A NOTE ON WEIGHTED COMPOSITION OPERATORS ON L^p-SPACES

M. R. JABBARZADEH AND E. POURREZA

ABSTRACT. In this paper we will consider the weighted composition operators uC_{φ} between two different $L^p(X, \Sigma, \mu)$ spaces, generated by measurable and non-singular transformations φ from X into itself and measurable functions u on X. We characterize the functions u and transformations φ that induce weighted composition operators between L^p -spaces by using some properties of conditional expectation operator, pair (u, φ) and the measure space (X, Σ, μ) . Also, some other properties of these types of operators will be investigated.

1. Preliminaries And Notation

Let (X, Σ, μ) be a sigma finite measure space. By L(X), we denote the linear space of all Σ -measurable functions on X. When we consider any subsigma algebra \mathcal{A} of Σ , we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the L^p -space $L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|.\|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. Here functions which are equal μ -almost everywhere are

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identical. An atom of the measure μ is an element $B \in \Sigma$ with $\mu(B) > 0$ such that for each $F \in \Sigma$, if $F \subset B$ then either $\mu(F) = 0$ or $\mu(F) = \mu(B)$. A measure with no atoms is called non-atomic. We can easily check the following well known facts (see [9]):

(a) Every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is a non-atomic over B and Z is a countable union of atoms of finite measure.

(b) For each $f \in L^r(\Sigma)$, there exist two functions $f_1 \in L^p(\Sigma)$ and $f_2 \in L^q(\Sigma)$ such that $f = f_1 f_2$ and $||f||_r^r = ||f_1||_p^p = ||f_2||_q^q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Associated with each sigma algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$, which is called *conditional expectation* operator, on the set of all non-negative measurable functions f or for each $f \in L^p$ for any $p, 1 \leq p \leq \infty$, and is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is \mathcal{A} - measurable, and

(ii) if A is any \mathcal{A} - measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$.

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^{\mathcal{A}}(f.g \circ T) = E^{\mathcal{A}}(f)(g \circ T).$ E2. $E^{\mathcal{A}}(1) = 1.$ E3. $|E^{\mathcal{A}}(fg)|^2 \leq E^{\mathcal{A}}(|f|^2)E^{\mathcal{A}}(|g|^2).$ E4. If f > 0 then $E^{\mathcal{A}}(f) > 0.$

Properties E1. and E2. imply that $E^{\mathcal{A}}(\cdot)$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma))$

= $L^p(\mathcal{A})$. Suppose that φ is a mapping from X into X which is measurable, (i.e., $\varphi^{-1}(\Sigma) \subseteq \Sigma$) such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual). Let h be the Radon-Nikodym derivative $h = \frac{d\mu \circ \varphi^{-1}}{d\mu}$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ -measurable function f or for each $f \in L^p(\Sigma)$ $(p \ge 1)$, there exists a Σ -measurable function g such that $E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$, though we make no assumptions regarding the invertibility of φ (see [2]). For a deeper study of the properties of E see the paper [6].

2. Some Results On Weighted Composition Operators Between Two L^p -Spaces

Let $1 \leq q \leq p < \infty$ and we define $\mathcal{K}_{p,q}$ or $\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ as follows:

$$\mathcal{K}_{p,q} = \{ u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma) \}.$$

 $\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ is a vector subspace of L(X). Also note that if $1 \leq q = p < \infty$, then $L^{\infty}(\Sigma) \subseteq \mathcal{K}_{p,p}(\mathcal{A}, \Sigma)$ and $\mathcal{K}_{p,p}(\Sigma, \Sigma) = L^{\infty}(\Sigma)$ (see [3]; problem 64, 65).

For $u \in L(X)$, let M_u from $L^p(\mathcal{A})$ into L(X) defined by $M_u f = u.f$ be the corresponding linear transformation. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each L^p convergent sequence assures us that for each $u \in \mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$, the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is a multiplication operator (bounded linear transformation).

We shall find the relationship between a sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and the set of multiplication operators which map $L^p(\mathcal{A})$ into $L^q(\Sigma)$. Our first task is the description of the members of $\mathcal{K}_{p,q}$ in terms of the conditional expectation induced by \mathcal{A} .

Theorem 1.1. Suppose $1 \leq q and <math>u \in L(X)$. Then $u \in \mathcal{K}_{p,q}$ if and only if $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

Proof. To prove the theorem, we adopt the method used by Axler [1]. Suppose $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, so $E^{\mathcal{A}}(|u|^q) \in L^{\frac{r}{q}}(\mathcal{A})$. For each $f \in L^p(\mathcal{A})$, we have $|f|^q \in L^{\frac{p}{q}}(\mathcal{A})$. Since $\frac{q}{p} + \frac{q}{r} = 1$, Hölder's inequality yields

$$||u.f||_{q} = \left\{ \int |u|^{q} |f|^{q} d\mu \right\}^{\frac{1}{q}} = \left\{ \int E^{\mathcal{A}}(|u|^{q}) |f|^{q} d\mu \right\}^{\frac{1}{q}}$$

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$$\leq \left\{ \left(\int \left(E^{\mathcal{A}}(|u|^{q}) \right)^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} \left(\int \left(|f|^{q} \right)^{\frac{p}{q}} d\mu \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} = \| (E^{\mathcal{A}}(|u|^{q}))^{\frac{1}{q}} \|_{r} \|f\|_{p}.$$

Hence $u \in \mathcal{K}_{p,q}$. Now suppose only that $u \in \mathcal{K}_{p,q}$. So the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ given by $M_u f = u.f$ is a bounded linear operator. Let φ be a nonnegative integrable simple function then

$$\int E^{\mathcal{A}}\left(\left|u\right|^{q}\right)\varphi d\mu \leq \left\|M_{u}\right\|^{q}\left(\int\varphi^{\frac{p}{q}}d\mu\right)^{\frac{q}{p}} = \left\|M_{u}\right\|^{q}\left\|\varphi\right\|_{\frac{p}{q}}$$

It follows that $E^{\mathcal{A}}(|u|^{q}) \in L^{\left(\frac{p}{q}\right)'}(X, \mathcal{A}, \mu_{|\mathcal{A}}) \simeq L^{\frac{r}{q}}(X, \mathcal{A}, \mu_{|\mathcal{A}}).$

Corollary 2.2. Suppose $1 \le q and <math>u \in L(X)$. Then M_u from $L^p(\Sigma)$ into $L^q(\Sigma)$ is bounded linear operator if and only if $u \in L^{\frac{pq}{p-q}}(\Sigma)$. In this case $||M_u|| = ||u||_{\frac{pq}{p-q}}$.

Proof. Put $\mathcal{A} = \Sigma$ in the previous theorem. Then we will have $E^{\mathcal{A}} = I$ (identity operator). Then the proof holds.

Let $u \in L(X)$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. If p = q then r must be ∞ . So $M_u(L^p(\Sigma)) \subseteq L^p(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$. In this case $||M_u|| = ||u||_{\infty}$. This fact is well-known. For the direct proof, see [3].

Take a function u in L(X) and let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair (u, φ) induces a linear operator uC_{φ} from $L^p(\Sigma)$ into L(X) defined by

$$uC_{\varphi}(f) = u.f \circ \varphi \qquad (f \in L^p(\Sigma)).$$

Here, the non-singularity of φ guarantees that uC_{φ} is well defined as a mapping of equivalence classes of functions on support u. If uC_{φ} takes $L^{p}(\Sigma)$ into $L^{q}(\Sigma)$, then we call uC_{φ} a weighted composition operator $L^{p}(\Sigma)$ into $L^{q}(\Sigma)$ $(1 \leq q \leq \infty)$.

Boundedness of composition operators in $L^p(\Sigma)$ spaces $(1 \leq p \leq \infty)$ where measure spaces are sigma finite appeared already in Singh paper [7] and for two different $L^p(\Sigma)$ spaces in the paper [8]. Also boundedness of weighted composition operators on

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 $L^{p}(\Sigma)$ spaces has already been studied in [4]. Namely, for a nonsingular measurable transformation φ and complex valued measurable weight function u on X, uC_{φ} is bounded if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^{p}) \circ \varphi^{-1} \in L^{\infty}(\Sigma)$. In the following theorem we give a necessary and sufficient condition for boundedness of weighted composition operators from $L^{p}(\Sigma)$ into $L^{q}(\Sigma)$, where p > q as follows:

Theorem 2.3. Suppose $1 \leq q and <math>\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Let $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair (u, φ) induces a weighted composition operator uC_{φ} from $L^{p}(\Sigma)$ into $L^{q}(\Sigma)$ if and only if $J = hE^{\varphi^{-1}(\Sigma)}(|u|^{q}) \circ \varphi^{-1} \in L^{\frac{r}{q}}(\Sigma)$.

Proof. Let $f \in L^p(\Sigma)$. We will have

$$\begin{aligned} \|uC_{\varphi}f\|_{q}^{q} &= \int |u.f \circ \varphi|^{q} d\mu = \int hE^{\varphi^{-1}(\Sigma)}(|u|^{q}) \circ \varphi^{-1}|f|^{q} d\mu \\ &= \int |\sqrt[q]{J}f|^{q} d\mu = \|M_{\sqrt[q]{J}}f\|_{q}^{q}. \end{aligned}$$

So by Corollary 2.2, uC_{φ} is a weighted composition operator from $L^{p}(\Sigma)$ into $L^{q}(\Sigma)$ if and only if $\sqrt[q]{J} \in L^{r}(\Sigma)$ or equivalently $J \in L^{\frac{r}{q}}(\Sigma)$. \Box

Corollary 2.4. Suppose $1 \leq p \leq \infty$, $u \in L(X)$ and $\varphi : X \to X$ be a non-singular measurable transformation. Then the pair (u, φ) induces a weighted composition operator uC_{φ} from $L^{p}(\Sigma)$ into $L^{p}(\Sigma)$ if and only if $hE^{\varphi^{-1}(\Sigma)}(|u|^{p}) \circ \varphi^{-1} \in L^{\infty}(\Sigma)$.

Corollary 2.5. Under the same assumptions as in theorem 2.3, φ induces a composition operator $C_{\varphi} : L^p(\Sigma) \to L^q(\Sigma)$ if and only if $h \in L^{\frac{r}{q}}(\Sigma)$.

Remark 2.6. One of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let X be [0, 1], Σ the Borel sets, and μ Lebesgue measure. Let φ be the map $\varphi(x) = x^3$ on [0,1]. A simple computation shows that $h = 1/3x^{-2/3} \notin L^3(\Sigma)$. Then C_{φ} dos not define a bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$. However with u(x) = x, we have $\varphi^{-1}(\Sigma) = \Sigma$ (so E = I) and $J = 1/3 \in L^3(\Sigma)$. Hence $uC_{\varphi} = M_u \circ C_{\varphi}$ is bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$.

The procedure which Axler has used for the case p < q in [1], when X is the interval $[-\pi, \pi]$, can also be used here.

At this stage we investigate a necessary and sufficient condition for a multiplication operator to be fredholm. For a bounded linear operator A on a Banach space, we use the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote the kernel and the range of A, respectively. We recall that A is said to be a Fredholm operator if $\mathcal{R}(A)$ is closed and if $\dim \mathcal{N}(A) < \infty$ and $\operatorname{codim} \mathcal{R}(A) < \infty$. Now we attempt to prove a theorem which is likely to be found elsewhere.

Theorem 2.7. Suppose that μ is a non-atomic measure on $L^2(\Sigma)$. Then the following conditions are equivalent:

- (a) M_u is an invertible operator.
- (b) M_u is a Fredholm operator.
- (c) $\mathcal{R}(M_u)$ is closed and $codim \mathcal{R}(M_u) < \infty$.
- (d) $|u| \ge \delta$ almost everywhere on X for some $\delta > 0$.

Proof. The implications $(d) \Longrightarrow (a) \Longrightarrow (b) \Longrightarrow (c)$ are obvious. We show $(c) \Longrightarrow (d)$.

Suppose that $\mathcal{R}(M_u)$ is closed and codim $\mathcal{R}(M_u) < \infty$. Then there exists a $\delta > 0$ such that $|u| \ge \delta$ on $\sigma(u)$. So it is enough to show that $\mu(\sigma(u)^c) = 0$. First of all we prove that M_u is onto. Let $0 \ne f_0 \in \mathcal{R}(M_u)^{\perp}$, therefore, for any $f \in L^2(\Sigma)$ we have $(M_u f, f_0) = 0$. Now we choose t > 0 such that the set

$$Z_t = \{ s \in X : |f_0|^2(x) \ge t \}$$

is of positive measure. Since μ is a non-atomic measure we may choose a sequence of disjoint subsets Z_n of Z_t such that $0 < \mu(Z_n) < \infty$. Now let $g_n = \chi_{Z_n} \cdot f_0$. It is clear that each g_n is non-zero element of $L^2(\Sigma)$, and for $n \neq m$, $(g_n, g_m) = 0$. Therefore, for $f \in L^2(\Sigma)$ On Some operators of L^p -Spaces

we have

$$(f, M_u^*g_n) = (M_u f, \chi_{Z_n} f_0) = (M_u \chi_{Z_n f}, f_0) = 0.$$

So $g_n \in \mathcal{N}(M_u^*)$ for any n. Therefore, $\{g_n\}$ is a linearly independent subset of $\mathcal{N}(M_u^*)$, which is a contradiction to dim $\mathcal{N}(M_u^*) = \operatorname{codim} \mathcal{R}(M_u) < \infty$. If $\mu(\sigma(u)^c) > 0$, then there exists a set $Z \subset \sigma(u)^c$ such that $0 < \mu(Z) < \infty$, so we conclude that $\chi_Z \in L^2(\Sigma) \setminus \mathcal{R}(M_u)$, which contradicts the fact that M_u is onto. Therefore $\mu(\sigma(u)^c) = 0$. \Box

Corollary 2.8. M_u is a Fredholm operator if and only if M_u^n (= M_{u^n}) is also Fredholm.

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M. R. Jabbarzadeh

Department of Mathematics University of Tabriz Tabriz Iran e-mail:mjabbar@tabrizu.ac.ir

E. Pourreza

Department of Mathematics University of Tabriz Tabriz Iran e-mail:e_pourreza@tabrizu.ac.ir