psi(\omega)\})$, *for a geometric stopping time $\tau(q)$ with parameter $q \in (0,1)$.*

Proof. Conditioning on stopping time $\tau(q)$, one may restates the characteristic function of $X_{\tau(q)}$ as:

For part (i):

$$\begin{aligned} E(e^{i\omega X_{\tau(q)}}) &= \int_0^\infty E(e^{i\omega X_{\tau(q)}}|\tau(q)=t)f_{\tau(q)}(t)dt = \int_0^\infty E(e^{i\omega X_t})qe^{-qt}dt \\ &= \int_0^\infty e^{\psi(\omega)t}qe^{-qt}dt = \frac{q}{q-\psi(\omega)}; \end{aligned}$$

For part (ii):

$$\begin{aligned} E(e^{i\omega X_{\tau(q)}}) &= \sum_{n=0}^{\infty} E(e^{i\omega X_{\tau(q)}}|\tau(q)=n)P(\tau(q)=n) = \sum_{n=0}^{\infty} E(e^{i\omega X_n})(1-q)q^n \\ &= \sum_{n=0}^{\infty} e^{\psi(\omega)n}(1-q)q^n = \frac{1-q}{1-q\exp\{-\psi(\omega)\}}, \end{aligned}$$

where for both cases, the second equality arrives from the fact that X_t and $\tau(q)$ are independent and the third equality obtains from the definition of the characteristic exponent ψ and infinitely divisibility of Lévy process X_t. $\square$

The following theorem represents an error bound for approximating density functions of extrema of a Lévy process.

Theorem 3.2. *Suppose X_t is a Lévy process defined by a triple (μ, σ, ν). Moreover, suppose that:*

- **(I):** *The stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$;*
- **(II):** *The $r_n(dx)$ are a sequence of positive-definite rational functions which converge, in $L^{p^*}(\mathbb{R})$ (where $1/p^* + 1/p = 1$ and $1 < p \leq 2$), to characteristic exponent $q/(q - \psi(dx))$ (or $(1 - q)/(1 - q\exp\{-\psi(dx)\})$ for geometric stoping time).*

Then the density function of the suprema and infima of the Lévy process X_t, denoted f_q^+ and f_q^-, respectively, can be approximated, in $L^{p^*}(\mathbb{R})$, by a sequence of the density functions $f_{q,n}^+$ and $f_{q,n}^-$ where:

(i): *For exponentially distributed stopping time $\tau(q)$, with $q > 0$,*

$$||f_q^\pm - f_{q,n}^\pm||_{p^*} \leq \frac{1}{2}tan(\frac{\pi}{2p^*})||r_n - \frac{q}{q-\psi}||_{p^*}^2 + (tan(\frac{\pi}{2p^*}) + \frac{1}{2})||r_n - \frac{q}{q-\psi}||_{p^*};$$

(ii): *For geometric stopping time $\tau(q)$, with $q \in (0, 1)$,*

$$||f_q^\pm - f_{q,n}^\pm||_{p^*} \leq \frac{1}{2}tan(\frac{\pi}{2p^*})||r_n - \frac{1-q}{1-qe^{-\psi}}||_{p^*}^2 + (tan(\frac{\pi}{2p^*}) + \frac{1}{2})||r_n - \frac{1-q}{1-qe^{-\psi}}||_{p^*}.$$

Proof. From [2] and Lemma 3.1, one can observe that the Fourier transform of the density functions of random variables M_q and I_q, say Φ^+ and Φ^- respectively, satisfy *either* the multiplicative WHF $\Phi^+(\omega)\Phi^-(\omega) = q/(q - \psi(\omega))$, where $\omega \in \mathbb{R}$ (for exponentially distributed stopping time) *or* the multiplicative WHF $\Phi^+(\omega)\Phi^-(\omega) = (1-q)/(1-q\exp\{-\psi(\omega)\})$, where $\omega \in \mathbb{R}$ (for geometric stopping time). Now, from the fact that the expressions $q/(q-\psi(\cdot))$ and $(1-q)/(1-q\exp\{-\psi(\cdot)\})$ are the characteristic function of the Lévy process X_t, at exponential and geometric stopping time, respectively, we observe that both expressions are bounded above by 1 because of the property of the characteristic function given by Lemma (2.10, part ii). For part (i), from Theorem 2.4 observe that

$$||\Phi_n^{\pm} - \Phi^{\pm}||_p \quad \leq \quad \frac{1}{2}tan(\frac{\pi}{2p})||r_n - \frac{q}{q-\psi}||_p^2 + (tan(\frac{\pi}{2p}) + \frac{1}{2})||r_n - \frac{q}{q-\psi}||_p.$$

The rest of proof arrives from an application of the Hausdorff-Young Theorem. The proof of part (ii) is quite similar. □

Remark 3.3. In the case that the distribution of I_q or M_q has an atom at $x=0$, then it corresponding probability mass function at zero can be found, approximately, by

$$P(I_q = 0) = \lim_{\omega\to\infty} \Phi^-(-i\omega) \quad \text{and} \quad P(M_q = 0) = \lim_{\omega\to\infty} \Phi^+(i\omega).$$

Using the fact that the Compound Poisson has bounded characteristic exponent $\psi(\cdot)$. The following formulates result of the above theorem in terms of the jumps measure $\nu(dx)$.

Theorem 3.4 (Compound Poisson). *Suppose X_t is a Compound Poisson process defined by a triple (μ, σ, ν). Moreover, suppose that*

- **(I):** *the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$;*
- **(II):** *the $\nu_n(dx)$ are a sequence of the density functions which converge in $L^2(\mathbb{R})$, to jumps measure ν and $\int_{-1}^{1} x\nu_n(dx) = \int_{-1}^{1} x\nu(dx)$.*

Then the density functions of the suprema and infima of the Compound Poisson process X_t, denoted by $f_q^+(\cdot)$ and $f_q^-(\cdot)$, respectively, can be approximated by a sequence of the density functions $f_{q,n}^+(\cdot)$ and $f_{q,n}^-(\cdot)$ where:

- **(i):** *For exponentially distributed stopping time $\tau(q)$,*

$$||f_q^{\pm} - f_{q,n}^{\pm}||_2 \leq \frac{1}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3}{2q}||\nu_n - \nu||_2;$$

- **(ii):** *For geometric stopping time $\tau(q)$,*

$$||f_q^{\pm} - f_{q,n}^{\pm}||_2 \leq \frac{(1-q)^2}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3(1-q)}{2q}||\nu_n - \nu||_2.$$

Proof. Suppose that $\psi_n(\cdot)$ sequence of the characteristic exponent corresponding to $\nu_n(dx)$. For part (i) using Theorem 3.2, one may conclude that

$$\begin{aligned} ||\Phi_n^{\pm} - \Phi^{\pm}||_2 &\leq \frac{1}{2}||\frac{q}{q-\psi_n} - \frac{q}{q-\psi}||_2^2 + \frac{3}{2}||\frac{q}{q-\psi_n} - \frac{q}{q-\psi}||_2 \\ &\leq \frac{1}{2q^2}||\psi_n - \psi||_2^2 + \frac{3}{2q}||\psi_n - \psi||_2 \\ &\leq \frac{1}{4\pi q^2}||\nu_n - \nu||_2^2 + \frac{3}{q\sqrt{8\pi}}||\nu_n - \nu||_2. \end{aligned}$$

The second inequality arrives from the fact that the characteristic function $q/(q-\psi(\cdot))$ is bounded above by 1, while the third inequality comes from the Levy-Khintchine representation (1.1) along with conditions A_2 and the Hausdorff-Young Theorem. The rest of proof arrives from an application of the Hausdorff-Young Theorem. The proof of part (ii) is quite similar. $\square$

4. Application to the finite (infinite)-time ruin probability

Suppose surplus process of an insurance company can be restated as

$$U_t = u + X_t, \tag{4.1}$$

where Lévy process X_t and $u > 0$ stands for initial wealth/reserve of the process.

The finite-time ruin probability for the such surplus process is denoted by $\mathbf{R}^{(q)}(u)$ and defined by

$$\mathbf{R}^{(q)}(u) = P(T \leq \tau_q | U_0 = u),$$

where T is the hitting time, i.e., $T := \inf\{t : U_t \leq 0 | U_0 = u\}$ and τ_u is a random stoping time. Such the stoping time has been distributed corroding to *either* an exponential distribution (with mean $1/q$) *or* a geometric distribution (with mean $(1-q)/q$).

The infinite-time ruin probability for the surplus process (4.1) is denoted by $\mathbf{R}(u)$ and defined by

$$\mathbf{R}(u) = P(T < \infty | U_0 = u).$$

The infinite-time ruin probability $\mathbf{R}(u)$, can also be evaluated by $\mathbf{R}(u) = \lim_{q\to 0}\mathbf{R}^{(q)}(u)$.

Using [1, Lemma 1] with setting $\beta = 0$ and replacing X by $-X$: in a situation that the infima density function f_q^- of the Lévy process X_t is available, the finite-time ruin probability under the above surplus process can be restated as

$$\mathbf{R}^{(q)}(u) = P(I_q < -u) = \int_{-\infty}^{-u} f_q^-(y)dy.$$

Now using the $L_p(\mathbb{R})-$norm for an integral operator (see [7, Theorem 3.36]), one may restate results of Theorem 3.2 and Theorem 3.4 for approximating finite-time ruin probability under the surplus process (4.1) as the following two corollaries.

Corollary 4.1. *Suppose X_t in the surplus process (4.1) is a Lévy process defined by a triple (μ, σ, ν). Moreover, suppose that:*

- **(I):** *The stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$;*
- **(II):** *The $r_n(dx)$ is a sequence of positive-definite rational functions which converges, in $L^{p^*}(\mathbb{R})$ (where $1/p^* + 1/p = 1$ and $1 < p \leq 2$), to characteristic exponent $q/(q - \psi(dx))$ (or $(1-q)/(1 - q\exp\{-\psi(dx)\})$ for geometric stoping time).*

Then the finite-time ruin probability under the surplus process (4.1), say $\mathbf{R}^{(q)}(u)$, can be approximated, in $L^{p^*}(\mathbb{R})$, by a sequence of the ruin probability, say $\mathbf{R}_n^{(q)}(u)$, where:

- **(i):** *For exponentially distributed stopping time $\tau(q)$, for $q > 0$,*

$$||\mathbf{R}^{(q)} - \mathbf{R}_n^{(q)}||_{p^*} \leq \frac{1}{2} tan(\frac{\pi}{2p^*})||r_n - \frac{q}{q-\psi}||^2_{p^*} + (tan(\frac{\pi}{2p^*}) + \frac{1}{2})||r_n - \frac{q}{q-\psi}||_{p^*};$$

- **(ii):** *For geometric stopping time $\tau(q)$, for $q \in (0,1)$,*

$$||\mathbf{R}^{(q)} - \mathbf{R}_n^{(q)}||_{p^*} \leq \frac{1}{2} tan(\frac{\pi}{2p^*})||r_n - \frac{1-q}{1-qe^{-\psi}}||^2_{p^*} + (tan(\frac{\pi}{2p^*}) + \frac{1}{2})||r_n - \frac{1-q}{1-qe^{-\psi}}||_{p^*}.$$

Corollary 4.2. *(Compound Poisson) Suppose X_t in the surplus process (4.1) is a Compound Poisson process defined by a triple (μ, σ, ν). Moreover, suppose that*

- **(I):** *the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$;*
- **(II):** *the $\nu_n(dx)$ are a sequence of the density functions which converge in $L^2(\mathbb{R})$, to jumps measure ν and $\int_{-1}^{1} x\nu_n(dx) = \int_{-1}^{1} x\nu(dx)$.*

Then the finite-time ruin probability under the surplus process (4.1), say $\mathbf{R}^{(q)}(u)$, can be approximated, in $L^{p^*}(\mathbb{R})$ sense, by a sequence of the finite-time ruin probability, say $\mathbf{R}_n^{(q)}(u)$, where:

- **(i):** *For the exponentially distributed stopping time $\tau(q)$,*

$$||\mathbf{R}^{(q)} - \mathbf{R}_n^{(q)}||_2 \leq \frac{1}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3}{2q}||\nu_n - \nu||_2;$$

- **(ii):** *For the geometric stopping time $\tau(q)$,*

$$||\mathbf{R}^{(q)} - \mathbf{R}_n^{(q)}||_2 \leq \frac{(1-q)^2}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3(1-q)}{2q}||\nu_n - \nu||_2.$$

It is worth mentioning that the above results may be obtained for the infinite-time ruin probability by letting $q \to 0$.

The next section provides some practical applications of the above results.

5. **Examples**

In the first step, this section provides two particle procedures for the problem of finding the density functions of the suprema and infima of a Lévy process.

Using the fact that the characteristic exponent $\psi(i\omega)$, $\omega \in \mathbb{R}$, is a real-valued function, (see [2]) along with Lemma 2.8, we suggest the following two procedures to generate approximation density functions for M_q and I_q. Suppose X_t is a Meromorophic Lévy process[3] with the characteristic exponents $\psi(\cdot)$. Moreover, suppose that the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0)=\infty$. Then by the following steps, one can approximate, in $L^{p^*}(\mathbb{R})$ (where $1/p^*+1/p=1$ and $1<p\leq 2$), the density functions of the extrema random variables M_q and I_q.

Step (1-1): Find out all zeros and poles of $q/(q-\psi(\omega))$ (or $(1-q)/(1-q\exp\{-\psi(\omega)\})$);

Step (1-2): Define $f^+(\omega)$ as product over all zeros/poles lying in $\mathbb{C}^-$ and $f^-(\omega)$ as product over all zeros/poles lying in $\mathbb{C}^+$;

Step (2): Determine error of approximating $q/(q-\psi(\omega))$ (or $(1-q)/(1-q\exp\{-\psi(\omega)\})$) by $f^+(\omega)f^-(\omega)$;

Step (3): Obtain the density functions of M_q and I_q by the inverse Fourier transform of $f^+(\cdot)$ and $f^-(\cdot)$, respectively.

Proof. For an exponential stopping time, [15] showed that zeros and poles of $q/(q-\psi(\omega))$, respectively, appear as $\{-i\alpha_n, i\alpha_n\}$ and $\{-i\beta_n, i\beta_n\}$, where $\cdots < -\beta_1 < -\alpha_1 < 0 < \beta_1 < \alpha_1 < \cdots$. [14] proved that $f^+(\omega)f^-(\omega)$ where

$$f^+(\omega)=\prod_{n\geq 1}\frac{1+i\omega/\alpha_n}{1+i\omega/\beta_n} \quad \text{and} \quad f^-(\omega)=\prod_{n\geq 1}\frac{1-i\omega/\alpha_n}{1-i\omega/\beta_n}$$

uniformly approximates $q/(q-\psi(\omega))$. Now observe that, all terms of $f^+(\cdot)$ and $f^-(\cdot)$ (e.g. $\frac{1+i\omega/\alpha_n}{1+i\omega/\beta_n}$ or $\frac{1-i\omega/\alpha_n}{1-i\omega/\beta_n}$) are positive-definite rational functions. Therefore, $f^+(\cdot)$ and $f^-(\cdot)$ are two positive-definite rational functions and analytical in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. An application of the Paly-Winer theorem warranties that the inverse Fourier transform of $f^+(\cdot)$ and $f^-(\cdot)$ are two positive density functions vanishing on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively.

For the geometric stopping time, using the fact that $q<1$, one may show that all poles of $(1-q)/(1-q\exp\{-\psi(\cdot)\})$ evaluated by the equation $1-q\exp\{-\psi(\omega)\}=0$, or equivalently by $\ln(q)+\psi(\omega)=0$. Now, [15]'s findings shows that all poles will appear as $\{-i\beta_n, i\beta_n\}$. On the other hand, zeros of $(1-q)/(1-q\exp\{-\psi(\cdot)\})$ are points where $\psi(\omega)=\infty$. Therefore, the zeros appeared as $\{-i\alpha_n, i\alpha_n\}$. The rest of proof is similar to [14] and [15]. □

---

[3] Lévy process X_t belongs to the meromorophic class of Lévy process if and only if $\bar{\nu}^+(x)=\nu(x,\infty)$ and $\bar{\nu}^-(x)=\nu(-\infty,-x)$ are two completely monotone functions and characteristic exponents $\psi(\cdot)$ is a meromorophic function, see [15] for more details.

The following examples shows application of the above procedure.

Example 5.1. Stable processes have been successfully fitted to stock returns, excess bond returns, foreign exchange rates, commodity price returns, real estate return data (see, e.g., [19] and [28], financial data (see, e.g., [6]), Market- and Credit-Value-at-Risk, Value-at-Risk, credit risk management (see, e.g., [27]). With the exception of the normal distribution ($\alpha = 2$), stable distribution are the heavy tailed distributions which paly an important role in heavy-tail modeling of economic data (see, e.g., [20] and [21]) and finance data (see, e.g., [28]).

Now consider a symmetric stable process X_t with the homomorphic characteristic exponent function $\psi(\omega) = 1/(i\mu\omega - \lambda^\alpha|\omega|^\alpha)$, where $\alpha \in (0, 2]$.

Using the fact that the real value α, in the above characteristic exponent, can be constructed from the rational numbers m/n, where m and n respectively are even and odd numbers. Now, an expression $q/(q - \psi(\omega^n))$ can be restated as

$$\frac{qi\mu\omega^n - q\lambda^{m/n}\omega^m}{1 - qi\mu\omega^n + q\lambda^{m/n}\omega^m} \quad = \quad (qi\mu\omega^n - q\lambda^{m/n}\omega^m)\prod_{i=1}^{n^+}\frac{1}{\omega - z_i^+}\prod_{i=1}^{n^-}\frac{1}{\omega - z_i^-},$$

where $n^+ + n^-$ is number of solutions for equation $1 - qi\mu\omega^n + q\lambda^{m/n}\omega^m = 0$ in ω. Moreover, z_i^+ and z_i^- are solutions of the recent equation where belong to $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Therefore, approximate solutions for the density function of extrema, $f_q^\pm$, are the inverse Fourier transform of $\phi_n^\pm(\omega) := \sqrt{qi\mu\omega - q\lambda^{m/n}\omega^{m/n}}/\prod_{i=1}^{n^\mp}(\omega^{1/n} - z_i^\mp)$.

To implement Procedure (5) for the Meromorophic Lévy process, one has to determine all zeros and poles of $q/(q - \psi(\cdot))$ (or $(1 - q)/(1 - q\exp\{-\psi(\cdot)\})$) which is a difficult task in may cases. Moreover, in the case where zeros or poles of $q/(q - \psi(\cdot))$ (or $(1 - q)/(1 - q\exp\{-\psi(\cdot)\})$) appear as $\{\alpha_n \pm \beta_n i\}$ (where at least one of $\alpha_n > 0$). Some terms of decomposition $f^+(\cdot)$ (or $f^-(\cdot)$) are not positive-definite rational function. Therefore, the inverse Fourier transform of $f^+(\cdot)$ and $f^-(\cdot)$ can be negative in some interval. The following procedure extents result of Procedure (5) for such cases and the non-homomorphic Lévy processes.

Before stating the second procedure, we need the following lemma.

Lemma 5.2. *Suppose $\psi(\cdot)$ stands for the characteristic exponent of a Lévy process. Moreover, suppose that $\alpha_0 + \beta_0 i$ is a root of $q - \psi(\lambda) = 0$, $\lambda \in \mathbb{C}$. Then $-\alpha_0 + \beta_0 i$ also is root of $q - \psi(\lambda) = 0$.*

Proof. Using the Lévy Khintchine formula (Equation, (1.1)), equation of $q - \psi(\lambda) = 0$ at point $\alpha_0 + \beta_0 i$ can be restated as

$$\begin{cases} -\sigma^2\alpha_0\beta_0 i + \alpha_0\mu i + i\displaystyle\int_{\mathbb{R}} \left(e^{-\beta_0}\sin(\alpha_0 x) - \alpha_0 x I_{[-1,1]}(x)\right)\nu(dx) = 0; \\ -\dfrac{1}{2}\sigma^2(\alpha_0^2 - \beta_0^2) - \mu\beta_0 + \displaystyle\int_{\mathbb{R}} \left(e^{-\beta_0}\cos(\alpha_0 x) - 1 + \beta_0 x I_{[-1,1]}(x)\right)\nu(dx) = q. \end{cases}$$

Since $\sin(\cdot)$ and $\cos(\cdot)$, respectively, are odd and even functions. Therefore, one may conclude that point $-\alpha_0 + \beta_0 i$ satisfies the above system of equations, as well. □

Suppose X_t is a Lévy process with characteristic exponents $\psi(\cdot)$. Moreover, suppose that the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$. Then by the following steps, one can approximate, in $L^{p^*}(\mathbb{R})$ (where $1/p^* + 1/p = 1$ and $1 < p \leq 2$) sense, the density functions of the extrema random variables M_q and I_q.

Step 1: Approximating $h(\omega) := q/(q - \psi(\omega))$, for the exponential stopping time, (or $h(\omega) := (1-q)/(1 - q\exp\{-\psi(\omega)\})$, for the geometric stopping time) by a positive-definite rational function by the following steps:
  1): Find out all poles of $h(\omega)$;
  2): Based upon such poles pick up some positive-definite rational functions given in Lemma 2.7;
  3): Approximate $h(\omega)$ by positive-definite rational function $r(\omega)$, given by Lemma 2.7;
  4): Set $A_0 := \lim_{\omega\to\infty} h(\omega)$ and m_k equal to order of k^{th} pole;
  5): Determine positive coefficients C_{lk} by a visual investigation or

$$C_{lk} = \max\left\{0, argmin \int_{\mathbb{R}} (h(\omega) - r(\omega))^p d\omega\right\}$$

Step 2): Determine error of approximating $h(\omega)$ by $r(\omega)$;
Step 3-: Decompose the positive-definite rational function $r(\omega)$ as a product of two functions, say $f^{\pm}(\omega)$, which are sectionally analytic and bounded in $\mathbb{C}^{\pm}$;
Step 4): Obtain the density functions of M_q and I_q by the inverse Fourier transform of $f^{+}(\cdot)$ and $f^{-}(\cdot)$, respectively.

Proof. Since $h(\omega)$ is a characteristic function. Lemma 2.10 warranties that, it is a positive-definite function and consequently its limit at infinity, say A_0, is a positive real number. Moreover Lemma 5.2 warranties that, one may use positive rational functions $r_{3k}(\cdot)$ and $r_{4k}(\cdot)$ whenever pole with form $\alpha \pm \beta i$ has been observed. The rest of proof is similar to Procedure (5). □

Example 5.3. Suppose X_t is a Lévy process with independent and continuous $\tau(q)$ and a jumps measure $\nu(dx) = \exp\{\alpha x\}cosech^2(x/2)dx$. The characteristic exponent for such Lévy process is given by

$$\psi(\omega) = -\frac{\sigma^2\omega^2}{2} - i\rho\omega - 4\pi(\omega - i\alpha)\coth(\pi(\omega - i\alpha)) + 4\gamma,$$

where $\gamma = \pi\alpha\cot(\pi\alpha)$, $\rho = 4\pi^2\alpha + \frac{4\gamma(\gamma-1)}{\alpha} - \mu$, $\omega \in \mathbb{R}$, and α, μ, and σ are given. Note that it is impossible to solve equation $q - \psi(\omega) = 0$ in the general case. Now consider special cases, whenever $\sigma = \mu = 2$ and $\alpha = 0$. Now, we compute the Wiener-Hopf factorization for $q = 5$. Finding all poles of $5/(5 - \psi(\omega))$ is difficult task. Using Maple 15, one may readily compute the three first poles as $\{-0.4781i,\ 0.5658i,\ 1.4921i\}$. On the other hands $A_0 = \lim_{\omega\to\infty} 5/(5 - \psi(i\omega)) = 0$. Now, we approximate $5/(5 - \psi(\omega))$ by

$$r(\omega) = \frac{C_1}{-i\omega + 0.4781} + \frac{C_2}{i\omega + 0.5658} + \frac{C_3}{i\omega + 1.4921}.$$

A graphical illustration shows that, one may readily chose $C_1 = C_2 = C_3 = 1/4.5$ see Figure 1-a. Error of this approximation is about 0.08719956902. $r(\omega)$ can be restarted as

$$\begin{aligned} r(\omega) &= \frac{(i\omega + 0.9560)(-i\omega + 1.9123)}{4.5(-i\omega + 0.4781)(i\omega + 0.5658)(i\omega + 1.4921)} \\ &= \frac{i\omega + 0.9560}{\sqrt{4.5}(i\omega + 0.5658)(i\omega + 1.4921)} \frac{-i\omega + 1.9123}{\sqrt{4.5}(-i\omega + 0.4781)} \\ &= f^-(\omega)f^+(\omega). \end{aligned}$$

Therefore, the density function of $I_{\tau(5)}$ and $M_{\tau(5)}$ can be approximated by

$$\begin{aligned} f_{I_{\tau(5)}}(x) &= 0.5110841035e^{1.4921x} + 0.3719983876e^{0.5658x}, \quad \text{for } x \le 0; \\ f_{M_{\tau(5)}}(x) &= 0.2857404120Dirac(x) + 0.4097937547e^{-0.4781x} \quad \text{for } x \ge 0, \end{aligned}$$

where $Dirac(x)$ stands for the dirac delta at point $x = 0$. Figures 1-b and 1-c illustrate behavior of $f_{I_{\tau(5)}}(\cdot)$ and $f_{M_{\tau(5)}}(\cdot)$, respectively.

Example 5.4. Suppose X_t in the surplus process (4.1) is the Lévy process in Example 5.3. Moreover, suppose that the random stoping time $\tau(q)$ has an exponential distribution with mean 0.2. Using result of Example 5.3, Figure 2 illustrates behavior of the finite-time ruin probability for different initial value u.

The following example explores situation that roots of $q - \psi(\omega) = 0$ appears in form of $\alpha + i\beta$.

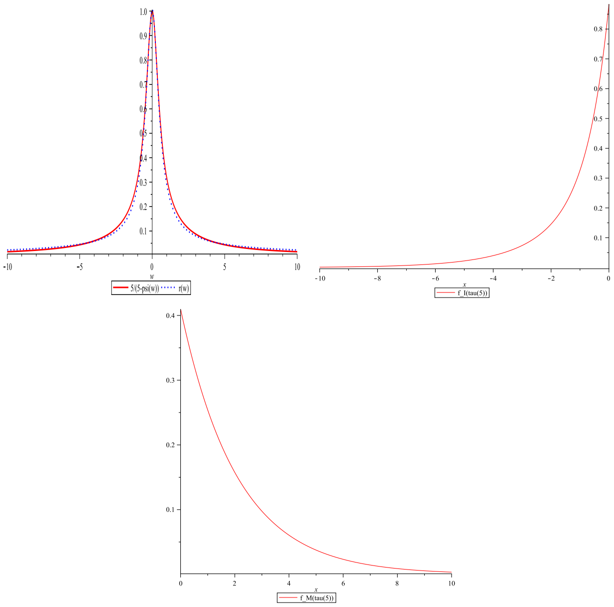

FIGURE 1. Graphical illustration of: (a) $\frac{5}{5-\psi(\omega)}$ and its approximation $r(\omega)$; (b) $f_{I_{\tau(5)}}$; and (c) $f_{M_{\tau(5)}}$.

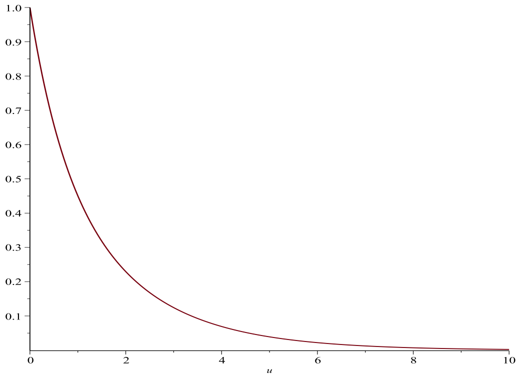

FIGURE 2. Behavior of the finite-time ruin probability for different initial value u.

Example 5.5. Consider a generalized hyperbolic process with the characteristic function

$$\phi(\omega) = e^{\psi(\omega)} = e^{i\mu\omega}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+i\omega)^2}\right)^{\lambda/2}\frac{\mathcal{K}_\lambda(\delta\sqrt{\alpha^2-(\beta+i\omega)^2})}{\mathcal{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2})},$$

where $\lambda, \mu \in \mathbb{R}$, $\alpha, \delta > 0$, $\beta \in (-\alpha, \alpha)$, and $\mathcal{K}_\lambda(\cdot)$ is the Modified Bessel functions of the third kind with index λ. Many well known processes are member of the class of generalized hyperbolic Lévy processes. For $\lambda > 0$ and $\delta \to 0$ one gets a Variance-Gamma process. The case $\lambda = -1/2$ corresponds to the normal inverse Gaussian process, see [10] for some analytic facts and applications about the generalized hyperbolic processes. The generalized hyperbolic process X_t is a pure jump process which can be considered as a Brownian motion with drift that evolves according to an increasing Levy process (i.e., subordinator). Such properties make the generalized hyperbolic process is an appealing process to model the financial returns, see [22] for more details.

Note that it is impossible to solve Equation $q-\psi(\omega) = 0$ in the general case. Now consider special cases, whenever $\alpha = \mu = 2$, $\beta = -\lambda = 1$, and $\delta = 3$. Now, we compute the Wiener-Hopf factorization for $q = 5$. Finding all poles of $5/(5-\psi(\omega))$ is difficult task. Using Maple 15, one may readily compute the sixth first poles as $\{\pm 0.4809389066 + 4.280110446i; \pm 0.9037063690 + 2.340695867i; \pm 2.516794346 + 0.4442175550i; \pm 3.756731426 - 0.9399774855i; \pm 4.853043564 - 2.318278971i; \pm 5.894258220 - 3.713000684i; \pm 6.909960755 - 5.121014155i; \pm 7.912034845 - 6.538310520i; \pm 8.905992610 - 7.962083725i; \pm 9.894699100 - 9.390493630i\}$. On the other hands $A_0 = \lim_{\omega \to \infty} 5/(5-\psi(i\omega)) = 0$.

[29] established that the generalized hyperbolic process has completely monotone jump density. Therefore, one has to approximate $q/(q-\psi(\omega))$ by function class $\mathcal{D}^*$, given by Lemma 2.7. Therefore, $5/(5-\psi(\omega))$ can be approximated by

$$\begin{aligned} r(\omega) \quad = \quad & \frac{C_1}{i\omega + 4.280110446} + \frac{C_2}{i\omega + 2.340695867} + \frac{C_3}{i\omega + 0.4442175550} \\ & + \frac{C_4}{-i\omega + 0.9399774855} + \frac{C_5}{-i\omega + 2.318278971} + \frac{C_6}{-i\omega + 3.713000684} \\ & + \frac{C_7}{-i\omega + 5.121014155} + \frac{C_8}{-i\omega + 6.538310520} + \frac{C_9}{-i\omega + 7.962083725} \\ & + \frac{C_{10}}{-i\omega + 9.390493630}. \end{aligned}$$

A graphical illustration shows that, one may readily chose $C_1 = \cdots = C_{10} = 0.4$, see Figure 2-a. An $L_2(\mathbb{R})$ error of this approximation is about 0.000002527687170, which can be improved by choosing more appropriate coefficients. $r(\omega)$ can be restated as

$$\begin{aligned} r(\omega) \quad = \quad & \frac{0.4}{i\omega + 4.280110446} + \frac{0.4}{i\omega + 2.340695867} + \frac{0.4}{i\omega + 0.4442175550} \\ & + \frac{0.4}{-i\omega + 0.9399774855} + \frac{0.4}{-i\omega + 2.318278971} + \frac{0.4}{-i\omega + 3.713000684} \\ & + \frac{0.4}{-i\omega + 5.121014155} + \frac{0.4}{-i\omega + 6.538310520} + \frac{0.4}{-i\omega + 7.962083725} \\ & + \frac{0.4}{-i\omega + 9.390493630} \\ = \quad & f^-(\omega) f^+(\omega), \end{aligned}$$

Therefore, the density function of $I_{\tau(5)}(\cdot)$ and $M_{\tau(5)}(\cdot)$ can be approximated by

$$\begin{aligned} f_{I_{\tau(5)}}(x) &= 0.3268288347e^{0.9399774846x} + 0.6308685531e^{9.390493235x} \\ &\quad +0.6059253078e^{7.962085726x} + 0.4905298019e^{3.713000220x} \\ &\quad +0.5383620597e^{5.121016047x} + 0.5757366525e^{6.538307461x} \\ &\quad +0.4259214977e^{2.318279006x}, \quad \text{for } x \leq 0; \\ f_{M_{\tau(5)}}(x) &= 0.2367700968Dirac(x) + 0.4078345184e^{-2.340695867x} \\ &\quad +0.5390740986e^{-4.280110443x} + 0.2582813546e^{-0.4442175554x}, \\ &\quad \text{for } x \geq 0. \end{aligned}$$

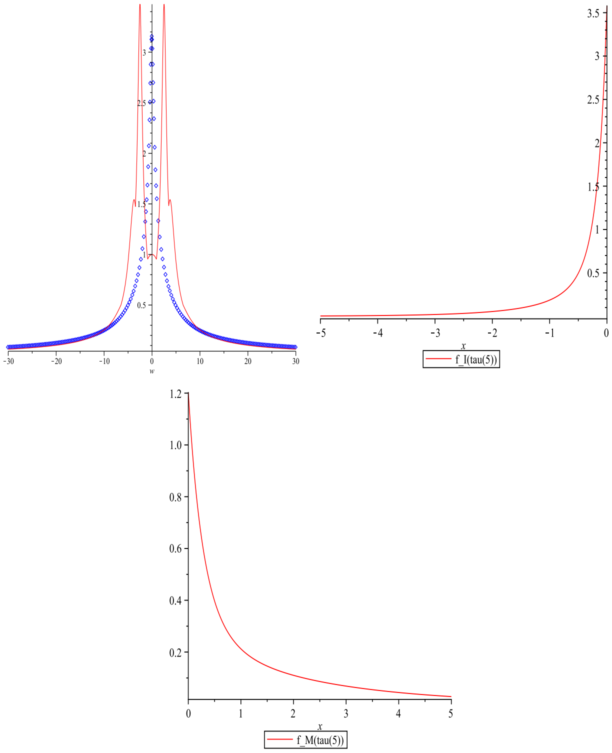

FIGURE 3. Graphical illustration of: (a) $\frac{5}{5-\psi(\omega)}$ and its approximation $r(\omega)$; (b) $f_{I_{\tau(5)}}$; and (c) $f_{M_{\tau(5)}}$.

Figures 3-b and 3-c illustrate behavior of $f_{I_{\tau(5)}}(\cdot)$ and $f_{M_{\tau(5)}}(\cdot)$, respectively. Since the generalized hyperbolic process has completely monotone jump density. Using [29]'s findings, one may conclude that the extrema's density functions should be completely monotone functions which cannot observe from Figures 3-b and 3-c. Such inconsistency may be interpreted by the fact that approximations of a completely monotone function is not completely monotone. On the other hand, since, we have $L^2(\mathbb{R})$ norm approximation. Then our approximation should be closed, in $L^2(\mathbb{R})$ sense, to some completely monotone functions in $\mathbb{R}$. In general, small oscillations are not a big problem, but we hope not to see functions that look like $xsin(x)$, for example, with increasingly large oscillations.

Example 5.6. Suppose X_t in the surplus process (4.1) is a generalized hyperbolic process, given by Example 5.5. Moreover, suppose that the random stoping time $\tau(q)$ has an exponential distribution with mean 0.2. Using result of Example 5.5, Figure 4 illustrates behavior of the finite-time ruin probability for different initial value u.

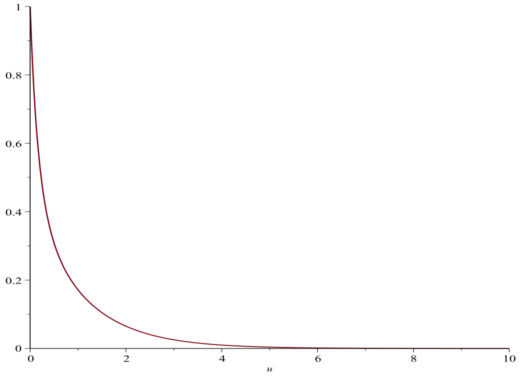

FIGURE 4. Behavior of the finite-time ruin probability for different initial value u.

6. Conclusion and suggestion

This article considers approximately the extrema's density functions of a class of Lévy processes. It provides two approximation techniques for approximating such the density functions. Namely, it suggests to replace $q/(q-\psi(\cdot))$ (or $(1-q)/(1-q\exp\{-\psi(\cdot)\})$) by a sequence of positive-definite rational functions. Two practical approximation procedures along several examples are given. The methods presented in this article can be generalized to other situations where the multiplicative WHF is applicable, such as finding first/last passage time and the overshoot, the last time the extrema was archived, several kind of option pricing, etc. Using [26]'s findings, result of this article may be generalized to a class of multivariate Lévy processes.

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