A NOTE ON WEIGHTED COMPOSITION OPERATORS ON $L^p$-SPACES

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Abstract. In this paper we will consider the weighted composition operators $uC\varphi$ between two different $L^p(X, \Sigma, \mu)$ spaces, generated by measurable and non-singular transformations $\varphi$ from $X$ into itself and measurable functions $u$ on $X$. We characterize the functions $u$ and transformations $\varphi$ that induce weighted composition operators between $L^p$-spaces by using some properties of conditional expectation operator, pair $(u, \varphi)$ and the measure space $(X, \Sigma, \mu)$. Also, some other properties of these types of operators will be investigated.

1. Preliminaries And Notation

Let $(X, \Sigma, \mu)$ be a sigma finite measure space. By $L(X)$, we denote the linear space of all $\Sigma$-measurable functions on $X$. When we consider any subsigma algebra $\mathcal{A}$ of $\Sigma$, we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the $L^p$-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ to $L^p(\mathcal{A})$, and denote its norm by $\|\cdot\|_p$. We define the support of a measurable function $f$ as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. Here functions which are equal $\mu$-almost everywhere are

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An atom of the measure \( \mu \) is an element \( B \in \Sigma \) with \( \mu(B) > 0 \) such that for each \( F \in \Sigma \), if \( F \subset B \) then either \( \mu(F) = 0 \) or \( \mu(F) = \mu(B) \). A measure with no atoms is called non-atomic.

We can easily check the following well known facts (see [9]):

(a) Every sigma finite measure space \((X, \Sigma, \mu)\) can be decomposed into two disjoint sets \( B \) and \( Z \), such that \( \mu \) is a non-atomic over \( B \) and \( Z \) is a countable union of atoms of finite measure.

(b) For each \( f \in L^r(\Sigma) \), there exist two functions \( f_1 \in L^p(\Sigma) \) and \( f_2 \in L^q(\Sigma) \) such that \( f = f_1f_2 \) and \( \|f\|_r = \|f_1\|_p = \|f_2\|_q \) where \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \).

Associated with each sigma algebra \( \mathcal{A} \subseteq \Sigma \), there exists an operator \( E(\cdot|\mathcal{A}) = E^\mathcal{A}(\cdot) \), which is called conditional expectation operator, on the set of all non-negative measurable functions \( f \) or for each \( f \in L^p \) for any \( p \), \( 1 \leq p \leq \infty \), and is uniquely determined by the conditions

(i) \( E^\mathcal{A}(f) \) is \( \mathcal{A} \)- measurable, and

(ii) if \( A \) is any \( \mathcal{A} \)- measurable set for which \( \int_A f d\mu \) exists, we have \( \int_A f d\mu = \int_A E^\mathcal{A}(f) d\mu \).

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. \( E^\mathcal{A}(f.g \circ T) = E^\mathcal{A}(f)(g \circ T) \).

E2. \( E^\mathcal{A}(1) = 1 \).

E3. \( |E^\mathcal{A}(fg)|^2 \leq E^\mathcal{A}(|f|^2)E^\mathcal{A}(|g|^2) \).

E4. If \( f > 0 \) then \( E^\mathcal{A}(f) > 0 \).

Properties E1. and E2. imply that \( E^\mathcal{A}(\cdot) \) is idempotent and \( E^\mathcal{A}(L^p(\Sigma)) = L^p(\mathcal{A}) \). Suppose that \( \varphi \) is a mapping from \( X \) into \( X \) which is measurable, (i.e., \( \varphi^{-1}(\Sigma) \subseteq \Sigma \)) such that \( \mu \circ \varphi^{-1} \) is absolutely continuous with respect to \( \mu \) (we write \( \mu \circ \varphi^{-1} \ll \mu \), as usual). Let \( h \) be the Radon-Nikodym derivative \( h = \frac{d\mu \circ \varphi^{-1}}{d\mu} \). If we put \( \mathcal{A} = \varphi^{-1}(\Sigma) \), it is easy to show that for each non-negative \( \Sigma \)-measurable function \( f \) or for each \( f \in L^p(\Sigma) (p \geq 1) \), there exists a \( \Sigma \)-measurable function \( g \) such that \( E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi \). We can assume that the support of \( g \) lies in the support of \( h \), and there exists only one \( g \) with this property. We then write \( g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1} \), though we
make no assumptions regarding the invertibility of \( \varphi \) (see [2]). For a deeper study of the properties of \( E \) see the paper [6].

2. Some Results On Weighted Composition Operators Between Two \( L^p \)-Spaces

Let \( 1 \leq q \leq p < \infty \) and we define \( \mathcal{K}_{p,q} \) or \( \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) \) as follows:

\[
\mathcal{K}_{p,q} = \{ u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma) \}.
\]

\( \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) \) is a vector subspace of \( L(X) \). Also note that if \( 1 \leq q = p < \infty \), then \( L^\infty(\Sigma) \subseteq \mathcal{K}_{p,p}(\mathcal{A}, \Sigma) \) and \( \mathcal{K}_{p,p}(\Sigma, \Sigma) = L^\infty(\Sigma) \) (see [3]; problem 64, 65).

For \( u \in L(X) \), let \( M_u \) from \( L^p(\mathcal{A}) \) into \( L(X) \) defined by \( M_u f = u.f \) be the corresponding linear transformation. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each \( L^p \) convergent sequence assures us that for each \( u \in \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) \), the operator \( M_u : L^p(\mathcal{A}) \rightarrow L^q(\Sigma) \) is a multiplication operator (bounded linear transformation).

We shall find the relationship between a sigma finite algebra \( \mathcal{A} \subseteq \Sigma \) and the set of multiplication operators which map \( L^p(\mathcal{A}) \) into \( L^q(\Sigma) \). Our first task is the description of the members of \( \mathcal{K}_{p,q} \) in terms of the conditional expectation induced by \( \mathcal{A} \).

**Theorem 1.1.** Suppose \( 1 \leq q < p < \infty \) and \( u \in L(X) \). Then \( u \in \mathcal{K}_{p,q} \) if and only if \( \left( E^\mathcal{A}(|u|^q) \right)^{\frac{1}{q}} \in L(\mathcal{A}) \), where \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} \).

**Proof.** To prove the theorem, we adopt the method used by Axler [1]. Suppose \( \left( E^\mathcal{A}(|u|^q) \right)^{\frac{1}{q}} \in L(\mathcal{A}) \), so \( E^\mathcal{A}(|u|^q) \in L^{\frac{q}{r}}(\mathcal{A}) \). For each \( f \in L^p(\mathcal{A}) \), we have \( |f|^q \in L^{\frac{q}{r}}(\mathcal{A}) \). Since \( \frac{q}{p} + \frac{q}{r} = 1 \), Hölder’s inequality yields

\[
\|u.f\|_q = \left( \int |u|^q |f|^q d\mu \right)^{\frac{1}{q}} = \left( \int E^\mathcal{A}(|u|^q) |f|^q d\mu \right)^{\frac{1}{q}}
\]
\[
\begin{aligned}
\left\{ \left( \int \left( E^A(|u|^q) \right)^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} \left( \int \left( |f|^q \right)^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} \right\}^{\frac{1}{q}} = \| (E^A(|u|^q))^{\frac{1}{q}} \|_r \| f \|_p.
\end{aligned}
\]

Hence \( u \in K_{p,q} \). Now suppose only that \( u \in K_{p,q} \). So the operator \( M_u : L^p(A) \to L^q(\Sigma) \) given by \( M_u f = u \cdot f \) is a bounded linear operator. Let \( \varphi \) be a nonnegative integrable simple function then

\[
\int E^A(|u|^q) \varphi d\mu \leq \| M_u \|_q^{\frac{q}{p}} \left( \int \varphi^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} = \| M_u \|_q^{\frac{q}{p}} \| \varphi \|_\frac{r}{q}.
\]

It follows that \( E^A(|u|^q) \in L(\frac{r}{q})'(X, A, \mu|_A) \simeq L^r(X, A, \mu|_A). \)

**Corollary 2.2.** Suppose \( 1 \leq q < p < \infty \) and \( u \in L(X) \). Then \( M_u \) from \( L^p(\Sigma) \) into \( L^q(\Sigma) \) is bounded linear operator if and only if \( u \in L^{\frac{mr}{p(1-r/q)}}(\Sigma) \). In this case \( \| M_u \| = \| u \|_{\frac{mr}{p(1-r/q)}} \).

**Proof.** Put \( A = \Sigma \) in the previous theorem. Then we will have \( E^A = I \) (identity operator). Then the proof holds.

Let \( u \in L(X) \) and \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} \). If \( p = q \) then \( r \) must be \( \infty \). So \( M_u(L^p(\Sigma)) \subseteq L^q(\Sigma) \) if and only if \( u \in L^{\infty}(\Sigma) \). In this case \( \| M_u \| = \| u \|_{\infty} \). This fact is well-known. For the direct proof, see [3].

Take a function \( u \) in \( L(X) \) and let \( \varphi : X \to X \) be a non-singular measurable transformation; i.e. \( \mu(\varphi^{-1}(A)) = 0 \) for all \( A \in \Sigma \) such that \( \mu(A) = 0 \). Then the pair \((u, \varphi)\) induces a linear operator \( uC_{\varphi} \) from \( L^p(\Sigma) \) into \( L(X) \) defined by

\[
uC_{\varphi}(f) = u \cdot f \circ \varphi \quad (f \in L^p(\Sigma)).
\]

Here, the non-singularity of \( \varphi \) guarantees that \( uC_{\varphi} \) is well defined as a mapping of equivalence classes of functions on support \( u \). If \( uC_{\varphi} \) takes \( L^p(\Sigma) \) into \( L^q(\Sigma) \), then we call \( uC_{\varphi} \) a weighted composition operator \( L^p(\Sigma) \) into \( L^q(\Sigma) \) \((1 \leq q \leq \infty)\).

Boundedness of composition operators in \( L^p(\Sigma) \) spaces \((1 \leq p \leq \infty)\) where measure spaces are sigma finite appeared already in Singh paper [7] and for two different \( L^p(\Sigma) \) spaces in the paper [8]. Also boundedness of weighted composition operators on
$L^p(\Sigma)$ spaces has already been studied in [4]. Namely, for a non-singular measurable transformation $\varphi$ and complex valued measurable weight function $u$ on $X$, $uC_\varphi$ is bounded if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$. In the following theorem we give a necessary and sufficient condition for boundedness of weighted composition operators from $L^p(\Sigma)$ into $L^q(\Sigma)$, where $p > q$ as follows:

**Theorem 2.3.** Suppose $1 \leq q < p < \infty$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Let $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair $(u, \varphi)$ induces a weighted composition operator $uC_\varphi$ from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $J = hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1} \in L^\frac{r}{q}(\Sigma)$.

**Proof.** Let $f \in L^p(\Sigma)$. We will have

$$\|uC_\varphi f\|_q^q = \int |u \circ \varphi|^q d\mu = \int hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1} |f|^q d\mu = \int |\sqrt{J} f|^q d\mu = \|M_{\sqrt{J}} f\|_q^q.$$ 

So by Corollary 2.2, $uC_\varphi$ is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $\sqrt{J} \in L^r(\Sigma)$ or equivalently $J \in L^\frac{r}{q}(\Sigma)$. □

**Corollary 2.4.** Suppose $1 \leq p \leq \infty$, $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair $(u, \varphi)$ induces a weighted composition operator $uC_\varphi$ from $L^p(\Sigma)$ into $L^p(\Sigma)$ if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$.

**Corollary 2.5.** Under the same assumptions as in theorem 2.3, $\varphi$ induces a composition operator $C_\varphi : L^p(\Sigma) \to L^q(\Sigma)$ if and only if $h \in L^\frac{r}{q}(\Sigma)$.

**Remark 2.6.** One of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let $X$ be $[0, 1]$, $\Sigma$ the Borel sets, and $\mu$ Lebesgue measure.
Let $\varphi$ be the map $\varphi(x) = x^3$ on $[0, 1]$. A simple computation shows that $h = 1/3x^{-2/3} \notin L^3(\Sigma)$. Then $C_{\varphi}$ does not define a bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$. However with $u(x) = x$, we have $\varphi^{-1}(\Sigma) = \Sigma$ (so $E = I$) and $J = 1/3 \in L^2(\Sigma)$. Hence $uC_{\varphi} = M_u \circ C_{\varphi}$ is bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$.

The procedure which Axler has used for the case $p < q$ in [1], when $X$ is the interval $[-\pi, \pi]$, can also be used here.

At this stage we investigate a necessary and sufficient condition for a multiplication operator to be Fredholm. For a bounded linear operator $A$ on a Banach space, we use the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote the kernel and the range of $A$, respectively. We recall that $A$ is said to be a Fredholm operator if $\mathcal{R}(A)$ is closed and if $\dim\mathcal{N}(A) < \infty$ and $\text{codim}\mathcal{R}(A) < \infty$. Now we attempt to prove a theorem which is likely to be found elsewhere.

**Theorem 2.7.** Suppose that $\mu$ is a non-atomic measure on $L^2(\Sigma)$. Then the following conditions are equivalent:

(a) $M_u$ is an invertible operator.

(b) $M_u$ is a Fredholm operator.

(c) $\mathcal{R}(M_u)$ is closed and $\text{codim}\mathcal{R}(M_u) < \infty$.

(d) $|u| \geq \delta$ almost everywhere on $X$ for some $\delta > 0$.

**Proof.** The implications (d) $\implies$ (a) $\implies$ (b) $\implies$ (c) are obvious. We show (c) $\implies$ (d).

Suppose that $\mathcal{R}(M_u)$ is closed and $\text{codim}\mathcal{R}(M_u) < \infty$. Then there exists a $\delta > 0$ such that $|u| \geq \delta$ on $\sigma(u)$. So it is enough to show that $\mu(\sigma(u)^c) = 0$. First of all we prove that $M_u$ is onto. Let $0 \neq f_0 \in \mathcal{R}(M_u)^\perp$, therefore, for any $f \in L^2(\Sigma)$ we have $(M_u f, f_0) = 0$. Now we choose $t > 0$ such that the set

$$Z_t = \{s \in X : |f_0|^2(x) \geq t\}$$

is of positive measure. Since $\mu$ is a non-atomic measure we may choose a sequence of disjoint subsets $Z_n$ of $Z_t$ such that $0 < \mu(Z_n) < \infty$. Now let $g_n = \chi_{Z_n} f_0$. It is clear that each $g_n$ is non-zero element of $L^2(\Sigma)$, and for $n \neq m$, $(g_n, g_m) = 0$. Therefore, for $f \in L^2(\Sigma)$
we have
\[(f, M_u^* g_n) = (M_u f, \chi_{Z_n} f_0) = (M_u \chi_{Z_n} f, f_0) = 0.\]

So \(g_n \in \mathcal{N}(M_u^*)\) for any \(n\). Therefore, \(\{g_n\}\) is a linearly independent subset of \(\mathcal{N}(M_u^*)\), which is a contradiction to \(\dim \mathcal{N}(M_u^*) = \text{codim } \mathcal{R}(M_u) < \infty\). If \(\mu(\sigma(u)^c) > 0\), then there exists a set \(Z \subset \sigma(u)^c\) such that \(0 < \mu(Z) < \infty\), so we conclude that \(\chi_Z \in L^2(\Sigma) \backslash \mathcal{R}(M_u)\), which contradicts the fact that \(M_u\) is onto. Therefore \(\mu(\sigma(u)^c) = 0\).

\textbf{Corollary 2.8.} \(M_u\) is a Fredholm operator if and only if \(M_u^n (= M_u^n)\) is also Fredholm.

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\textbf{References}

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