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APPROXIMATING INITIAL-VALUE PROBLEMS WITH TWO-POINT BOUNDARY-VALUE PROBLEMS: BBM-EQUATION

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Abstract. The focus of the present study is the BBM equation which models unidirectional propagation of small amplitude, long waves in dispersive media. This evolution equation has been used in both laboratory and field studies of water waves. The principal new result is an exact theory of convergence of the two-point boundary-value problem to the initial-value problem posed on an infinite stretch of the medium of propagation. In addition to their intrinsic interest, our results provide justification for the use of the two-point boundary-value problem in numerical studies of the initial-value problem posed on the entire line.

1. **Introduction**

Considered here are small amplitude, long waves on the surface of an ideal fluid of finite depth over a featureless, horizontal bottom under the force of gravity. When such wave motion is long crested, it may propagate essentially in, say, the x-direction and without significant variation in the y-direction of a standard xyz-Cartesian frame in which gravity acts in the negative z-direction. Waves approaching a beach and in canals often have this long-crested structure. For such waves, the full

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three-dimensional Euler equations can be reduced to approximate models featuring only one independent spatial variable. Such models go back to at least the early part of the 19^{th} century and are included in works by Airy (1845) and Stokes (1847) in the first half of the century. The model featured in the present study has its roots in the work of Boussinesq (1871, 1872 and 1877) and later, Korteweg and de Vries (1895). More detailed historical accounts and derivations can be found in recent modern works (see e.g., the references in Bona et al., 2002 and 2004) and the wide-ranging historical discussions of Craik (2003, 2004 and 2005).

It seems worthwhile to embark on a slightly extended discussion of the context before entering exact theory. For describing the issue at hand, it sufficeis to remind the reader that if x denotes the coordinate in the direction of propagation and h_0 the undisturbed depth, assumed constant in the present study, then the crucial dependent variable is $\eta(x,t) = h(x,t) - h_0$, where t is proportional to elapsed time and $h(x,t)$ is the depth of the water column over the spatial point x at time t. It is assumed that the waves propagate in the direction of increasing values of x, that the amplitude a of the waves is small compared to the undisturbed depth h_0, that typical wavelengths λ of the motion are long compared to h_0, so that

$$\alpha = \frac{a}{h_0} << 1 \quad \text{and} \quad \beta = \frac{h_0}{\lambda} << 1,$$

and that the Stokes number

$$S = \frac{\alpha}{\beta^2} = \frac{a\lambda^2}{h_0^3} \tag{1.1}$$

is of order one. The latter presumption implies a balance is struck between nonlinear and dispersive effects. Under these assumptions, the evolution equations

$$\eta_t + \sqrt{gh_0}\Big(\eta_x + \frac{3}{2h_0}\eta\eta_x + \frac{h_0^2}{6}\eta_{xxx}\Big) = 0 \tag{1.2}$$

and

$$\eta_t + \sqrt{gh_0}\Big(\eta_x + \frac{3}{2h_0}\eta\eta_x - \frac{h_0^2}{6}\eta_{xxt}\Big) = 0 \tag{1.3}$$

are formal reductions of the two-dimensional Euler equations. Here, g is the gravity constant and subscripts connote partial differentiations. If x is scaled by λ, η by a and time by $\sqrt{h_0/g}$, these take the non-dimensional forms

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta^2\eta_{xxx} = 0 \tag{1.4}$$

and

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta^2\eta_{xxt} = 0. \tag{1.5}$$

In these variables, η, x and t are non-dimensional, but scaled so that η and its first few partial derivatives are of order one. A further numerical rescaling yields the familiar equations

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \tag{1.6}$$

and

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0. \tag{1.7}$$

Equations (1.2) and (1.6) are the classical Korteweg-de Vries (1895) equation first derived by Boussinesq (1877), while (1.3) and (1.7) are the regularized long wave or BBM equation written by Peregrine (1967) in his study of bore propagation and first analyzed by Benjamin *et al.* (1972). In (1.6) and (1.7), up to numerical factors, all lengths are scaled by h_0. This amounts to taking $h_0 = 1$ in (1.2) and (1.3) and scaling time appropriately. Thus, equations (1.6) and (1.7) are nondimensional, but the small amplitude, long wavelength assumptions reside implicitly in η, and hence should be explicit in the auxiliary data attached to the evolution equation if physically relevant solutions are to be considered. Once $\eta = \eta(x,t)$ is determined from equation (1.4) or (1.5), say, then the horizontal velocity $u(x,y,t)$ in the x-direction at height y above the bottom is determined, at this level of approximation, by the formulas

$$u(x,y,t) = \eta(x,t) - \frac{1}{4}\alpha\eta^2(x,t) + \frac{1}{3}\beta^2\Big(1 - \frac{3}{2}y^2\Big)\eta_{xx}(x,t),$$

(see the derivation in Bona and Chen's unpublished lecture note, or the formulas in Bona et al., 2002 and 2004, and Bona et al., 2005). The vertical velocity field is of order $O(\beta^4)$, of course.

Attention is turned to the just mentioned auxiliary data. It is standard in mathematical studies of these equations to focus upon the pure initial-value problem in which η is specified for all the relevant values of x at a given instant of time t, normally taken to be $t = 0$. That is, the wave profile

$$\eta(x,0) = f(x) \qquad \text{for} \qquad x \in \mathbb{R} \tag{1.8}$$

is specified for all values of x. Thus, values of $t > 0$ represent time elapsed since the inception of the motion as described by (1.8). Of course, if one wishes to be more explicit about the small amplitude, long wavelength assumption, then f can be taken in the form $f(x) = \alpha F(\beta x)$, where F is independent of α and β. The formulations (1.6)-(1.8) and (1.7)-(1.8) do not inquire as to how the motion was truly initiated, but imagine a snapshot taken of a disturbance already generated and then uses (1.6) or (1.7) to predict the further evolution of the waves. The initial-value problems (1.6)-(1.8) and (1.7)-(1.8) have a distinguished history both analytically and in experimental studies and applications, some of which will be discussed presently. It deserves remark that while 19^{th} century derivations of equations like (1.2) and (1.3) were purely formal, exact theory has recently appeared showing that, in fact, these equations provide faithful approximations of both Boussinesq systems, which allow for two-way propagation of the disturbances (see Alazman *et al.,* 2006) and of the full, two-dimensional Euler equations for free surface flow (see Bona et al., 2005) on the relevant time scales.

Another natural formulation for both (1.6) and (1.7) is the quarter-plane or half-line problem. This problem, first put forward by Bona and Bryant (1973), is concerned with waves propagating into an undisturbed stretch of the medium of propagation. One imagines measuring the waves as they come into the relevant portion of the medium at some fixed spatial point, say $x = 0$, as was done in the experimental study of Bona et al. (1981 and 1985). This leads to the boundary condition

$$\eta(0,t) = g(t) \qquad \text{for} \quad t \geq 0. \tag{1.9}$$

As in (1.8), if one wishes to make the small amplitude, long wavelength presumption apparent, one might take $g(t) = \alpha G(\beta t)$, where G is independent of α and β. Since both (1.6) and (1.7) are written to describe waves propagating in the positive direction along the x-axis, it is not particularly desirable to impose a boundary condition at a finite point to the right of $x = 0$. To do so can lead to reflected waves which neither (1.6) nor (1.7) is capable of approximating accurately. This point leads one to pose the problem for all $x \geq 0$, and thus placing the issue of a boundary condition at the right-hand end-point at $+\infty$. The equations (1.6) or (1.7) along with the boundary condition (1.9) must be supplemented by an initial condition as in (1.8), *viz.*

$$\eta(x,0) = f(x) \qquad \text{for} \quad x \geq 0. \tag{1.10}$$

In practice, it is often the case that $f \equiv 0$, corresponding to an initially undisturbed medium, but the mathematical theory does not require this. Function class restrictions on f which imply at least a weak form of boundedness as $x \to +\infty$ suffice to guarantee that (1.6)-(1.9)-(1.10) and (1.7)-(1.9)-(1.10) constitute well-posed problems (see Bona et al., 2005 and 2007 and Bona et al., 2003, 2006 and the references therein).

The initial-boundary-value problems (1.6)-(1.9)-(1.10) and (1.7)-(1.9)-(1.10), sometimes in a modified form that includes some kind of dissipation, have been used to test the predictive power of (1.6) and (1.7) in laboratory settings (see, for example, Hammack, 1973, Hammack and Segur, 1974 and Bona et al., 1981). However, when comparison between experimentally produced waves are made with model predictions, one often has to resort to numerical approximation of its solution. For this, a bounded domain is normally used, though there is theory for numerical schemes approximating the initial-boundary-value problem (1.7)-(1.9)-(1.10) set on the half line (see e.g., Guo and Shen, 2000). There is also available analytical theory for the two-point boundary value problem wherein (1.6) or (1.7) is posed on a finite spatial interval with an initial condition and suitable boundary conditions. In the case of (1.7), this was first developed by Bona and Dougalis (1980) who showed that (1.7) is globally well posed with the auxiliary specifications

$$\begin{aligned} &\eta(x,0) = f(x), \qquad \text{for} \quad 0 \le x \le L, \\ &\eta(0,t) = g(t), \qquad \eta(L,t) = h(t), \qquad \text{for} \quad t \ge 0, \end{aligned} \tag{1.11}$$

when f, g and h are suitably restricted. In the comparisons with experiments mentioned above, f and h are taken to be zero and both the experiments and the numerical simulations are only carried out on a time interval during which there is no appreciable motion at the right-hand end of the domain of propagation. (In the experiments, the waves were generated by a flap-type wavemaker and the boundary data g in (1.9) was determined by measurement.) Numerical schemes for this problem were put forward and tested in Bona, Pritchard and Scott (1985). More recent work appears in Bona and Chen (1998). Theory based directly on the motion of the wavemaker rather than on an auxiliary measurement has recently been developed by Rosier (2004) and by Bona and Varlamov (2005).

Study of the KdV-equation posed on a finite interval began with the work of Bubnov (1979). A review may be found in the recent paper of Bona, Sun and Zhang (2003). For the Korteweg-de Vries equation (1.6), well-posdness holds for the auxiliary specifications

$$\begin{aligned} &\eta(x,0) = f(x), \qquad \text{for} \quad 0 \le x \le L, \\ &\eta(0,t) = g(t), \quad \eta(L,t) = h(t), \quad \eta_x(L,t) = r(t), \qquad \text{for } t \ge 0, \end{aligned} \tag{1.12}$$

where f, g, h and r are drawn from reasonable function classes. It is also the case that the problem (1.6) posed with

$$\begin{aligned} &\eta(x,0) = f(x), \qquad \text{for} \quad 0 \le x \le L, \\ &\eta(0,t) = g(t), \quad \eta_x(L,t) = h(t), \quad \eta_{xx}(L,t) = r(t), \qquad \text{for } t \ge 0 \end{aligned} \tag{1.13}$$

is well posed in Sobolev classes as Colin and Ghidaglia (2001) showed.

A natural question arises within the circle of ideas just reviewed. What is the relationship between the two-point boundary value problems for (1.6) or (1.7) and the quarter-plane problem for the same equations? It has been assumed, in using a finite interval for numerical simulations, that these problems yield essentially the same answer in the appropriate part of space-time, if $h \equiv 0$ (and $r \equiv 0$ in the case of (1.6)). Bona *et al.* (2005 and 2007) recently have answered this question. For the reader's convenience, their results are stated here in a rough form.

Theorem 1.1. *Let $u_\infty = u_\infty(x,t)$ be the solution of the BBM-equation* (1.7) *posed for $x, t \ge 0$ with zero initial condition and the boundary condition described in* (1.9) *and let $u_L = u_L(x,t)$ be the solution of the two-point boundary-value problem for the BBM-equation* (1.7) *posed for $0 \le x \le L$ and $t \ge 0$ with zero initial condition and the boundary condition described in* (1.11) *with $h \equiv 0$ and g a continuous function satisfying the compatibility condition $g(0) = 0$. Then, for any $\lambda \in (0,1)$, there is a positive increasing function $c(t)$ dependent on λ which is of the order of $\int_0^t |g(s)|\,ds$, $\int_0^t g^2(s)\,ds$, $g(t)$ and $h(t)$ such that*

$$\|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^1[0,L]} \le e^{-\lambda L + c(t)}.$$

Theorem 1.2. *Let $u_\infty = u_\infty(x,t)$ be the solution of the KdV equation* (1.6) *posed for $x, t \ge 0$ with zero initial condition and the boundary condition $u_\infty(0,t) = g(t) \in H^s[0,T]$ for any $s > \frac{2}{3}$. Let $u_L(x,t)$ be the solution of the KdV-equation* (1.6) *posed for $0 \le x \le L$ and $t \ge 0$ with the initial condition and boundary condition described in* (1.12)*, where $f = h = r \equiv 0$. Assume that the compatibility condition $g(0) = f(0) = 0$ is satisfied. Then, $u_\infty(x,t)$ and $u_L(x,t)$ exist for $t \in [0,T]$, and for any*

$b > 0$, *the inequality*

$$\sup_{t\in[0,T]} \|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^1[0,L]} \leq Ce^{-bL}\,,$$

holds, where C only depends on the norm of the boundary data g and on T in the form $e^{\gamma T}$. The constant γ, which is dependent on b, the norm of g, is of order one.

Following the same perspective, our purpose here is to bring forward exact theory comparing the pure initial-value problem with the two-point boundary-value problems in view. While this problem is not relevant to experiments as described above, it is directly applicable to many numerical simulations of initial-value problems posed on all of the real line $\mathbb{R}$. Such simulations invariably rely upon computations made on a bounded interval with either homogeneous Dirichlet boundary conditions or periodic boundary conditions. The present paper deals with the BBM-equation (1.7), as the title suggests.

The plan of the paper is as follows. In Section 2, existing theory is briefly reviewed and then extended in a way that is useful for the present goals. Similar theory is worked out for the two-point boundary value problem in Section 3, while the main comparison results are derived in Section 4.

To give the study focus, the main result is here stated informally. Detailed assumptions can be found spelled out in Section 4.

Theorem 1.3. *Let $u_\infty = u_\infty(x,t)$ be the solution of the BBM-equation* (1.7) *posed for $x \in \mathbb{R}$ and $t \geq 0$ with initial condition* (1.8) *which lies in $H^1(\mathbb{R})$ and decays to zero exponentially as $x \to \pm\infty$. Let $u_L = u_L(x,t)$ be the solution of the two-point boundary-value problem for the BBM-equation* (1.7) *posed for $-L \leq x \leq R$ and $t \geq 0$ with the same initial data f, but restricted to $[-L,R]$, and with the boundary specifications $v(-L,t) = g(t)$ and $v(R,t) = h(t)$, where g and h are required to be sufficiently small (for details, see Corollary 4.2). Then, for some $\lambda \in (0,1)$ there is a positive increasing function $c_1(t)$ dependent on the values of λ, $\|f\|_{H^1(\mathbb{R})}$, and $\int_0^t(|g(s)| + |h(s)| + g^2(s) + h^2(s))\,ds$ such that*

$$\|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^1_{LR}} \leq c_1(t)e^{-\lambda\min\{L,R\}+c_2 t},$$

where the constant c_2 *is of order* $(4+\sqrt{2}\|f\|_{H^1(\mathbb{R})})/(4(1-\lambda^2))$ *and* $H^1_{LR}=\{f,f'\in L_2(-L,R)\}$ *with the norm* $\|f\|_{H^1_{LR}}=\Big\{\int_{-L}^{R}\Big(f^2(x)+(f'(x))^2\Big)\,dx\Big\}^{\frac{1}{2}}$.

2. **The pure initial-value problem**

Notation

For the reader's convenience, we commence by collecting together the main notation to be used throughout. The positive real axis $[0,\infty)$ is denoted by $\mathbb{R}^+$. Through this paper, I is used to denote the interval $[0,T]$ if T is finite and $[0,\infty)$ if $T=\infty$. The class $C(I)$ is the continuous functions defined on I, $C_b(I)$ is the subset of $C(I)$ consisting of all bounded continuous functions on I, while $C_0(\mathbb{R}^+)$ is the subset of bounded and continuous functions that vanish at $+\infty$. For $p\geq 1$, $L_p=L_p(\mathbb{R})$ is the Lebesgue space with its usual norm; the notation $|\cdot|_p=\|\cdot\|_{L_p}$ will be followed throughout. The norm on $C_b(\mathbb{R}^+)$ and $C_0(\mathbb{R})$ is $|\cdot|_\infty$. For any real number s, $H^s=H^s(\mathbb{R})$ is the usual L_2–based Sobolev space with its norm abbreviated by $\|f\|_s=\|f\|_{H^s}$ and $H^s_\Omega=H^s(\Omega)$, where Ω is a subset of $\mathbb{R}$, with its usual quotient norm denoted by $\|f\|_{H^s_\Omega}$. If $L,R>0$ and Ω is the interval $[-L,R]$, and H^s_Ω is also denoted by H^s_{LR}. If J is an interval in $\mathbb{R}$ and is X a Banach space, then $C(J;X)$ consists of all continuous functions defined on J with images in X and $C_b(J;X)$ is the subspace of functions $f\in C(J;X)$ such that $\sup_{t\in J}\|f(t)\|_X<\infty$. If $j\geq 1$ is an integer, then $C^j(J;X)$ is the subset of $C(J;X)$ whose functions are j-times differentiable with respect to the variable t. These spaces carry their usual norms.

Considered now is the pure initial-value problem

$$\left.\begin{aligned} &u_t+u_x+uu_x-u_{xxt}=0, && x\in\mathbb{R},\quad t>0,\\ &u(x,0)=\varphi(x), && x\in\mathbb{R}, \end{aligned}\right\}\tag{2.1}$$

for the BBM-equation. Write the BBM-equation as

$$u_t-u_{xxt}=-u_x-uu_x,$$

and formally solve for u_t (see Benjamin *et al.,* 1972) to obtain:

$$u_t(x,t)=-\int_{-\infty}^{\infty}P(x,y)\big(u_y(y,t)+u(y,t)u_y(y,t)\big)\,dy,\tag{2.2}$$

where,

$$P(x,y) = \frac{1}{2}e^{-|x-y|}. \tag{2.3}$$

Since $\lim_{y\to\pm\infty} P(x,y) = 0$, a formal integration by parts on the right-hand side of (2.2) yields

$$u_t(x,t) = \int_{-\infty}^{\infty} K(x,y)\Big(u(y,t) + \frac{1}{2}u^2(y,t)\Big)\,dy \tag{2.4}$$

where,

$$K(x,y) = \frac{1}{2}\,\operatorname{sgn}(x-y)e^{-|x-y|}. \tag{2.5}$$

Integrating with respect to the temporal variable over $[0,t]$, one thus obtains the integral equation

$$u(x,t) = \varphi(x) + \int_0^t \int_{-\infty}^{\infty} K(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\,dy\,ds. \tag{2.6}$$

Theorem 2.1. *(Benjamin et al., 1972, and Bona and Tzvetkov, 2008) If the initial data φ lies in H^k for some $k \geq 0$, then the Cauchy problem (2.1) for the BBM-equation is globally well-posed in H^k in the sense that there is a unique solution u in $C(\mathbb{R}^+, \mathcal{S}')$ that necessarily lies in $C(\mathbb{R}^+; H^k)$, and which depends continuously in $C(\mathbb{R}^+; H^k)$ on $\varphi \in H^k$. If φ also lies in $C^m(\mathbb{R})$ for some $m \geq 1$, then the solution u is a classical solution. Moreover, $\partial_t^i u \in C(\mathbb{R}^+; H^{k+1})$ for $i \geq 1$.*

Remark: If $u \in C(\mathbb{R}^+; H^1)$ solves (2.1), then its H^1-norm is independent of time $t \geq 0$. That is,

$$\|u(\cdot,t)\|_1^2 = \|\varphi\|_1^2 = \int_{\mathbb{R}} \Big(\varphi^2(x) + (\varphi'(x))^2\Big)\,dx.$$

This implies that u is bounded and continuous; in fact,

$$|u(\cdot,t)|_\infty \leq \frac{1}{\sqrt{2}}\|\varphi\|_1.$$

Theorem 2.2. *Suppose $\varphi \in H^1$ and there is $\lambda \in (0,1)$ such that $e^{\lambda|x|}\varphi(x)$ is uniformly bounded on $\mathbb{R}$. Then, the solution u of (2.1) satisfies*

$$|u(x,t)| \leq \gamma \exp\Big\{-\lambda|x| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,t\Big\} \tag{2.7}$$

and

$$|u_t(x,t)| \le \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\gamma \exp\Big\{-\lambda|x| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,t\Big\} \tag{2.8}$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, *where,*

$$\gamma = \sup_{x\in\mathbb{R}}\{e^{\lambda|x|}|\varphi(x)|\}. \tag{2.9}$$

Proof. Write $u(x,t) = e^{-\lambda|x|}U(x,t)$, so that $u^2(x,t) = e^{-\lambda|x|}u(x,t)U(x,t)$. The integral equation (2.6) is equivalent to

$$U(x,t) = e^{\lambda|x|}\varphi(x) + \int_0^t\int_{-\infty}^{\infty} e^{\lambda|x|-\lambda|y|}K(x,y)\Big(U(y,s) + \frac{1}{2}u(y,s)U(y,s)\Big)dyds.$$

For $x>0$, divide $\mathbb{R}$ into three subintervals, $(-\infty,0]$, $(0,x]$ and (x,∞), and make estimates as follows:

$$\begin{aligned}
\Big|U(x,t)\Big| &\le e^{\lambda x}|\varphi(x)| \\
&\quad + \frac{1}{2}\int_0^t\int_{-\infty}^0 e^{-(x-y)+\lambda(x+y)}\Big|U(y,s) + \frac{1}{2}u(y,s)U(y,s)\ \Big|\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t\int_0^x e^{-(x-y)+\lambda(x-y)}\Big|U(y,s) + \frac{1}{2}u(y,s)U(y,s)\Big|\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t\int_x^\infty e^{x-y+\lambda(x-y)}\Big|U(y,s) + \frac{1}{2}u(y,s)U(y,s)\Big|\,dy\,ds \\
&\le e^{\lambda x}|\varphi(x)| \\
&\quad + \frac{1}{2}\int_0^t\int_{-\infty}^0 e^{-(x-y)+\lambda(x+y)}\,dy\Big|U(\cdot,s) + \frac{1}{2}u(\cdot,s)U(\cdot,s)\Big|_\infty\,ds \\
&\quad + \frac{1}{2}\int_0^t\int_0^x e^{-(x-y)+\lambda(x-y)}\,dy\Big|U(\cdot,s) + \frac{1}{2}u(\cdot,s)U(\cdot,s)\Big|_\infty\,ds \\
&\quad + \frac{1}{2}\int_0^t\int_x^\infty e^{x-y+\lambda(x-y)}\,dy\Big|U(\cdot,s) + \frac{1}{2}u(\cdot,s)U(\cdot,s)\Big|_\infty\,ds.
\end{aligned}$$

Direct calculation reveals:

$$\begin{aligned}\Big|U(x,t)\Big| &\le e^{\lambda x}|\varphi(x)| \\ &\quad + \int_0^t \Big(\frac{1}{1-\lambda^2} - \frac{\lambda e^{-(1-\lambda)x}}{1-\lambda^2}\Big)\Big|U(\cdot,s) + \frac{1}{2}u(\cdot,s)U(\cdot,s)\Big|_\infty \, ds \\ &\le e^{\lambda x}|\varphi(x)| + \frac{1}{1-\lambda^2}\int_0^t \Big|U(\cdot,s) + \frac{1}{2}u(\cdot,s)U(\cdot,s)\Big|_\infty \, ds.\end{aligned}$$

Since the H^1-norm of the solution u of (2.1) is constant and $|u(\cdot,t)|_\infty \le \frac{\sqrt{2}}{2}\|\varphi(\cdot,t)\|_1$, it transpires that for $x \ge 0$,

$$\Big|U(x,t)\Big| \le e^{\lambda x}|\varphi(x)| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\int_0^t \Big|U(\cdot,s)\Big|_\infty \, ds.$$

Similarly, it can be shown that for $x < 0$,

$$\Big|U(x,t)\Big| \le e^{-\lambda x}|\varphi(x)| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\int_0^t \Big|U(\cdot,s)\Big|_\infty \, ds.$$

In consequence, for all $x \in \mathbb{R}$ and $t \ge 0$,

$$\Big|U(x,t)\Big| \le e^{\lambda|x|}|\varphi(x)| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\int_0^t \Big|U(\cdot,s)\Big|_\infty \, ds.$$

Taking the supremum with respect to x over $\mathbb{R}$ of both sides of this latter inequality yields:

$$\Big|U(\cdot,t)\Big|_\infty \le \gamma + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\int_0^t \Big|U(\cdot,s)\Big|_\infty \, ds.$$

Applying Gronwall's Lemma to the last inequality leads to the estimate

$$\Big|U(\cdot,t)\Big|_\infty \le \gamma \, \exp\Big\{\frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\, t\Big\}. \tag{2.10}$$

Because $U(x,t) = e^{\lambda|x|}u(x,t)$, it follows that

$$|u(x,t)| \le e^{-\lambda|x|}|U(\cdot,t)\Big|_\infty \le \gamma \, \exp\Big\{-\lambda|x| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\, t\Big\}.$$

Since

$$U_t(x,t) = \int_{-\infty}^{\infty} e^{\lambda|x|-\lambda|y|}K(x,y)\Big(U(y,t) + \frac{1}{2}u(y,t)U(y,t)\Big)\, dy,$$

the bound (2.10) implies that

$$|U_t(x,t)| \le \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,\gamma\,\exp\Big\{\frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,t\Big\},$$

and thus

$$|u_t(x,t)| \le \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,\gamma\,\exp\Big\{-\lambda|x| + \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,t\Big\}.$$

Hence, the theorem is established.

Corollary 2.3. *If $\varphi \in H^k \bigcap C^k$ for some $k \ge 1$ and there is a $\lambda \in (0,1)$ such that $e^{\lambda|x|}\varphi^{(j)}(x)$ is uniformly bounded for $j = 1, \cdots, k$, then*

$$e^{\lambda|x|}\partial_x^j u(x,t) \quad \textit{and} \quad e^{\lambda|x|}\partial_x^j u_t(x,t)$$

are bounded uniformly for $x \in \mathbb{R}$ and t in any compact subset of $\mathbb{R}^+$

Proof. Taking the derivative with respect to x on both sides of (2.6), there appears the formula

(2.11)
$$\begin{aligned} u_x(x,t) = \varphi'(x) + &\int_0^t \Big(u(x,s) + \frac{1}{2}u^2(x,s)\Big)\,ds \\ &- \frac{1}{2}\int_0^t\int_{-\infty}^{\infty} e^{-|x-y|}\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\,dy\,ds. \end{aligned}$$

Taking the t-derivative of (2.11) gives

$$\begin{aligned} u_{tx}(x,t) = &\, u(x,t) + \frac{1}{2}u^2(x,t) \\ &- \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big(u(y,t) + \frac{1}{2}u^2(y,t)\Big)\,dy. \end{aligned} \tag{2.12}$$

Inequality (2.7) then yields:

$$
\begin{aligned}
e^{\lambda|x|}|u_x(x,t)| \leq & e^{\lambda|x|}|\varphi'(x)| + \int_0^t \gamma(s)\Big(1+\frac{1}{2}|u(\cdot,s)|_\infty\Big)\,ds \\
& + \frac{1}{2}e^{\lambda|x|}\int_0^t \gamma(s)\int_{-\infty}^{\infty} e^{-|x-y|}e^{-\lambda|y|}\Big(1+\frac{1}{2}|u(\cdot,s)|_\infty\Big)\,dy\,ds \\
\leq & e^{\lambda|x|}|\varphi'(x)| + \int_0^t \gamma(s)\,ds\,\Big(1+\frac{\sqrt{2}}{4}\|\varphi\|_1\Big) \\
& + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}e^{\lambda|x|-\lambda|y|}\,dy\int_0^t \gamma(s)\,ds\,\Big(1+\frac{\sqrt{2}}{4}\|\varphi\|_1\Big),
\end{aligned}
$$

where $\gamma(t) = \gamma\exp\{(4+\sqrt{2}\|\varphi\|_1)/(4(1-\lambda^2))\,t\}$. Elementary calculations reveal:

$$
\int_{-\infty}^{\infty} e^{-|x-y|}e^{\lambda|x|-\lambda|y|}\,dy = \frac{2}{1-\lambda^2} - \frac{2\lambda}{1-\lambda^2}e^{-(1-\lambda|x|)} \leq \frac{2}{1-\lambda^2}.
$$

In consequence,

$$
e^{\lambda|x|}|u_x(x,t)| \leq e^{\lambda|x|}|\varphi'(x)| + \gamma(2-\lambda^2)\exp\Big\{\frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)}\,t\Big\},
$$

and so $e^{\lambda|x|}|u_x(x,t)|$ is uniformly bounded on $\mathbb{R}\times[0,T]$ for any T finite. Similarly, one sees that

$$
e^{\lambda|x|}u_{tx}(x,t)
$$

is uniformly bounded on $\mathbb{R}\times[0,T]$ for any $T<+\infty$. Note that

$$
u_{xx}(x,t) = \varphi''(x) + u(x,t) - \varphi(x) + \int_0^t (u_x(x,t) + u(x,t)u_x(x,s))\,ds
$$

and

$$
u_{xxt}(x,t) = u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t).
$$

It immediately follows that

$$
e^{\lambda|x|}u_{xx}(x,t) \quad \text{and} \quad e^{\lambda|x|}u_{xxt}(x,t)
$$

are bounded uniformly on $x\in\mathbb{R}$ and t on a compact subset of $\mathbb{R}^+$. The proof of the corollary finishes with a straightforward induction.

3. The two-point boundary-value problem

Considered in this section is the two-point boundary-value problem

$$\left.\begin{aligned} &v_t + v_x + vv_x - v_{xxt} = 0, && -L \le x \le R, \quad t > 0,\\ &v(-L,t) = g(t),\ v(R,t) = h(t), && t \ge 0,\\ &v(x,0) = \varphi(x), && -L \le x \le R, \end{aligned}\right\} \tag{3.1}$$

where $L, R > 0$, together with the compatibility conditions

$$g(0) = \varphi(-L), \quad h(0) = \varphi(R). \tag{3.2}$$

The main result is the following.

Theorem 3.1. *If $\varphi \in H^1_{LR}$ and $g, h \in C(I)$ satisfy the compatibility condition* (3.2)*, then there is a unique distributional solution v of* (3.1) *which lies in the space $C(I; H^1_{LR})$. The solution v depends continuously on φ, g and h. If $\varphi \in H^1_{LR} \bigcap C^1([-L,R])$ and $g, h \in C^k(I)$ for some $k \ge 1$, then v satisfies* (3.1) *in the classical sense on $[-L,R] \times I$.*

This theorem is a consequence of the last corollary in this section. Its proof is the object of the rest of the section.

Solving for v_t in (3.1), as in Bona and Dougalis (1980), leads to:

(3.3)
$$v_t(x,t) = g'(t)\phi_1(x) + h'(t)\phi_2(x) - \int_{-L}^{R} P_{LR}(x,y)\big(v_y(y,t) + v(y,t)v_y(y,t)\big)\,dy,$$

where,

$$\phi_1(x) = \frac{e^{R-x} - e^{-R+x}}{e^{L+R} - e^{-(R+L)}}, \quad \phi_2(x) = \frac{e^{L+x} - e^{-L-x}}{e^{L+R} - e^{-(L+R)}} \tag{3.4}$$

and

(3.5)
$$P_{LR}(x,y) = \frac{1}{2(e^{2(R+L)} - 1)}\Big(-e^{2L+x+y} + e^{|x-y|} - e^{2R-(x+y)} + e^{2(L+R)-|x-y|}\Big).$$

Since $P_{LR}(x,-L) = P_{LR}(x,R) = 0$, integrating by parts on the right-hand side of (3.3) yields:

(3.6)
$$v_t(x,t) = g'(t)\phi_1(x) + h'(t)\phi_2(x) + \int_{-L}^{R} K_{LR}(x,y)\big(v(y,t) + \frac{1}{2}v^2(y,t)\big)dy$$

where,

(3.7)

$$K_{LR}(x,y) = \frac{1}{2(e^{2(L+R)}-1)}\Big(-e^{2L+x+y} - \operatorname{sgn}(x-y)e^{|x-y|} + e^{2R-(x+y)} + \operatorname{sgn}(x-y)e^{2(L+R)-|x-y|}\Big).$$

Integrate both sides of (3.6) with respect to the temporal variable t and use the facts that $g(0) = \varphi(-L)$ and $h(0) = \varphi(R)$ to determine that

$$\begin{aligned} v(x,t) = &\varphi(x) + \big(g(t) - \varphi(-L)\big)\phi_1(x) + \big(h(t) - \varphi(R)\big)\phi_2(x) \\ &+ \int_0^t \int_{-L}^{R} K_{LR}(x,y)\big(v(y,\tau) + \frac{1}{2}v^2(y,\tau)\big)\, dy\, d\tau. \end{aligned} \tag{3.8}$$

The following two results are found in Bona and Dougalis (1980).

Theorem 3.2. *(Local Existence) If $\varphi \in C([-L,R])$, $g, h \in C(I)$ satisfy the compatibility condition* (3.2)*, then there exists $I_0 = [0, T_0] \subset I$ such that the integral equation* (3.8) *has a unique solution v, say, lying in $C([-L,R] \times I_0)$. Moreover,*

$$\lim_{t \to 0^+} v(x,t) = \varphi(x)$$

in $C([-L,R])$ and

$$\lim_{x \to -L^+} v(x,t) = g(t) \qquad \text{and} \qquad \lim_{x \to R^-} v(x,t) = h(t)$$

in $C(I_0)$. The solution depends continuously in $C([-L,R] \times I_0)$ on $\varphi \in C([-L,R])$ and $g, h \in C(I_0)$.

Theorem 3.3. *(Regularity) If $v \in C([-L,R] \times I_0)$ is the solution of the integral equation* (3.8)*, where $g, h \in C(I_0)$, then the function v is a distributional solution of the BBM-equation on $[-L,R] \times I_0$. If $\varphi \in C^m([-L,R])$ and $g, h \in C^k(I_0)$ for some $m, k \geq 1$, then u comprises a classical solution of the BBM-equation* (3.1) *on $[-L,R] \times I_0$ and $\partial_t^i \partial_x^j v \in C([-L,R] \times I_0)$ for $0 \leq i \leq k$ and $0 \leq j \leq m$.*

Introduce an intermediate variable V defined by the formula

$$V(x,t) = v(x,t) - \big[g(t)\phi_1(x) + h(t)\phi_2(x)\big] = v(x,t) - \mu(x,t) \tag{3.9}$$

where,

$$\mu(x,t) = g(t)\phi_1(x) + h(t)\phi_2(x) \tag{3.10}$$

and ϕ_1 and ϕ_2 are defined in (3.4). Then, $V(-L,t) = V(R,t) = 0$ for all $t \in I$ and V satisfies the equations

(3.11)
$$\left.\begin{aligned}
&V_t + V_x + VV_x - V_{xxt} = -\Big(\mu + \mu V + \frac{1}{2}\mu^2\Big)_x, \quad -L < x < R, \quad t \in I_0,\\
&V(-L,t) = V(R,t) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in I_0,\\
&V(x,0) = \varphi(x) - \varphi(-L)\phi_1(x) - \varphi(R)\phi_2(x), \quad -L \le x \le R, \quad t \in I_0.
\end{aligned}\right\}$$

To extend the existence time interval from $I_0 = [0, T_0]$ to I, a standard energy method is used. Multiply both sides of (3.11) by $2V$ and integrate over $[-L, R]$ with respect to x to obtain:

$$\begin{aligned}
&\frac{d}{dt}\int_{-L}^{R}\Big(V^2(x,t) + V_x^2(x,t)\Big)\,dx\\
&= -\int_{-L}^{R} 2V(x,t)\Big(\mu + \mu V + \frac{1}{2}\mu^2\Big)_x\,dx \qquad (3.12)\\
&\le |\mu(\cdot,t)|_\infty \|V(\cdot,t)\|_{H^1_{LR}}^2 + 2\|\mu(\cdot,t)\|_{L_2(-L,R)}\|V(\cdot,t)|_{L_2(-L,R)}\\
&\quad + |\mu(\cdot,t)|_\infty \|\mu(\cdot,t)\|_{L_2(-L,R)}\,\|V_x(\cdot,t)\|_{L_2(-L,R)}.
\end{aligned}$$

Elementary considerations reveal that

$$|\mu(\cdot,t)|_\infty \le |g(t)| + |h(t)|,$$

$$\|\mu(\cdot,t)\|_{L_2(-L,R)} \le \frac{\sqrt{2}}{2}(|g(t)| + |h(t)|)$$

provided that $L + R \ge 1$. For $t \ge 0$, let $c(t)$ denote the quantity

$$c(t) = |g(t)| + |h(t)| \ge |\mu(t)|. \qquad (3.13)$$

Then, (3.12) reduces to:

$$\frac{d}{dt}\|V(\cdot,t)\|_{H^1_{LR}} \le \frac{1}{2}c(t)\|V(\cdot,t)\|_{H^1_{LR}} + c(t) + c^2(t). \qquad (3.14)$$

Solving this inequality yields the upper bound

(3.15)
$$\begin{aligned}
\|V(\cdot,t)\|_{H^1_{LR}} &\le \|V(\cdot,0)\|_1\, e^{\frac{1}{2}\int_0^t c(\tau)\,d\tau} + \int_0^t \Big(c(s) + c^2(s)\Big)e^{\frac{1}{2}\int_s^t (c(\tau)\,d\tau}\,ds\\
&\le \|V(\cdot,0)\|_1\, e^{\frac{1}{2}\int_0^t c(\tau)\,d\tau} + \int_0^t \Big(c(s) + c^2(s)\Big)\,ds\, e^{\frac{1}{2}\int_0^t c(\tau)\,d\tau},
\end{aligned}$$

where $V(x,0) = \varphi(x) - \varphi(-L)\phi_1(x) - \varphi(R)\phi_2(x)$. Multiply (3.11) by V_t and integrate over $[-L,R]$ with respect to x to reach the inequality

$$\begin{aligned}
&\|V_t(\cdot,t)\|^2_{H^1_{LR}} - V_t(-L,t)V_{xt}(-L,t) + V_t(R,t)V_{xt}(R,t)\\
&= -\int_{-L}^{R} V_t\Big(V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\Big)_x\, dx\\
&\le \|V_{xt}\|_{L_2(-L,R)}\|V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\|_{L_2(-L,R)}\\
&\le \|V_t(\cdot,t)\|_{H^1_{LR}}\|V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\|_{L_2(-L,R)},
\end{aligned}$$

where $\mu = \mu(x,t)$ is as in (3.10). Note that on $[-L,R]$, $\mu(x,t) \le c(t)$. Since $K_{LR}(-L,y) = K_{LR}(R,y) = 0$ for every $y \in (-L,R)$, $V_t(-L,t) = V_t(R,t) = 0$ for all $t \in I$, it follows that

(3.16)

$$\|V_t(\cdot,t)\|_{H^1_{LR}(} \le \Big(1 + c(t)\Big)\|V(\cdot,t)\|_{H^1_{LR}} + \frac{1}{2}\|V(\cdot,t)\|^2_{H^1_{LR}} + c(t) + \frac{1}{2}c^2(t).$$

Applying (3.15) in (3.16) yields an *a priori* bound on $\|V_t(\cdot,t)\|_{H^1_{LR}}$. The associated *a priori* bounds in $C(I;H^1_{LR})$ allow iteration of the local theory to obtain a solution defined on all of I. The regularity Theorem 3.3 then immediately allows inference of the following result.

Theorem 3.4. *The initial-boundary-value problem* (3.11) *is globally well-posed in* H^1_{LR} *if the boundary data* g,h *lie in* $C(I)$ *and satisfy the compatibility condition* (3.2). *That is, corresponding to such* g,h, *there is a unique solution* $V \in C(I;H^1_{LR})$. *The solution* V *respects the bounds in* (3.15) *and* (3.16) *and depends continuously on variations of* φ, g *and* h *within their function classes. In addition, if* $g,h \in C_b(I) \cap L_1(I)$, *then the* H^1_{LR}*-norm of* V *is uniformly bounded, viz.*

$$\|V(\cdot,t)\|_{H^1_{LR}} \le \Big(\|V(\cdot,0)\|_{H^1_{LR}} + |g|_1 + |h|_1 + 2|g|_2^2 + 2|h|_2^2\Big)\, e^{\frac{1}{2}(|g|_1+|h|_1)},$$

for all $t \in I$.

Corollary 3.5. *(Global Well-posedness) The initial-boundary-value problem* (3.8) *is globally well-posed if* $\varphi \in H^1_{LR}$ *and* $g,h \in C(I)$ *satisfy the compatibility condition* (3.2). *That is, there is a unique solution* v *of*

(3.8) *in* $C(I;H^1_{LR})$ *which depends continuously in this class upon variations of* $\varphi \in H^1_{LR}$ *and* g, h *in* $C(I)$ *and respects the inequality*

$$\|v(\cdot,t)\|_{H^1_{LR}} \le (\|\varphi\|_1 + 2|\varphi(-L)| + 2|\varphi(R)|)e^{\frac{1}{2}\int_0^t c(\tau)\,d\tau} + \int_0^t \Big(c(s) + c^2(s)\Big)\,ds\, e^{\frac{1}{2}\int_0^t c(\tau)\,d\tau}$$

provided $L+R$ *is sufficiently large, where* $c(t) = |g(t)|+|h(t)|$*. Moreover, if* $g, h \in C_b(I) \cap L_1(I)$*, then* v *satisfies the time-independent bound*

$$\|v(\cdot,t)\|_{H^1_{LR}} \le$$

$$\Big(\|\varphi\|_1 + 2|\varphi(-L)| + 2|\varphi(R)| + |g|_1 + |h|_1 + 2|g|_2^2 + 2|h|_2^2\Big)\, e^{\frac{1}{2}(|g|_1+|h|_1)}.$$

Theorem 3.1 now follows.

4. **Comparison results**

Let u be the solution of the pure initial-value problem for the BBM-equation (2.1). Then, both the traces $u(-L,t)$ and $u(R,t)$ are well defined for any $L, R > 0$. Let v be the solution of the two-point boundary-value problem (3.1), where the initial data φ is understood as the initial data in (2.1) restricted to the interval $[-L, R]$ and the boundary data is, in the first instance, $g(t) = u(-L,t)$ and $h(t) = u(R,t)$. The goal of this section is to develop estimates of the difference between u and v on the spatial interval $[-L, R]$.

To begin, introduce a new dependent variable

(4.1)
$$U(x,t) = u(x,t) - u(-L,t)\phi_1(x) - u(R,t)\phi_2(x) = u(x,t) - \mu_{LR}(x,t)$$

where,

(4.2)
$$\mu_{LR}(x,t) = u(-L,t)\phi_1(x) + u(R,t)\phi_2(x)$$

and ϕ_1 and ϕ_2 are defined in (3.4). A simple calculation shows that U satisfies the initial-boundary-value problem

(4.3)
$$\left.\begin{aligned} &U_t + U_x + UU_x - U_{xxt} = -\Big(\mu_{LR} + \mu_{LR}U + \frac{1}{2}\mu_{LR}^2\Big)_x, && -L < x < R, t > 0,\\ &U(-L,t) = U(R,t) = 0, && t > 0,\\ &U(x,0) = \varphi(x) - \varphi(-L)\phi_1(x) - \varphi(R)\phi_2(x), && -L < x < R. \end{aligned}\right\}$$

Because of the theory developed for u, there is a unique classical solution U of (4.3). The difference between U and V, where V is defined in (3.9) and is the solution of (3.12), is a useful quantity to understand. Once this is appropriately bounded, the identity $v - u = V - U - (\mu_{LR} - \mu)$ allows one to make a further estimate of the desired sort. Denote by W the difference

$$W = V - U = (v - u) + (\mu_{LR} - \mu). \tag{4.4}$$

Then, W is differentiable in both $t \in I$ and $x \in [-L, R]$ provided that $g, h \in C(I)$ satisfy (3.2). Moreover, W satisfies the initial-boundary-value problem

(4.5)
$$\left.\begin{aligned}
&W_t + W_x + WW_x - W_{xxt} = \Big(\mu_{LR} + \mu_{LR}U + \frac{1}{2}\mu_{LR}^2\Big)_x - \Big(\mu + \mu U + \frac{1}{2}\mu^2\Big)_x \\
&\qquad\qquad - \Big((U+\mu)W\Big)_x, \quad -L < x < R,\ t > 0, \\
&W(-L,t) = W(R,t) = 0, \qquad t \geq 0, \\
&W(x,0) = 0, \qquad -L \leq x \leq R.
\end{aligned}\right\}$$

Multiply (4.5) by $2W$ and integrate over $[-L, R]$; after integrations by parts, there appears

$$\begin{aligned}
&\frac{d}{dt}\int_{-L}^{R} \Big(W^2(x,t) + W_x^2(x,t)\Big)\,dx \\
&= 2\int_{-L}^{R} (U+\mu)WW_x\,dx \\
&\quad + 2\int_{-L}^{R} W\Big((\mu_{LR} - \mu) + (\mu_{LR} - \mu)U + \frac{1}{2}(\mu_{LR}^2 - \mu^2)\Big)_x\,dx,
\end{aligned} \tag{4.6}$$

implying that

$$\begin{aligned}
&\frac{d}{dt}\int_{-L}^{R} \Big(W^2(x,t) + W_x^2(x,t)\Big)\,dx \leq |U(\cdot,t) + \mu(\cdot,t)|_\infty \|W(\cdot,t)\|_{H^1_{LR}}^2 \\
&\quad + 2\Big|1 + U(\cdot,t) + \frac{1}{2}\Big(\mu_{LR}(\cdot,t) + \mu(\cdot,t)\Big)\Big|_\infty \big|\mu_{LR}(\cdot,t) - \mu(\cdot,t)\big|_2 \big|W_x(\cdot,t)\big|_2
\end{aligned}$$

where,

$$\begin{aligned} U(x,t)+\mu(x,t) &= u(x,t)-\mu_{LR}(x,t)+\mu(x,t)\\ &= u(x,t)+\Big(g(t)-u(-L,t)\Big)\phi_1(x)+\Big(h(t)-u(R,t)\Big)\phi_2(x). \end{aligned}$$

Notice that

$$\begin{aligned} |U(\cdot,t)+\mu(\cdot,t)|_\infty &\le 3|u(\cdot,t)|_\infty+|g(t)|+|h(t)|\\ &\le \frac{3\sqrt{2}}{2}\,\|\varphi\|_1+|g(t)|+|h(t)|, \end{aligned}$$

$$\begin{aligned} 1+&U(x,t)+\frac{1}{2}\Big(\mu_{LR}(x,t)+\mu(x,t)\Big)\\ &=1+u(x,t)-\mu_{LR}(x,t)+\frac{1}{2}\Big(\mu_{LR}(x,t)+\mu(x,t)\Big)\\ &=1+u(x,t)+\frac{1}{2}\big[g(t)-u(-L,t)\big]\phi_1(x)+\frac{1}{2}\big[h(t)-u(R,t)\big]\phi_2(x), \end{aligned}$$

whence,

$$\begin{aligned} \Big|1+U(\cdot,t)&+\frac{1}{2}\Big(\mu_{LR}(\cdot,t)+\mu(\cdot,t)\Big)\Big|_\infty\\ &\le 1+2|u(\cdot,t)|_\infty+\frac{1}{2}|g(t)|+\frac{1}{2}|h(t)|\\ &\le 1+\sqrt{2}\|\varphi\|_1+\frac{1}{2}|g(t)|+\frac{1}{2}|h(t)| \end{aligned}$$

and

$$\mu_{LR}(\cdot,t)-\mu(\cdot,t)=\big[u(-L,t)-g(t)\big]\phi_1(x)+\big[u(R,t)-h(t)\big]\phi_2(x).$$

In consequence, assuming that $L+R\ge\frac{1}{2}$, say, it follows that

$$\|\mu_{LR}(\cdot,t)-\mu(\cdot,t)\|\le\frac{\sqrt{2}}{2}\,\big|u(-L,t)-g(t)\big|+\frac{\sqrt{2}}{2}\,\big|U(R,t)-h(t)\big|.$$

Define the quantities

$$C(t)=\frac{3\sqrt{2}}{4}\|\varphi\|_1+\frac{1}{2}|g(t)|+\frac{1}{2}|h(t)|, \tag{4.7}$$

$$D(t)=2+2\sqrt{2}\|\varphi\|_1+|g(t)|+|h(t)| \tag{4.8}$$

and

$$E(t)=\frac{\sqrt{2}}{4}\,\big|u(-L,t)-g(t)\big|+\frac{\sqrt{2}}{4}\,\big|u(R,t)-h(t)\big|. \tag{4.9}$$

With these definitions, the preceding ruminations can be combined to yield the differential inequality

$$\frac{d}{dt}\|W(\cdot,t)\|_{H^1_{LR}} \le C(t)\|W(\cdot,t)\|_{H^1_{LR}} + D(t)E(t).$$

Solving this inequality gives:

$$\tag{4.10} \begin{aligned} \|W(\cdot,t)\|_{H^1_{LR}} &\le \int_0^t D(s)E(s)e^{\int_s^t C(\tau)\,d\tau}\,ds \\ &\le \int_0^t D(s)E(s)\,ds\,\exp\Big\{\int_0^t C(\tau)\,d\tau\Big\}, \end{aligned}$$

and so

$$\|u(\cdot,t)-v(\cdot,t)\|_{H^1_{LR}} \le \|W(\cdot,t)\|_{H^1_{LR}} + \|\mu-\mu_{LR}\|_{H^1_{LR}}.$$

Straightforward calculation reveals:

$$\|\phi_1\|^2_{H^1_{LR}} = \|\phi_2\|^2_{H^1_{LR}} = \frac{e^{L+R}+e^{-(L+R)}}{e^{L+R}-e^{-(L+R)}}.$$

These considerations are used to establish the following result.

Theorem 4.1. *Let u be the solution of* (2.1) *corresponding to a given $\varphi \in H^1(\mathbb{R})$. Let v be the solution of* (3.1) *assuming that $g, h \in C(I)$ and that they satisfy the compatibility condition* (3.2)*. Then, for any $t \in I$, the difference between u and v satisfies the inequality*

$$\tag{4.11} \begin{aligned} &\|u(\cdot,t)-v(\cdot,t)\|_{H^1_{LR}} \\ &\le \|W(\cdot,t)\|_{H^1_{LR}} + \|\mu-\mu_{LR}\|_{H^1_{LR}} \\ &\le \int_0^t D(s)E(s)\,ds\,\exp\Big\{\int_0^t C(\tau)\,d\tau\Big\} \\ &\quad + \Big(|u(-L,t)-g(t)|+|u(R,t)-h(t)|\Big)\Big\{\frac{e^{L+R}+e^{-(L+R)}}{e^{L+R}-e^{-(L+R)}}\Big\}^{\frac{1}{2}}, \end{aligned}$$

where $C(t)$, $D(t)$ and $E(t)$ are given in (4.7)*,* (4.8) *and* (4.9)*, respectively.*

Corollary 4.2. *In* (2.1)*, suppose the initial data $\varphi \in H^1(\mathbb{R})$ and that for some $\lambda \in (0,1)$, $e^{\lambda|x|}\varphi(x)$ is uniformly bounded on $\mathbb{R}$. Consider* (3.1)*, where the initial data is understood as the restriction of φ to $[-L,R]$, and where it is assumed that $L, R > \frac{1}{2}$. Choose boundary data $g(t) \equiv \varphi(-L)$*

and $h(t) \equiv \varphi(R)$*. Then, there is a constant* c_1 *dependent only on* λ, $\|\varphi\|_1$ *and* $\gamma = \sup_{x\in\mathbb{R}} e^{\lambda|x|}|\varphi(x)|$ *such that*

$$\|u(\cdot,t) - v(\cdot,t)\|_{H^1_{LR}} \le c_1 e^{-\lambda \min\{L,R\}+c_2 t}$$

where,

$$c_2 = \frac{4+\sqrt{2}\|\varphi\|_1}{4(1-\lambda^2)} + 2\|\varphi\|_1.$$

Corollary 4.3. *In the last corollary, if the initial data* φ *has a compact support and* $L, R > 0$ *are chosen sufficiently large so that support of* φ *is enclosed in* $(-L, R)$*, then the estimate of the difference between* u *and* v *holds with* $g(t) \equiv h(t) \equiv 0$.

Corollary 4.4. *Let* $\varphi = \varphi(x) \in H^1(\mathbb{R})$ *decay exponentially as* $x \to \pm\infty$*. Suppose* $g, h \in C(I)$ *satisfy the compatibility condition* (3.2)*. View* $v(x,t) = v_{LR}(x,t)$ *as function of* L *and* R *as well. Then, for any fixed point* $(x,t) \in \mathbb{R}^+ \times I$,

$$\lim_{L,R\to+\infty} v_L(x,t) = u(x,t)$$

where u *is the solution of initial value problem* (2.1)*.*

The latter convergence is uniform on compact sets. More precisely, we have the following.

Corollary 4.5. *Let* $\varphi \in H^1(\mathbb{R})$ *be of order* $e^{-\lambda|x|}$ *as* $x \to \pm\infty$*, for some* $\lambda \in (0,1)$*. Suppose* $g, h \in C(I)$ *satisfy* (3.2)*. Then, for any* $\epsilon > 0$ *and any finite time interval* $[0, T_0] \subset I$*, if both LandR are chosen greater than* $\frac{1}{\lambda}\ln\frac{\gamma(T_0)}{\epsilon}$*, then*

$$|u(x,t) - v(x,t)| \le \epsilon$$

uniformly on $[-L, R] \times [0, T_0]$.

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