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SELF-SIMILAR FRACTALS AND ARITHMETIC DYNAMICS

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Abstract. The concept of self-similarity on subsets of algebraic varieties is defined by considering algebraic endomorphisms of the variety as 'similarity' maps. Self-similar fractals are subsets of algebraic varieties which can be written as a finite and disjoint union of 'similar' copies. Fractals provide a framework in which, one can unite some results and conjectures in Diophantine geometry. We define a well-behaved notion of dimension for self-similar fractals. We also prove a fractal version of Roth's theorem for algebraic points on a variety approximated by elements of a fractal subset. As a consequence, we get a fractal version of Siegel's theorem on finiteness of integral points on hyperbolic curves and a fractal version of Faltings' theorem on Diophantine approximation on abelian varieties.
Keywords: Self-similarity, Diophantine approximation, arithmetic dynamics.
MSC(2010): Primary: 14G40; Secondary: 11D45.

1. Introduction

Self-similar fractals are very basic geometric objects which presumably could have been defined as early as Euclid. By self-similar fractals, we mean objects which are (almost) disjoint union of pieces 'similar' to the whole object. In Euclidean context, one can think of Euclidean plane as the ambient space and Euclidean similarities as 'similarity' maps. There are several interesting examples of such fractals in the literature. Sierpinski carpet, Koch snowflake, and Cantor set are among the typical examples of Euclidean fractals. In a more modern geometric context, the ambient space of an affine fractal could be a real vector space, and 'similarity' maps could be chosen to be affine maps, which are usually assumed to be distance decreasing.

In the algebraic context, ambient space of an affine fractal could be a vector space over arbitrary field and polynomial self-maps of the vector space with coefficients in the base field could be taken as 'similarity' maps. Ideals in the

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ring of integers of a number field are examples of affine fractals in this context. Self-similar fractals in a ring could be much more complicated. For example, integers missing a number of digits in their decimal expansion form a fractal. The algebraic concept of self-similar fractals could also be extended to subsets of algebraic varieties, if we take algebraic endomorphisms as 'similarity' maps.

In this paper, we assume that a fractal is a finite union of its similar images except for finitely many points and a self-similar fractal has the extra condition that this fractal is a finite union of its similar images and its similar images are disjoint or at most with finite intersection. For example, rational points on a projective space could be thought of as a self-similar set, but not as a fractal, since it is union of infinitely many similar copies of itself.

The first important question about self-similar fractals is how to define their dimension. One can introduce a notion of dimension which is independent of the representation of the self-similar fractal as union of similar images. We use arithmetic height-functions to introduce such a concept of dimension for self-similar fractals. In fact, this notion of fractal-dimension turns out to be related to the growth of the number of points of bounded height in our fractal. This way, we recover some classical computations in this direction.

One can think of Diophantine approximation of algebraic points by a fractal whose elements are algebraic over $\mathbb{Q}$. Self-similarity of fractals imply a strong version of Roth's theorem in this case.

One shall note that, fractals are not necessarily dense in the ambient space with respect to complex topology. Therefore, such approximation theorems are only interesting if we are approximating a limiting point with respect to some Riemannian metric.

As a reward, we get fractal versions of Siegel's theorem on finiteness of integral points and Faltings' theorem on diophantine approximation on abelian varieties. Here are special cases, which could be formulated without any reference to fractals. We have treated these special cases separately in [19] and [20]:

Theorem 1.1. *Let X be an affine open subcurve of a connected smooth projectuve curve of genus ≥ 1 defined over $\mathbb{C}$ in the ambient affine space $\mathbb{A}^n(\mathbb{C})$ and let $F \subset \mathbb{A}^n(\mathbb{C})$ denote any finitely generated subgroup of $\mathbb{C}^n$. Then $X(K) \cap F$ is finite.*

This implies that Siegel's theorem is an algebro-geometric fact, not an Arithmetic one.

Theorem 1.2. *Let A be an abelian variety defined over a finitely generated subfield K of $\mathbb{C}$. Let E be a geometrically irreducible subvariety of A defined over K and F be a finitely generated subgroup of $A(K)$. Let w be a valuation on K and $H(x)$ a height function on K coming from a choice of projective model for K over the algebraic closure of $\mathbb{Q}$ in K. If $d_w(x, E)$ denotes the w-adic distance from x to E, and κ and c are positive constants, then, there are only* *finitely many points in F satisfying the following inequality*

$$d_w(x, E) < cH(x)^{-\kappa}.$$

This, in turn, implies that Faltings' theorem on Diophantine approximation on abelian varieties is also an algebro-geometric fact, not an Arithmetic one.

There are quite a few classical objects in arithmetic geometry which can be considered as self-similar fractals. For example, for an abelian variety A defined over a number-field as ambient space, the set of rational points $A(\mathbb{Q})$ or any finitely generated subgroup of $A(\bar{\mathbb{Q}})$ and the set of torsion points A^{tor} can be thought of as self-similar fractals with respect to endomorphisms of A.

In fact, fractals provide a common framework in which similar theorems about objects in arithmetic geometry could be united in a single context. For example, similarity between Manin-Mumford conjecture on torsion points on an abelian variety (proved by Raynaud [21]), and Lang's conjecture on finitely generated subgroups of rational points on an abelian variety (proved by Faltings [6]), made us propose the following general conjecture about fractals (Defined in section 2):

Conjecture 1.3. *Let V be an irreducible variety defined over a finitely generated field K and let $F \subset V(K)$ denote a fractal in V. Then, for any reduced subscheme Z of V defined over K the Zariski closure of $Z(K) \cap F$ is union of finitely many points and finitely many components B_j such that $B_j(K) \cap F$ is a fractal in B_j with respect to some of the same self-similarity maps for each j.*

A generalized version of Lang's conjecture is covered by the above conjecture. Some of our results in this paper also can be considered as its special cases. Detailed evidences are presented in the final section. We will also present a conjecture extending the above covering Andre-Oort conjecture, proved by Pila and Tsimerman ([17] and [18]).

2. Fractals in $\mathbb{Z}$

The idea of considering fractal subsets of $\mathbb{Z}$ is due to Omid Naghshineh [13] who proposed the following problem for "International Mathematics Olympiad" held in United Kingdom in July 2002.

Problem 2.1. Let F be an infinite subset of $\mathbb{Z}$ such that $F = \bigcup_{i=1}^{n} a_i.F + b_i$ for integers a_i and b_i where $a_i.F + b_i$ and $a_j.F + b_j$ are disjoint for $i \neq j$ and $|a_i| > 1$ for each i. Prove that

$$\sum_{i=1}^{n} \frac{1}{|a_i|} \leq 1.$$

In [13], he explains his ideas about fractals in $\mathbb{Z}$ and suggests how to define their dimension and how to prove this notion is independent of the choice of self-similarity maps. His suggestions are carried out by Hessam Mahdavifar [13]. In this section, we present their results and ideas.

Definition 2.2. Let $\phi_i : \mathbb{Z} \to \mathbb{Z}$ for $i = 1$ to n denote linear maps of the form $\phi_i(x) = a_i.x + b_i$ where a_i and b_i are integers with $|a_i| > 1$. A subset $F \subseteq \mathbb{Z}$ is called a self-similar fractal with respect to ϕ_i if F is disjoint union of its images under the linear map ϕ_i. In this case, we write $F = \sqcup_i \phi_i(F)$ and define dimension of F to be the real number s such that

$$\sum_{i=1}^{n} |a_i|^{-s} = 1.$$

The basic example for self-similar fractals in $\mathbb{Z}$ is the set of integers which miss a number of digits in their decimal expansion. This definition of dimension is motivated by the notion of box dimension for fractals on real vector spaces, which coincides with Hausdorff dimension. The challenge is to prove that, this notion of dimension is independent of all the choices made, and depends only on self-similar fractal itself as a subset of $\mathbb{Z}$. Also, smaller self-similar fractals must have smaller dimension. Now, it is easy to solve the above IMO problem. Note that $\mathbb{Z}$ is a self-similar fractal of dimension one. A self-similar $F \subseteq \mathbb{Z}$ is of dimension ≤ 1 which solves the problem.

Theorem 2.3. *Let $F \subseteq \mathbb{Z}$ satisfy $F \subseteq \cup_i \phi_i(F)$ (ϕ_i and a_i as in Definition 2.2). If s is a real number such that $\sum_i |a_i|^{-s} < 1$ then the number of elements of F in the ball $B(x)$ is bounded above by cx^s for some constant c and for large x.*

Proof. Let $F_i = \phi_i(F)$, and let $N(x)$ and $N_i(x)$ denote the number of elements of F and F_i in the ball $B(x)$, respectively. We have

$$N(x) \leq \sum_i N_i(x)$$

and since for $f \in F_i$ and $\phi_i^{-1}(f) \in F$ we have $|\phi_i^{-1}(f)| \leq (|f| + |b_i|)/|a_i|$ we can write

$$N_i(x) \leq N(\frac{x + |b_i|}{|a_i|})$$

If we let $t = Max_i\{|b_i|/|a_i|\}$ then we get the following estimate

$$N(x) \leq \sum_i N(\frac{x}{|a_i|} + t)$$

We define a function $h : [1, \infty] \to \mathbb{R}$ by $h(x) = x^{-s}N(x)$ and we shall show that h is a bounded function. The above estimate will give:

$$h(x) \leq \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s h(\frac{x}{|a_i|} + t)$$

There exists a constant M such that for $x > M$ we have $(x/|a_i|) + t < x$ for all i and

$$\sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s < 1$$

Now, assume $|a_1| \leq ... \leq |a_n|$ and define $x_0 = |a_n|(M-t)$ and $x_j = |a_1|(x_{j-1}-t)$ for $j \geq 1$. Then x_j is an unbounded decreasing sequence. The function h is bounded on $[M, x_0]$ and we inductively show that it has the same bound on $[x_j, x_{j+1}]$: for if $x \in [x_j, x_{j+1}]$ then $(x/|a_i|) + t \in [(x_j/|a_i|) + t, x - j + 1/|a_i|) + t] \subset [M, x_j]$ and since by induction hypothesis we have $h(x/|a_i|) + t) \leq c$ for all i,

$$h(x) \leq \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s h(\frac{x}{|a_i|} + t) < c \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s < c$$

It remains to notice that h is also bounded on $[1, M]$. □

Theorem 2.4. *Let $F \subseteq \mathbb{Z}$ satisfy $F \supseteq \sqcup_i \phi_i(F)$ where ϕ_i are as above. If r is a real number such that $\sum_i Norm(a_i)^{-r} > 1$ then the number of elements of F in the ball $B(x)$ is bounded below by cx^r for some constant c and for large x.*

Proof. We use the notation in the proof of the previous lemma. Since for $f \in F_i$ and $\phi_i^{-1}(f) \in F$ we have $|\phi_i^{-1}(f)| \geq (|f| - |b_i|)/|a_i|$ and we get

$$N_i(x) \geq N(\frac{x - |b_i|}{|a_i|}) \geq N(\frac{x}{|a_i|} - t)$$

where $t = Max_i\{|b_i|)/|a_i|\}$. Now, it remains to show that $h : [1, \infty] \to \mathbb{R}$ defined by $h(x) = x^{-r}N(x)$ is bounded below, which can be proved along the same line as the previous lemma. □

Proposition 2.5. *Let $F_1 \subseteq F_2 \subseteq \mathbb{Z}$ be fractals. Then the notion of fractal dimension is well-defined and $dim(F_1) \leq dim(F_2)$.*

Proof. Suppose $F = \sqcup_i \phi_i(F) = \sqcup_j \psi_j(F)$ where ϕ_i and ψ_i are linear functions $\phi_i(x) = a_i.x + b_i$ and $\psi_j(x) = c_j.x + d_j$. Assume $\sum_i |a_i|^{-\alpha} = 1$ and $\sum_i |c_j|^{-\beta} = 1$. We must show that $\alpha = \beta$. Suppose $\alpha < \beta$. Insert real numbers $\alpha < s < r < \beta$. Since $F \subset \cup_i \phi_i(F)$ and $\sum_i |a_i|^{-s} < 1$, we get $N(x) \leq cx^s$ for large x and since $F \supseteq \sqcup_i \psi_j(F)$ and $\sum_i |c_j|^{-r} > 1$, we get $N(x) \geq cx^r$ for large x which is a contradiction. Thus $\alpha = \beta$.

Now, for fractals $F_1 \subseteq F_2$ suppose that $F_1 = \sqcup_i \phi_i(F)$ and $F_2 = \sqcup_j \psi_j(F)$ where ϕ_i and ψ_i functions as above, and let $\sum_i |a_i|^{-\alpha} = 1$ and $\sum_i |c_j|^{-\beta} = 1$. We must show that $\alpha \leq \beta$. Suppose $\alpha > \beta$ and insert real numbers $\alpha > r > s > \beta$. Then one can get a contradiction as above. □

Naghshineh and Mahdavifar also suggest that the same calculations work for $\mathbb{Z}[i]$ if we use norm of a complex number instead of absolute value for a real number. The same arguments indicate that, the notion of dimension of a fractal is linked to asymptotic behavior of the number of points of bounded norm.

3. Affine fractals

It would be more convenient for the reader, if we formulate the most general form of an affine fractal, and then treat special cases.

Definition 3.1. Let X be an affine algebraic variety defined over a finitely generated field K, and let f_i for $i = 1$ to n, denote nonconstant polynomial endomorphisms of X with coefficients in K. A subset $F \subset X(K)$ is called an affine self-similar fractal with respect to $f_1, ..., f_n$ if F is an almost disjoint (i.e. finite intersection) union of its images under the polynomial endomorphisms f_i for $i = 1$ to n, in which case, by abuse of notation, we write $F = \sqcup_i f_i(F)$. An affine fractal in X is a subset which is affine fractal with respect to some polynomial endomorphisms $f_1, ..., f_n$. Note that such a representation is not unique. In case we only have $F = \cup_i f_i(F)$ except for finitely many points of F which are outside $\cup_i f_i(F)$ we simply call F a fractal.

Example 3.2. Let K be a number field and let O_K denote its ring of integers. One can take O_K as ambient space and polynomial maps $\phi_i : O_K \to O_K$ with coefficients in O_K as self-similarities. Let a_i denote the leading coefficient of ϕ_i, and n_i denote the degree of ϕ_i. Fix an embedding $\rho : K \hookrightarrow \mathbb{C}$. Assume $Norm_\rho(a_i) > 1$ in case ϕ_i is linear. Let $F \subseteq O_K$ be an affine fractal with respect to ϕ_i for $i = 1$ to n. One can define the fractal-dimension of F to be the real number s for which

$$\sum_{i=1}^{n} Norm(a_i)^{-\frac{s}{n_i}} = 1.$$

Arguments of the previous section hold almost line by line, if one replaces the absolute value of an integer with the product of various archemidian norms of an algebraic integer in O_K. Therefore, we have the following result:

Proposition 3.3. *The above notion of dimension for affine fractals in O_K is well-defined and well-behaved with respect to inclusion of affine fractals, i.e. dimension of an affine fractal is independent of the choice of self-similarities and compatible with inclusion of fractal subsets.*

For a fractal generated by finitely many points we have proved the following:

Proposition 3.4. *Let F be a fractal with respect to f_i as above, the number of points of norm bounded by X is $O(X^s)$ where s is determined by*

$$\sum_{i=1}^{n} Norm(a_i)^{-\frac{s}{n_i}} = 1.$$

Example 3.5. Start from a linear semi-simple algebraic group G and a rational representation $\rho : G \to GL(W_{\mathbb{Q}})$ defined over $\mathbb{Q}$. Let $w_0 \in W_{\mathbb{Q}}$ be a point whose orbit $V = w_0\rho(G)$ is Zariski closed. Then the stabilizer $H \subset G$ of w_0 is reductive and V is isomorphic to $H\backslash G$. By a theorem of Borel-Harish-Chandra $V(\mathbb{Z})$ breaks up to finitely many $G(\mathbb{Z})$ orbits [1]. Thus the points of $V(\mathbb{Z})$ are parametrized by cosets of $G(\mathbb{Z})$. Fix an orbit $w_0G(\mathbb{Z})$ with w_0 in $G(\mathbb{Z})$. Then the stabilizer of w_0 is $H(\mathbb{Z}) = H \cap G(\mathbb{Z})$.

The additive structure of G allows one to define self-similar subsets of $V(\mathbb{Z})$ and study their asymptotic behavior using the idea of fractal dimension. For example, one can define self-similarities to be maps $\phi : V(\mathbb{Z}) \to V(\mathbb{Z})$ of the form

$$\phi(\omega_0\gamma) = \omega_0([n]\gamma + g_0)$$

where $[n]$ denotes multiplication by n in $G(\mathbb{Z})$ and g_0 is an element in $G(\mathbb{Z})$. These similarity maps are expansive if $n > 1$ and lead to a notion of dimension for self-similar fractals in $V(\mathbb{Z})$. Upper bound similar to above holds for fractals in $V(\mathbb{Z})$.

Duke-Rudnick-Sarnak [4] putting some extra technical assumptions, have determined the asymptotic behavior of

$$N(V(\mathbb{Z}), x) = \sharp\{\gamma \in H(\mathbb{Z}) \setminus G(\mathbb{Z}) : ||w_0\gamma|| \leq x\}.$$

They prove that there are constants $a \geq 0, b > 0$ and $c > 0$ such that

$$N(V(\mathbb{Z}), x) \sim cx^a(logx)^b.$$

Note that, the whole set $V(\mathbb{Z})$ could not be a fractal, since the asymptotic behavior of its points is not polynomial.

Example 3.6. Here is an example of an affine self-similar fractal with respect to nonlinear polynomial maps. The subset

$$\{(2^i, 2^j) \in \mathbb{Q}^2 | i, j \in \mathbb{Z}\}$$

is an affine self-similar fractal with respect to $f_1(x_1, x_2) = (x_1^2, x_2^2)$, $f_2(x_1, x_2) = (2x_1^2, x_2^2)$, $f_3(x_1, x_2) = (x_1^2, 2x_2^2)$ and $f_4(x_1, x_2) = (2x_1^2, 2x_2^2)$. Notice that, after projectivization, we still get a self-similar set in the projective line $P^1(\mathbb{Q})$. The subset

$$\{(2^i; 2^j) \in P^1(\mathbb{Q}) | i, j \in \mathbf{N} \cup \{0\}\}$$

is a self-similar fractal with respect to $f_1(x_1; x_2) = (x_1^2; x_2^2)$ and $f_2(x_1; x_2) = (2x_1^2; x_2^2)$.

4. Fractals in arithmetic geometry

In general, there is no global norm on the set of points in a fractal to motivate us how to define the notion of fractal-dimension. In special cases, arithmetic height functions are appropriate replacements for the norm of an algebraic integer, particularly because finiteness theorems hold in this context.

Northcott associated a height function to points on the projective space which are defined over number fields [15]. In course of his argument for the fact that, the number of periodic points of an endomorphism of a projective space which are defined over a given number-field are finite, he proved that the number of points of bounded height is finite. Therefore, one can study the asymptotic behavior of rational points on a fractal hosted by a projective variety. Let us formulate a general definition.

Definition 4.1. Let V be a projective variety defined over a finitely generated field K and let f_i for $i = 1$ to n denote finite surjective endomorphisms of V defined over K, which are of degrees > 1. A subset $F \subset V(K)$ is called a self-similar fractal with respect to f_i, if F is almost disjoint union of its images under the endomorphisms f_i, i.e. $F = \sqcup_i f_i(F)$. F is called a fractal if $F = \cup_i f_i(F)$

Example 4.2. Let f_i for $i = 1, ..., n$ denote homogeneous endomorphisms of a projective space defined over a global field K with each homogeneous component of degree m_i. Let $F \subseteq P^n(K)$ be a fractal with respect to f_i: $F = \sqcup_i f_i(F)$. One can define the fractal-dimension of F to be the real number s for which $\sum_i m_i^{-s} = 1$.

Proposition 4.3. *In the context of projective spaces, dimension of a self-similar fractal F is well-defined and well-behaved with respect to fractal embeddings.*

Proof. Indeed, for the number-field case, we use the logarithmic height h to control the height growth of points under endomorphisms. Again we claim that if $\sum_i m_i^{-s} < 1$ and $F \subseteq \cup_i f_i(F)$ then the number of elements of F of logarithmic height less than x, which we denote again by $N(x)$, is bounded above by cx^s for some constant c and large x. Let $F_i = f_i(F)$, and $N_i(x)$ denote the number of elements of F_i of logarithmic height less than x. We have

$$N(x) \leq \sum_i N_i(x)$$

and for $f \in F_i$ and $f_i^{-1}(f) \in F$ we have $h(f_i(f)) = m_i.h(f) + O(1)$. Therefore

$$N(x) \leq \sum_i N(m_i^{-1}x + t)$$

for some t. We define a function $\bar{h} : [1, \infty] \to \mathbb{R}$ by $\bar{h}(x) = x^{-s}N(x)$. The argument of Theorem 2.3 implies that $\bar{h}$ is bounded, and hence the claim follows. By a similar argument, if $F \supseteq \sqcup_i f_i(F)$ and if r is a real number such that $\sum_i m_i^{-r} > 1$ then $N(x)$ is bounded below by cx^s for some constant c and large x. One can follow the argument of Proposition 2.5 to finish the proof. $\square$

Example 4.4. For the function field case, one could use another appropriate height function. Let $\bar{\mathbb{F}}_q(X)$ denote the function field of an absolutely irreducible projective variety X which is non-singular in codimension one, defined over a finite field $\bar{\mathbb{F}}_q$ of characteristic p. One can use the logarithmic height on $P^n(\bar{\mathbb{F}}_q(X))$ defined by Neron [10]. Finiteness theorem holds for this height function as well.

Example 4.5. Let $h, R, w, r_1, r_2, d_K, \zeta_K$ denote class number, regulator, number of roots of unity, number of real and complex embeddings, absolute discriminant and the zeta function associated to the number field K. Schanuel proved that [23] the asymptotic number of points in $P^n(K)$ of logarithmic height bounded by $log(x)$ is given by

$$\frac{hR}{w\zeta_K(n+1)}\left(\frac{2^{r_1}(2\pi)^{r_2}}{d_K^{1/2}}\right)^{n+1}(n+1)^{r_1+r_2-1}x^{n+1}.$$

This proves that rational points on projective space can not be regarded as a fractal of finite dimension.

Example 4.6. Schmidt in case $K = \mathbb{Q}$ [24] and Thunder for general number field K [26] generalized the estimate of Shanuel to Grassmanian varieties, and proved that

$$C(G(m,n)(K), log(x)) \sim c_{m,n,K}x^n$$

where C denotes the number of points of bounded logarithmic height and $c_{m,n,K}$ is an explicitly given constant. Also, Franke-Manin-Tschinkel provided a generalization to flag manifolds [8]. Let G be a semi-simple algebraic group over K and P a parabolic subgroup and $V = P\backslash G$ the associated flag manifold. Choose an embedding of $V \subset P^n$ such that the hyperplane section H is linearly equivalent to $-sK_V$ for some positive integer s, then there exists an integer $t \geq 0$ and a constant c_V such that

$$C(V(K), x)^s \sim c_V x(logx)^t.$$

All of these spaces are self-similar objects which have the potential to be ambient spaces for fractals, but they are too huge to be fractals themselves.

Example 4.7. Wan [27] proved that in the function field case, the asymptotic behavior of points in $P^n(K)$ of logarithmic height bounded by d is given by

$$\frac{hq^{(n+1)(1-g)}}{(q-1)\zeta_X(n+1)}q^{(n+1)d}.$$

which shows that $P^n(\bar{\mathbb{F}}_q(X))$ can indeed be considered as a finite dimensional fractal.

Example 4.8. Let A be an abelian variety over a number-field K and let $F \subseteq A(\bar{\mathbb{Q}})$ be a fractal with respect to endomorphisms ϕ_i which are translations of multiplication maps $[n_i]$ by elements of $A(\bar{\mathbb{Q}})$. We define dimension of F to be the real number s for which $\sum_i n_i^{-s} = 1$. Then dimension of F is well-defined and well-bahaved with respect to fractal embeddings.

Proposition 4.9. *In the context of abelian varieties, dimension of a self-similar fractal F is well-defined and well-bahaved with respect to fractal embeddings.*

Proof. Indeed, in this case, we use the Neron-Tate logarithmic height $\hat{h}$ to control the growth of the heights of points under the action of endomorphisms ϕ_i. The same proof as before works except that

$$\hat{h}([n_i](f)) = (n_i)^2 \hat{h}(f)$$

does not hold for translations of the form $[n_i]$. One should use the fact that for the Néron-Tate height associated to a symmetric ample bundle on A and for every $a \in A(\bar{\mathbb{Q}})$ and $n \in \mathbf{N}$, we have

$$\hat{h}([n](f) + a) + \hat{h}([n](f) - a) = 2\hat{h}([n](f)) + 2\hat{h}(a).$$

This helps to get the right estimate. The rest of proof goes as before. □

The above notion of dimension implies that the number of points of bounded height defined over a fixed number-field has polynomial growth, which gives an immediate proof for a classical result of Néron [14].

Example 4.10. Analogous to abelian varieties, one also can define fractals on t-modules. By a t-module of dimension N and rank d defined over the algebraic closure $\bar{k} = \overline{\bar{\mathbb{F}}_q(t)}$ we mean, fixing an additive group $(\mathbf{GSp}(2g)_a)^N(\bar{k})$ and an injective homomorphism Phi from the ring $\bar{\mathbb{F}}_q[t]$ to the endomorphism ring of $(\mathbf{GSp}(2g)_a)^N$ which satisfies

$$Phi(t) = a_0 F^0 + ... + a_d F^d$$

with a_d non-zero, where a_i are $N \times N$ matrices with coefficients in $\bar{k}$, and F is a Frobenius endomorphism on $(\mathbf{GSp}(2g)_a)^N$. One can think of polynomials $P_i \in \bar{\mathbb{F}}_q[t]$ of degrees r_i for $i = 1$ to n as self-similarities of the t-module $(\mathbf{GSp}(2g)_a)^N$ and let $F \subseteq (\mathbf{GSp}(2g)_a)^N(\bar{k})$ be a fractal with respect to P_i , i.e. $F = \sqcup_i Phi(P_i)(F)$. We define the fractal dimension of F to be the real number s such that $\sum_i (r_i d)^{-s} = 1$. Then dimension of F is well-defined and well-bahaved with respect to inclusions.

Proposition 4.11. *In the context of t-modules dimension of a self-similar fractal F is well-defined and well-bahaved with respect to fractal embeddings.*

Proof. Indeed, Denis defines a canonical height $\hat{h}$ on t-modules which satisfies

$$\hat{h}[Phi(P)(\alpha)] = q^{dr}.\hat{h}[\alpha]$$

for all $\alpha \in (\mathbf{GSp}(2g)_a)^N$, where P is a polynomial in $\bar{\mathbb{F}}_q[t]$ of degree r [3]. This can be used to prove the result in the same lines as before. One can get information on the asymptotic behavior of $N(\mathbf{GSp}(2g)_a^N(\bar{k}), x)$ by representing $\mathbf{GSp}(2g)_a^N(\bar{k})$ as a fractal. □

5. **Diophantine approximation by fractals**

This section is devoted to proving theorems which were mentioned in the introduction. The arguments are along the same lines as analogous classical results.

Roth's theorem on Diophantine approximation of rational points on projective line implies a version on projective varieties defined over number-fields. Self-similarity of rational points on abelian varieties makes room to improve the estimates. This argument can be imitated in case of arithmetic fractals defined over finitely generated fileds.

Theorem 5.1 (Fractal version of Roth's thereom on diophantine approximation)**.** *Fix a finitely generated field of characteristic zero K and $\sigma : K \hookrightarrow \mathbb{C}$ a complex embedding. Let V be a smooth projective algebraic variety defined over K and let L be a very ample line-bundle on V. Denote the arithmetic height function associated to the line-bundle L by h_L. Suppose $F \subset V(K)$ is a fractal subset with respect to finitely many height-increasing self-endomorphisms $\phi_i : V \to V$ defined over K such that for all i we have*

$$h_L(\phi_i(f)) > m_i h_L(f) + 0(1)$$

where $m_i > 1$. Fix a Riemannian metric on $V_\sigma(\mathbb{C})$ and let d_σ denote the induced metric on $V_\sigma(\mathbb{C})$. Then, for every $\delta > 0$ and every choice of an algebraic point $\alpha \in V(\bar{K})$ which is not a critical value of any of the ϕ_i's and all choices of a constant C, there are only finitely many fractal points $\omega \in F$ approximating α such that

$$d_\sigma(\alpha, \omega) \le Ce^{-\delta h_L(\omega)}.$$

Proposition 5.2. *With assumptions of the above theorem, suppose for some $\delta_0 > 0$ we have that, for any choice of a constant C and every choice of an algebraic point $\alpha \in V(\bar{K})$ there are only finitely many fractal points $\omega \in F$ approximating α in the following manner*

$$d_\sigma(\alpha, \omega) \le Ce^{-\delta_0 h_L(\omega)}.$$

Then, for every $\delta > 0$ and every choice of an algebraic point $\alpha \in V(\bar{K})$ which is not a critical value of any of the ϕ_i's and all choices of a constant C, there are only finitely many fractal points $\omega \in F$ approximating α such that

$$d_\sigma(\alpha, \omega) \le Ce^{-\delta h_L(\omega)}.$$

Proof of Proposition 5.2. Note that, we have assumed that the above is true for some $\delta_0 > 0$ without any assumption on ϕ_i or on α. Let $\delta' > 0$ be infimum of such $\delta_0 > 0$.

Fix $\epsilon > 0$ such that $\epsilon < \delta' < m_i \epsilon$ for all i. Suppose that w_n is an infinite sequence of elements in F such that $\omega_n \to \alpha$ which satisfies the estimate

$$d_\sigma(\alpha, \omega_n) \le Ce^{-\epsilon h_L(\omega_n)}.$$

then infinitely many of them are images of elements of F under the same ϕ_i. By going to a subsequence, one can find a sequence ω'_n in F and an algebraic point α' in $V(\bar{K})$ such that $\omega'_n \to \alpha'$ and for a fixed ϕ_i we have $\phi_i(\alpha') = \alpha$ and $\phi_i(\omega'_n) = \omega_n$ for all n. Then

$$d_\sigma(\alpha, \omega_n) \leq Ce^{-\epsilon h_L(\omega_n)} \leq C' e^{-\epsilon m_i h_L(\omega'_n)}$$

for an appropriate constant C'. On the other hand,

$$d_\sigma(\alpha', \omega'_n) \leq C'' d_\sigma(\alpha, \omega_n)$$

holds for an appropriate constant C'' and large n by injectivity of $d\phi_i^{-1}$ on the tangent space of α. This contradicts our assumption on δ', since $\delta' < m_i \epsilon$. □

Proof of Theorem 5.1. If we assume that points of F and similarity maps are defined over some number-field, Roth's theorem implies that the assumption of theorem is true for any $\delta_0 > 2$. All such examples are forward orbits of finitely many height increasing self-similarities. The same is true for finitely generated field of characteristic zero by a result of Lang [9] generalizing Roth's theorem and height defined by Moriwaki [12]. □

Remark 5.3. The conditions of Thereom 5.1. do not hold for general fractals in $V(\bar{K})$. For example, torsion points of an abelian variety are dense in complex topology, and have vanishing height. Therefore, our fractal analogue of Roth's theorem could not hold in this case.

Let us state a more precise version of our version of Siegel's theorem.

Theorem 5.4 (Fractal version of Siegel's theorem on integral points)**.** *Fix a finitely generated field of characteristic zero K. Let V be a smooth affine algebraic variety defined over K with smooth projectivization $\bar{V}$ and let L be an very ample line-bundle on $\bar{V}$. Denote the arithmetic height function associated to the line-bundle L by h_L. Suppose $F \subset V(K)$ is a fractal subset with respect to finitely many height-increasing polynomial self-endomorphisms $\phi_i : V \to V$ defined over K such that for all i we have*

$$h_L(\phi_i(f)) > m_i h_L(f) + 0(1),$$

where $m_i > 1$. One could also replace this assumption with norm analogue replacing h_L with complex norm. For any affine hyperbolic algebraic curve X embedded in V defined over K, $X(K) \cap F$ is a finite set.

We borrow a lemma from [25] albeit in our notation.

Lemma 5.5. *Let K be a finitely generated field of characteristic zero. Let X be a curve defined over K of genus at least 1, where ϕ is a non-constant rational function on X defined over K and P_n are points on $X(K)$ whose height tend to ∞ and none is a pole of ϕ.*

Then for $z_n = \phi(P_n)$ *we have*

$$\lim_{n\to\infty} \frac{log|z_n|_v}{logH(z_n)} = 0,$$

where $log(H_L) = h_L$.

Proof of Theorem 5.4. Let $\sigma : K \hookrightarrow \mathbb{C}$ denote a complex embedding of K. Fix a Riemannian metric on $V_\sigma(\mathbb{C})$ and let d_σ denote the induced metric on $V_\sigma(\mathbb{C})$. Then by Theorem 5.1 (our version of Roth's theorem), for every $\delta > 0$ and every choice of an algebraic point $\alpha \in V(\bar{K})$ which is not a critical value of any of the ϕ_i's and all choices of a constant C, there are only finitely many fractal points $\omega \in F$ approximating α such that

$$d_\sigma(\alpha, \omega) \leq H_L(\omega)^{-\delta}.$$

In case K is transcendental, we have to pick a model for K over algebraic closure of $\mathbb{Q}$ in K following Lang [9]. Now if P_n is a sequence of distinct points in $X(K) \cap F$, their heights tends to infinity and if ϕ is a non-constant rational function on X from some point on no P_n is a pole of ϕ. Then by above proposition

$$\lim_{n\to\infty} \frac{log|z_n|_\sigma}{logH(z_n)} = 0.$$

On the other hand, one defines height of rational points by

$$H(z) = \prod_{v \in M_K} sup(1, |z|_v),$$

where $|.|_v$ are normalized according to a product formula. Since similarity maps of F are expanding, we know that F is forward orbit of finitely many points. So for a finite set of places S we have

$$H(z) = \prod_{v \in S} sup(1, |z|_v),$$

and therefore

$$logH(z) = \sum_{v \in S} log(sup(1, |z|_v)).$$

Then, we have

$$1 = \sum_{v \in S} sup(0, \frac{log|z_n|_\sigma}{logH(z_n)}) \leq \sum_{v \in S} \frac{log|z_n|_\sigma}{logH(z_n)}$$

which could not be true, because the above limit is zero. This implies the finiteness result we are seeking for. □

Remark 5.6. If F and its self-similarity maps are defined over K, then F is forward orbit of finitely many points which are not neccessarily algebraic, and the above result is not implied by Siegel's theorem for S-integral points.

Based on this, we expect the following version of Liouville's theorem on diophantine approximation holds:

Theorem 5.7 (Fractal version of Liouville's theorem on diophantine approximation)**.** *Fix a finitely generated field of characteristic zero K. Let V be a smooth projective algebraic variety defined over K and let L be a very ample line-bundle on V with arithmetic height function h_L. Then there exists a positive constant δ_0 such that for any positive constant c and any geometrically irreducible algebraic subvariety E of V defined over K, there are only finitely many points defined over K in $V(K)$ outside E satisfying the following inequality*

$$d_w(x, E) < cH(x)^{-\delta_0}$$

except for points in an algebraic variety $V(\delta_0)$ which is of strictly smaller dimension than V. Here $d_w(x, E)$ denotes the w-adic distance from x to E.

Proof. This is a weak form of Vojta conjectures. In the number field case, this is mentioned in Faltings-Wustholz [7] as a trivial result in case E is geometrically irreducible. In the case of finitely generated fields of characteristic zero the result is a consequence of theorem I' in seminal work of Lang [9]. □

Now, the following version of Faltings' theorem, can be proved using the methods of self-similarity and height expansion.

Theorem 5.8 (Fractal version of Faltings' theorem on diophantine approximation on abelian varieties)**.** *Fix a finitely generated field of characteristic zero K. Let V be a smooth projective algebraic variety defined over K and let L be a very ample line-bundle on V. Denote the arithmetic height function associated to the line-bundle L by h_L. Suppose $F \subset V(K)$ is a fractal subset with respect to finitely many height-increasing polynomial finite endomorphisms $\phi_i : V \to V$ defined over K such that for all i we have*

$$h_L(\phi_i(f)) > m_i h_L(f) + 0(1),$$

where $m_i > 1$. Fix any positive constants κ and c. For any irreducible algebraic subvariety E of V defined over K and $d_w(x, E)$ denoting the w-adic distance from x to E, there are only finitely many points defined over K in F outside E satisfying the following inequality

$$d_w(x, E) < cH(x)^{-\kappa}.$$

Proof. Reduction of the inequalty for some positive δ_0 to arbitrary $\delta > 0$ is done in the same manner as in Theorem 5.1. Getting rid of $V(\delta)$ is the result of the fact that $V(\delta)$ is invariant under ϕ_i and $F \cap V(\delta)$ is again a fractal. One can proceed by reducing the problem from V and E to $V(\delta)$ and $E \cap V(\delta)$ and applying induction on dimension. □

The following will be a special case:

Corollary 5.9. *Fix a number field K. Let E be an irreducible affine smooth algebraic variety defined over K. For any positive constants δ and c there are only finitely many points defined over ring of integers O_K outside E satisfying the following inequality*

$$d_w(x, E) < cH(x)^{-\delta},$$

where $d_w(x, E)$ denotes the w-adic distance from x to E.

In particular, we have the following:

Corollary 5.10. *Let D be an irreducible affine smooth divisor defined over $\mathbb{Q}$. If $d_w(x, E)$ denotes the w-adic distance from x to E, then for any positive constants δ and c there are only finitely many points defined defined over $\mathbb{Z}$ outside D satisfying the following inequality*

$$d(x, E) < c||x||^{-\delta}.$$

A simple implication would be the following:

Corollary 5.11. *Let f be an algebraic equation in two variables determining an irreducible algebraic curve C in $\mathbb{R}^2$. Then for positive constant δ and for any positive constant c there are only finitely many points defined in $\mathbb{Z}^2$ outside the curve C satisfying the following inequality*

$$d(x, C) < c||x||^{-\delta}.$$

6. **Fractal conjecture**

Conjecture 6.1 (Fractal conjecture)**.** *Let V be an irreducible variety defined over a finitely generated field K and let $F \subset V(K)$ denote a fractal on V with respect to finitely many height-increasing endomorphisms*

$$f_i : V \to V$$

defined over K. Then, for any reduced subscheme Z of V defined over K the Zariski closure of $Z(\bar{K}) \cap F$ is a union of finitely many points and finitely many components B_j such that $B_j(K) \cap F$ is a fractal in B_j for each j, with respect to some f_i's. If no B_j are pre-priodic with respect to any f_i then any $B_j(K) \cap F$ is a fractal in B_j with respect to all f_i's.

Remark 6.2. We can start with $F \subset V(\bar{K})$, but then we can not assume f_i are height increasing and instead we may join some B_j to make a fractal.

In particular, we have stated the following conjecture:

Conjecture 6.3. *For any algebraic curve C embedded in V defined over K which is not invariant under f_i, $C(\bar{K}) \cap F$ is at most a finite set.*

The following conjecture would be a corollary:

Conjecture 6.4. *For any hyperbolic projective curve C embedded in an abelian variety A and any finitely generated subgroup Γ in A (all defined over a finitely generated field K of characteristic zero), $C(\bar{K})\cap\Gamma$ is finite. Even $C(\bar{K})\cap Div(\Gamma)$ is finite, where $Div(\Gamma)$ is divisible group of Γ.*

Remark 6.5. In case A, X and Γ are defined over a number field, the above was content of a generalization a conjecture of Mordell, proved by G. Faltings [5].

There is also another implication of our conjecture 5.1:

Conjecture 6.6 (Forward orbit conjecture)**.** *Let V be an irreducible variety defined over a finitely generated field K and let $f_i : V \to V$ denote finitely many self maps of V defined over K. Let F denote the forward orbit with respect to f_i of finitely many points of V defined over K. Then, for any reduced subscheme Z of V defined over K the Zariski closure of $Z(\bar{K})\cap F$ is union of finitely many points and finitely many components B_j such that $B_j(\bar{K}) \cap F$ is the forward orbit with respect to some f_i of finitely many points of B_j defined over K, for each j.*

It is instructive to notice that, the common geometric structures appearing in the context of Diophantine geometry, is exactly the same as the objects appearing in dynamics of endomorphisms of algebraic varieties which was the original context that height functions were introduced.

Let us start by restating Raynaud's theorem on torsion points of abelian varieties lying on a subvariety [21], which is a special case of conjecture 5.1.

Theorem 6.7 (Raynaud)**.** *Let K be a number field and let A be an abelian variety over the algebraically closed field $\bar{K}$, and Z a reduced subscheme of A. Then every irreducible component of the Zariski closure of $Z(\bar{K}) \cap A(\bar{K})_{tor}$ is a translation of an abelian subvariety of A by a torsion point.*

Another special case is Faltings' theorem on finitely generated subgroups of abelian varieties which has a very similar feature [6].

Theorem 6.8 (Faltings)**.** *Let K be a number field and let A be an abelian variety over the algebraically closed field $\bar{K}$, and Γ be a finitely generated subgroup of $A(\bar{K})$. For a reduced subscheme Z of A, every irreducible component of the Zariski closure of $Z(\bar{K}) \cap \Gamma$ is a translation of an abelian subvariety of A.*

Another consequence of 6.1 would be the following version of generalized Lang's conjecture [28].

Conjecture 6.9 (S. Zhang)**.** *Let X be an algebraic variety defined over a number-field K and let $f : X \to X$ be a surjective endomorphism defined over K. Suppose that the subvariety Y of X is not pre-periodic in the sense that the orbit $\{Y, f(Y), f^2(Y), ...\}$ is not finite, then the set of pre-periodic points in Y is not Zariski-dense in Y.*

Lang's conjecture is confirmed by Raynaud's result mentioned above in the case of abelian varieties and by results of Laurent [11] and Sarnak [22] and S. Zhang [28] in the case of multiplicative groups.

7. Quasi-fractal conjecture

Andre-Oort conjecture on sub-varieties of Shimura varieties is motivated by conjectures of Lang and Manin-Mumford which were proved by Faltings and raynaud respectively as mentioned above. Motivated by the Andre-Oort conjecture, we also present another conjecture in the same lines for quasi-fractals in an algebraic variety X, where self-similarities are allowed to be induced by correspondences instead of maps. For quasi-fractals, we drop the requirement that similar images shall be almost-disjoint.

Conjecture 7.1. *(Quasi-fractal conjecture) Let V be an irreducible variety defined over a finitely generated field K and let $F \subset V(\bar{K})$ denote a quasi-fractal on V with respect to correspondences $Y_1, ..., Y_n$ on V living in $V \times V$ with both projections finite and surjective. F may contain a subvariety of V. Then, for any reduced subscheme Z of V defined over K the Zariski closure of $Z(\bar{K}) \cap F$ is union of finitely many points and finitely many components B_i such that for each i the intersection $B_i(\bar{K}) \cap F$ is a quasi-fractal in B_i with respect to some correspondences induced by Y_i.*

Conjecture 6.1 is a more sophisticated version of our previous conjecture, which implies fractal conjecture also absorbs Andre-Oort conjecture into the fractal formalism. This version utilizes the concept of quasi-fractals.

Definition 7.2. Let V be an algebraic variety and let $Y_i \hookrightarrow V \times V$ for $i = 1$ to n be correspondences on V whose incuced maps to V under canonical projections are finite and surjective. A subset $F \subseteq V(\bar{K})$ is called a quasi-fractal with respect to $Y_1, ..., Y_n$ if F is invariant under the action of correspondences $Y_1, ..., Y_n$.

The l-Hecke orbit of a point on the moduli-space of principally polarized abelian varieties is an example of a quasi-fractal with respect to the l-Hecke correspondences associated to l-isogenies.

J. Pila also gives an unconditional proof of the Andre-Oort conjecture for arbitrary products of modular curves [16]. Pila and Tsimerman eventually prove the full Andre-Oort conjecture [17, 18].

Conjecture 6.1 also covers a parallel version of Andre-Oort conjecture for l-Hecke orbit of a special point in the function field case [2].

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References

[1] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups, *Ann. of Math. (2)* **75** (1962) 485–535.
[2] F. Breuer, Special subvarieties of Drinfeld modular varieties, *J. Reine Angew. Math.* **668** (2012) 35–57.
[3] L. Denis, Hauteurs canoniques et modules de Drinfeld, [Canonical heights and Drinfeld modules], *Math. Ann. (French)* **294** (1992), no. 2, 213–223.
[4] W. Duke, Z. Rudnick and P. Sarnak, Density of integer points on affine homogeneous varieties, *Duke Math. J.* **71** (1993), no. 1, 143–179.
[5] G. Faltings, Finiteness theorems for abelian varieties over number fields, Translated from the German original [*Invent. Math.* **73** (1983), no. 3, 349-366; ibid. **75** (1984), no. 2, 381] by Edward Shipz, in: Arithmetic Geometry (Storrs, Conn., 1984), pp. 9–27, Springer, New York, 1986.
[6] G. Faltings, Diophantine approximation on abelian varieties, *Ann. of Math. (2)* **133** (1991), no. 3, 549–576.
[7] G. Faltings and G. Wstholz, Diophantine approximations on projective spaces, *Invent. Math.* **116** (1994), no. 1-3, 109–138.
[8] J. Franke, Y.I. Manin and Y. Tschinkel, Rational points of bounded height on Fano varieties, *Invent. Math.* **95** (1989), no. 2, 421–435.
[9] S. Lang, Integral points on curves, *Publ. Math. Inst. Hautes Études Sci.* **6** (1960) 27–43.
[10] S. Lang and A. Néron, Rational points of abelian varieties over function fields, *Amer. J. Math.* **81** (1959) 95–118.
[11] M. Laurent, Equations diophantiennes exponentielles, (French), [Exponential Diophantine equations], *Invent. Math.* **78** (1984), no. 2, 299–327.
[12] A. Moriwaki, Arithmetic height functions over finitely generated fields, *Invent. Math.* **140** (2000), no. 1, 101–142.
[13] O. Naghshineh, How to make nice problems, *Nashr-e-riazi* **13** (2003), no.2, 57–60.
[14] A. Néron, Quasi-fonctions et hauteurs sur les varits abliennes, (French) *Ann. of Math. (2)* **82** (1965) 249–331.
[15] D.G. Northcott, Periodic points on an algebraic variety, *Ann. of Math. (2)* **51** (1950) 167–177.
[16] J. Pila, O-minimality and the Andr-Oort conjecture for C^n, *Ann. of Math. (2)* **173** (2011), no. 3, 1779–1840.
[17] J. Pila and J. Tsimerman, The Andr-Oort conjecture for the moduli space of abelian surfaces, *Compos. Math.* **149** (2013), no. 2, 204–216.
[18] J. Pila and J. Tsimerman, Ax-Lindemann for A_g, *Ann. of Math. (2)* **179** (2014), no. 2, 659–681.

[19] A. Rastegar, A geometric formulation of Siegel's diophantine theorem, Arxiv:1504.03162 [math.NT].

[20] A. Rastegar, Approximation on abelian varieties by their subgroups, Arxiv:1504.03367 [math.NT].

[21] M. Raynaud, Sous-varits d'une varit ablienne et points de torsion, (French) [Subvarieties of an abelian variety and torsion points], in: Arithmetic and Geometry, Vol. I, pp. 327–352, Progr. Math. 35 , Birkhäuser Boston, Boston, MA, 1983.

[22] P. Sarnak and S. Adams, Betti numbers of congruence groups, With an appendix by Z. Rudnick. *Israel J. Math.* **88** (1994), no. 1-3, 31–72.

[23] S.H. Schanuel, Heights in number fields, *Bull. Soc. Math. France* **107** (1979), no. 4, 433–449.

[24] W.M. Schmidt, Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height, *Duke Math. J.* **35** (1968) 327–339.

[25] J.P. Serre, Lectures on the Mordell-Weil Theorem, Translated from the French and edited by M. Brown from notes by M. Waldschmidt, With a foreword by Brown and Serre, Aspects of Mathematics 15, Vieweg , 3rd edition, Braunschweig, 1997.

[26] J.L. Thunder, An asymptotic estimate for heights of algebraic subspaces, *Trans. Amer. Math. Soc.* **331** (1992), no. 1, 395–424.

[27] D.Q. Wan, Heights and zeta functions in function fields, in: The Arithmetic of Function Fields (Columbus, OH, 1991) pp. 455–463, Ohio State Univ. Math. Res. Inst. Publ. 2, de Gruyter, Berlin, 1992.

[28] S. Zhang, Positive line bundles on arithmetic varieties, *J. Amer. Math. Soc.* **8** (1995), no. 1, 187–221.

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