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THE RANKS OF THE CLASSES OF A_{10}

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(Communicated by Ali Reza Ashrafi)

Dedicated to Professor Jamshid Moori on the occasion of his 70th Birthday

ABSTRACT. Let G be a finite group and X be a conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G. In this paper we establish the ranks of all the conjugacy classes of elements for simple alternating group A_{10} using the structure constants method and other results established in [A.B.M. Basheer and J. Moori, On the ranks of the alternating group A_n, Bull. Malays. Math. Sci. Soc..
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1. Introduction

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [24] for details). Also Di Martino et al. [11] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [22], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions.

We are interested in generation of finite simple groups by the minimal number of elements from a given conjugacy class of the group. This motivates the following definition.

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Definition 1.1. Let G be a finite simple group and X be a conjugacy class of G. The rank of X in G, denoted by $rank(G{:}X)$ is defined to be the minimal number of elements of X generating G.

One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see Zisser [25]).

In [17–19], J. Moori computed the ranks of involutry classes of the Fischer sporadic simple group Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [16] implies that $rank(Fi_{22}{:}2A) = 6$. In a considerable number of publications (for example but not limited to, see [1–6] or [19]) Moori, Ali and Ibrahim explored the ranks for various sporadic simple groups. In this article we apply the structure constants method together with some results on generation to determine all the ranks of classes of elements for the simple alternating group A_{10}.

2. **Preliminaries**

Let G be a finite group and $C_1, C_2, \ldots, C_k$, $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ being the set of complex irreducible characters of G, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{2.1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta^*_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \ldots, g_{k-1} \rangle = G. \tag{2.2}$$

Definition 2.1. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, the group G is said to be $(C_1, C_2, \ldots, C_k)$**-generated**.

Furthermore if $H \leq G$ is any subgroup containing a fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$. The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that* $\gcd(o(g), [N_G(H){:}H]) = 1$. *Then the number $h(g,H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g,H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See for example Ganief and Moori [13, 14]. $\square$

The above number $h(g,H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \cdots, C_k)$, namely $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq \Theta_G(C_1, C_2, \ldots, C_k)$, where

$$\Theta_G(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \ldots, C_k), \tag{2.3}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups containing elements of all the classes $C_1, C_2, \ldots, C_k$.

Since we have all the maximal subgroups of the sporadic simple groups (except for $G = \mathbb{M}$ the Monster group), it is possible to build a small subroutine in GAP [15] or Magma [8] to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \ldots, C_k)$ for any collection of conjugacy classes of a sporadic simple group.

If $\Theta_G > 0$ then certainly G is $(C_1, C_2, \ldots, C_k)$-generated. In the case $C_1 = C_2 = \cdots = C_{k-1} = C$, G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \leq k-1$.

We now quote some results for establishing generation and non-generation of finite simple groups. These results are also important in determining the ranks of the finite simple groups.

Lemma 2.3 (See Ali and Moori [5] or Conder et al. [9])**.** *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is* $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$*-generated.*

Proof. Since G is (lX, mY, nZ)-generated group, it follows that there exists $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x, y\rangle = G$. Let

$$N := \left\langle x, x^y, x^{y^2}, \ \cdots, x^{y^{m-1}} \right\rangle.$$

Then $N \trianglelefteq G$. Since G is simple and N is non-trivial subgroup we obtain that $N = G$. Furthermore we have

$$\begin{aligned} xx^y x^{y^2} x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2})\cdots(y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

Corollary 2.4 (See Ali and Moori [5]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then $rank(G{:}lX) \leq m$.*

Proof. Follows immediately by Lemma 2.3. □

Lemma 2.5 (See Ali and Moori [5]). *Let G be a finite simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Proof. Since G is $(2X, mY, nZ)$-generated group, it is also $(mY, 2X, tK)$-generated group. The result follows immediately by Lemma 2.3. □

Corollary 2.6. *If G is a finite simple $(2X, mY, nZ)$-generated group. Then $rank(G{:}mY) = 2$.*

Proof. By Lemma 2.5 and Corollary 2.4 we have $rank(G{:}mY) \leq 2$. But a non-abelian simple group can not be generated by one element. Thus $rank(G{:}mY) = 2$. □

The following two results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.7 (See Ali and Moori [5] or Conder et al. [9]). *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Proof. We prove the contrapositive of the statement, that is if $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$ then $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq |C_G(g_k)|$, for a fixed $g_k \in C_k$. To do this let us assume that $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$. Thus there exists at least one $(k-1)$-tuple $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying Equation (2.2). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1 g_2 \cdots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1}) \cdots (xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \ldots, xg_{k-1}x^{-1})$ will generate G. Moreover if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples $(x_1g_1x_1^{-1}, x_1g_2x_1^{-1}, \ldots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \cdots, x_2g_{k-1}x_2^{-1})$ are also distinct since G is centerless. Thus we have at least $|C_G(g_k)|$ $(k-1)$-tuples $(g_1, \ldots, g_{k-1})$ generating G. Hence $\Delta^*_G(C_1, \ldots, C_k) \geq |C_G(g_k)|$. □

The following result is due to Ree [20].

Theorem 2.8. *Let G be a transitive permutation group generated by permutations $g_1, g_2, \ldots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Proof. See for example Ali and Moori [5]. □

The following result is due to Scott ([9] and [21]).

Theorem 2.9 (Scott's Theorem). *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with* $\dim \mathbb{V} = n \geq 2$. *Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([12]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^{G}_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{2.4}$$

3. Ranks of the classes of A_{10}

In this section we apply the general results discussed in Section 2, to the group A_{10}. We determine the ranks for all its conjugacy classes of elements.

The group A_{10} is a simple group of order $1814400 = 2^7 \times 3^4 \times 5^2 \times 7$. By the $\mathbb{ATLAS}$ the group A_{10} has exactly 24 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$H_1 = A_9$	$H_2 = S_8$	$H_3 = (A_7 \times 3){:}2$
$H_4 = (A_5 \times A_5){:}4$	$H_5 = (A_6 \times A_4){:}2$	$H_6 = 2^4{:}S_5$
$H_7 = M_{10}$		

In this section we let $G = A_{10}$. We firstly list in Table 1 the values of $h(g, H_i)$ for all the non-identity classes and maximal subgroups $H_i,\ 1 \leq i \leq 7$, of A_{10}.

We start our investigation on the ranks of the non-trivial classes of A_{10} by looking at the two classes of involutions $2A$ and $2B$. It is well-known that two involutions generate a dihedral group. Thus the lower bound of the rank of an involutry class in a finite group $G \neq D_{2n}$ (the dihedral group of order $2n$) is 3.

The group A_{10} has a 9-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using Equation (2.4) together with the power maps associated with the character table of A_{10} given in the $\mathbb{ATLAS}$, we were able to compute all the values of d_{nX} for all non-trivial classes nX of G, where we list these values in Table 2.

The above values of codimension of the fixed space, will help us much in determining the ranks of many non-trivial classes of G.

TABLE 1. The values $h(g, H_i)$, $1 \leq i \leq 7$ for non-identity classes and maximal subgroups of A_{10}

	H_1	H_2	H_3	H_4	H_5	H_6	H_7
$2A$	6	17	32	26	46	45	0
$2B$	2	5	8	6	10	25	24
$3A$	7	21	36	21	42	0	0
$3B$	4	6	6	6	9	9	0
$3C$	1	0	3	0	3	0	9
$4A$	0	3	0	8	4	7	0
$4B$	4	7	8	4	8	3	0
$4C$	2	1	0	2	2	5	12
$5A$	5	10	10	1	5	0	0
$5B$	0	0	0	1	0	5	5
$6A$	3	5	8	5	10	0	0
$6B$	0	2	2	2	1	9	0
$6C$	2	2	2	0	1	1	0
$7A$	3	3	1	0	0	0	0
$8A$	0	1	0	2	0	1	2
$9A$	1	0	0	0	0	0	0
$9B$	1	0	0	0	0	0	0
$10A$	1	2	2	1	1	0	0
$12A$	1	1	2	1	2	0	0
$12B$	0	0	0	2	1	1	0
$15A$	2	1	1	1	2	0	0
$21A$	0	0	1	0	0	0	0
$21B$	0	0	1	0	0	0	0

TABLE 2. $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 9$

nX	$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$5A$	$5B$	$6A$	$6B$
d_{nX}	2	4	2	4	6	6	4	6	4	8	4	6
nX	$6C$	$7A$	$8A$	$9A$	$9B$	$10A$	$12A$	$12B$	$15A$	$21A$	$21B$	
d_{nX}	6	6	8	8	8	6	6	8	6	8	8	

Lemma 3.1. $rank(G{:}2A) \notin \{3, 4\}$.

Proof. If G is $(2A, 2A, 2A, nX)$-generated group for a non trivial class nX of G, then by Theorem 2.9 we must have $d_{2A} + d_{2A} + d_{2A} + d_{nX} \geq 2 \times 9$. However, it is clear from Table 2 that $3 \times d_{2A} + d_{nX} < 18$, for each nX of G. Thus G is not $(2A, 2A, 2A, nX)$-generated group and it follows that $rank(G{:}2A) \neq 3$. Similarly G is not $(2A, 2A, 2A, 2A, nX)$-generated group and therefore $rank(G{:}2A) \neq 4$. Hence the result. □

Lemma 3.2. *The group G is $(2A, 5B, 21A)$-generated.*

Proof. Let $a := (7, 8)(9, 10) \in 2A$ and $b := (1, 2, 3, 4, 7)(5, 6, 9, 8, 10) \in 5B$. Then $\langle a, b\rangle = G$ with $ab = (1, 2, 3, 4, 7, 10, 8)(5, 6, 9) \in 21A$. Thus G is $(2A, 5B, 21A)$-generated. □

Proposition 3.3. $rank(G{:}2A) = 5$.

Proof. Since by Lemma 3.2, G is $(2A, 5B, 21A)$-generated group, it follows by applications of Lemma 2.3 that G is $(2A, 2A, 2A, 2A, 2A, (21A)^5)$-generated group. Thus $rank(G{:}2A) \leq 5$. Since $rank(G{:}2A) \notin \{3, 4\}$ by Lemma 3.1, it follows that $rank(G{:}2A) = 5$. □

We recall from the PhD thesis of J. Ward [22] that a finite non-abelian simple group G is said to have *Property 1* if G can be generated by 5 conjugate involutions whose product is the identity. Also G is said to satisfy *Property 2* if G can be generated by 3 conjugate involutions a, b and c, two of which, a and b, commute and such that ab is also conjugate to a, b and c.

Proposition 3.4. $rank(G{:}2B) = 3$.

Proof. From [22, Table 1.1] we see that G has both Property 1 and Property 2. By the latter one, we infer that $G = \langle a, b, c\rangle$ with a, b and c are all conjugate and $ab = ba$ and $ab \in [a]_G$. Now from Lemma 3.1 we know that these three involutions are not from class $2A$ (otherwise we would have $rank(G{:}2A) = 3$). Since G has only two classes of involutions, it follows that these three conjugate involutions generating G are contained in class $2B$. Hence $rank(G{:}2B) = 3$. □

Proposition 3.5. $rank(G{:}3A) = 5$.

Proof. Direct application of [7, Theorem 12]. □

Remark 3.6. The rank of class $3A$ in G can still be established using the structure constant method together with the results of Section 2.

Lemma 3.7. *For $nX \in S := \{3B, 4B, 5A, 6A\}$, the group G is $(3C, nX, 21A)$-generated.*

Proof. Let

$$\begin{array}{ll} a := (2,3,4)(5,6,7)(8,9,10) \in 3C, & b := (1,2,8)(4,5,10) \in 3B, \\ c := (1,2)(3,4,5,8) \in 4B, & d := (1,2,8,5,4) \in 5A \ \text{ and} \\ e := (1,2)(4,5,8)(7,10) \in 6A. & \end{array}$$

We note that $ab = (1,2,3,5,6,7,10)(4,8,9)$, $ac = (1,2,4)(3,5,6,7,8,9,10)$, $ad = (1,2,3)(4,8,9,10,5,6,7)$ and $ae = (1,2,3,5,6,10,4)(7,8,9)$ are all elements of order 21 and in fact are all contained in $21A$. Now it is not difficult to verify that

$$\langle a, b\rangle = \langle a, c\rangle = \langle a, d\rangle = \langle a, e\rangle = G = A_{10}$$

and thus G is $(3C, nX, 21A)$-generated group for $nX \in S$. □

Proposition 3.8. *With S being the previous set, we have $rank(G{:}nX) = 3$ for all $nX \in S$.*

Proof. If G is (nX, nX, mY)-generated group for $nX \in S$ and for any other non trivial class mY of G, then we must have $d_{nX} + d_{nX} + d_{mY} \geq 2 \times 9$. However, it is clear from Table 2 that $2 \times d_{nX} + d_{mY} < 18$, for $nX \in S$ and all non-trivial classes mY of G and therefore G is not (nX, nX, mY)-generated group. This establishes the non-generation of G by two elements from $nX \in S$ and thus $rank(G{:}nX) \neq 2$, $nX \in S$. Using Lemma 3.7 we know that G is $(3C, nX, 21A)$-generated group. This implies that G is also $(nX, 3C, 21A)$-generated group. It follows by applications of Lemma 2.3 that G is $(nX, nX, nX, (21A)^3)$-generated group; that is $(nX, nX, nX, 7A)$-generated group. Thus $rank(G{:}nX) \leq 3$. Since $rank(G{:}nX) \neq 2$ we deduce that $rank(G{:}nX) = 3$ for all $nX \in S$, completing the proof. □

Lemma 3.9. *$rank(A_n{:}5A) = 2$, for $5 \leq n \leq 9$.*

Proof. The cases A_5, A_8 and A_9 follows respectively by [7, Theorem 14, Propositions 20 and 29]. The group A_6 is generated by $(1, 2, 3, 4, 5)$ and $(1, 3, 4, 5, 6)$ both are elements of class $5A$ of A_6. We handle the case A_7. We claim that $A_7 = \langle (1, 2, 3, 4, 5), (1, 4, 5, 6, 7)\rangle$. We have $(1, 2, 3, 4, 5)$ $(1, 4, 5, 6, 7)$ $= (1, 5, 6, 7, 2, 3, 4)$, which has order 7. This implies that

$$35 || \langle (1, 2, 3, 4, 5), (1, 4, 5, 6, 7)\rangle |.$$

By looking at the maximal subgroups of A_7 (see the $\mathbb{ATLAS}$ [10] for example) we can see that there is no maximal subgroup of A_7 with order divisible by 35. It follows that $\langle (1, 2, 3, 4, 5), (1, 4, 5, 6, 7)\rangle = A_7$ and hence $rank(A_7{:}5A) = 2$, completing the result. □

Lemma 3.10. *$rank(A_n{:}5A) \neq 2$, $\forall\ n \geq 10$.*

Proof. Suppose that $x, y \in 5A$ of A_n, $n \geq 10$ and let $x = (a, b, c, d, e)$ and $y = (f, g, h, i, j)$. If $xy = yx$ then $\langle x, y\rangle \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. If $xy \neq yx$, then x and y are not disjoint cycles and have some common points, i.e, $\{a, b, c, d, e\} \bigcap \{f, g, h, i, j\} \neq \phi$. Thus the number of moved points by $\langle x, y\rangle$ is at most 9 and it follows that $\langle x, y\rangle \leq A_9$. Hence $rank(A_n{:}5A) \neq 2$ for $n \geq 10$. □

Note that the non-generation of G by two elements from class $5A$ was already shown in the proof of Proposition 3.8 using Scott's Theorem and Table 2.

We now have the following conjecture concerning the rank of class $5A$ of A_n for all $n \geq 6$.

Conjecture 1. For any $n \geq 6$, we speculate that

$$rank(A_n{:}5A) = \begin{cases} \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n = 4k \text{ or } 4k+1, \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n = 4k+2 \text{ or } 4k+3. \end{cases} \tag{3.1}$$

Note that the above conjecture was shown to be true theoretically for $6 \leq n \leq 9$ in Lemma 3.9. The author used GAP to verify the correctness of the conjecture up to $n \leq 50$. Moreover, the author is speculating that

$$
(3.2) \qquad A_n = \begin{cases} \langle(1,2,3,4,5),(1,5,6,7,8),(1,9,10,11,12),\ldots, \\ (1,n-6,n-5,n-4,n-3),(1,n-3,n-2,n-1,n)\rangle & \text{if } n=4k, \\ \langle(1,2,3,4,5),(1,6,7,8,9),(1,10,11,12,13),\ldots, \\ (1,n-7,n-6,n-5,n-4),(1,n-3,n-2,n-1,n)\rangle & \text{if } n=4k+1, \\ \langle(1,2,3,4,5),(1,3,4,5,6),(1,7,8,9,10),\ldots, \\ (1,n-4,n-3,n-2,n-1),(1,n-3,n-2,n-1,n)\rangle & \text{if } n=4k+2, \\ \langle(1,2,3,4,5),(1,4,5,6,7),(1,8,9,10,11),\ldots, \\ (1,n-5,n-4,n-3,n-2),(1,n-3,n-2,n-1,n)\rangle & \text{if } n=4k+3. \end{cases}
$$

Proposition 3.11. *$rank(G{:}9X) = 2$, for $X \in \{A, B\}$.*

Proof. Direct application of [7, Theorem 14]. □

The result of Proposition 3.11 can be established using the structure constant method. In the next proposition, we give the ranks of all the remaining non-trivial classes of G.

Proposition 3.12. *Let $T := \{3C, 4A, 4C, 5B, 6B, 6C, 7A, 8A, 10A, 12A, 12B, 15A, 21A, 21B\}$. Then $rank(G{:}nX) = 2$ for all $nX \in T$.*

Proof. The aim here is to show that G is an $(nX, nX, 21A)$-generated group for any $nX \in T$. We firstly note that all the maximal subgroups of G do not contain elements of order 21 except H_3. By $h(C, H)$ we mean the number of conjugate subgroups of H that contain a fixed element of the conjugacy class C. It follows that $h(21A, H_i) = 0$ for all $i \in \{1, 2, \cdots, 7\} \setminus \{3\}$, while $h(21A, H_3) = 1$. For all the classes $nX \in T$ we give in Table 3 some information about $\Delta_G(nX, nX, 21A)$, $h(21A, H_3)$, $\Sigma_{H_3}(nX, nX, 21A)$ and $\Theta_G(nX, nX, 21A)$. The last column of Table 3 establishes the generation of G

TABLE 3. Some information on the classes $nX \in T$

	$\Delta_G(nX,nX,21A)$	$h(21A,H_3)$	$\Sigma_{H_3}(nX,nX,21A)$	$\Theta_G(nX,nX,21A)$
$3C$	301	1	28	273
$4A$	189	1	0	189
$4C$	1512	1	0	1512
$5B$	2709	1	0	2709
$6B$	308	1	14	294
$6C$	11340	1	840	10500
$7A$	2541	1	0	2541
$8A$	25704	1	0	25704
$10A$	4977	1	252	4725
$12A$	13104	1	357	12747
$12B$	11340	1	0	11340
$15A$	7077	1	84	6993
$21A$	3936	1	72	3864
$21B$	3711	1	36	3675

by the triple $(nX, nX, 21A)$ for all $nX \in T$. It follows that $rank(G{:}nX) = 2$ for all nX. □

Remark 3.13. For all $nX \in T$ of Proposition 3.12, the result can also be proved using Lemma 2.6 together with the facts that G is $(2B, nX, 21B)$-generated group for $nX \in \{3C, 4C, 5B, 6C, 7A, 8A, 10A, 12B, 15A, 21A\}$ and is $(2B, nX, 21A)$-generated group for $nX \in \{4A, 6B, 12A, 21B\}$.

Now we gather the results on ranks of the non-trivial classes of G.

Theorem 3.14. *Let G be the alternating group A_{10}. Then*

(1) $rank(G{:}nA) = 5$, *for* $n \in \{2, 3\}$,
(2) $rank(G{:}nX) = 3$, *for* $nX \in \{2B, 3B, 4B, 5A, 6A\}$,
(3) $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 2B, 3A, 3B, 4B, 5A, 6A\}$.

Proof. The result follows from Propositions 3.3, 3.4, 3.5, 3.8, 3.11 and 3.12. □

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