

A REPRESENTATION FOR CHARACTERISTIC FUNCTIONALS OF STABLE RANDOM MEASURES WITH VALUES IN SAZONOV SPACES

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ABSTRACT. We deal with a Sazonov space (\mathcal{X} : real separable) valued symmetric α stable random measure Φ with independent increments on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. A pair (k, μ) , called here a control pair, for which $k : \mathcal{X} \times \mathbb{R}^k \rightarrow \mathbb{R}^+$, μ a positive measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, is introduced. It is proved that the law of Φ is governed by a control pair; and every control pair will induce such Φ . Moreover, k is unique for a given μ . Our derivations are based on the Generalized Bochner Theorem and the Radon- Nikodym Theorem for vector measures.

1. Introduction

Let \mathcal{X} be a real separable Banach space equipped with the norm $\|\cdot\|_{\mathcal{X}}$, on occasion $\|\cdot\|$, whenever there is no ambiguity. Also, let (Ω, Σ, P) be a probability space. An \mathcal{X} -valued random vector X is a measurable mapping from the probability space (Ω, Σ, P) into the Banach space \mathcal{X} equipped with its Borel σ -field $\mathcal{B}(\mathcal{X})$ generated by the open subsets in \mathcal{X} . Let \mathcal{X}' be the topological dual of \mathcal{X} ; i.e., the space of all bounded linear functionals on \mathcal{X} . For two Banach spaces \mathcal{X} and \mathcal{K} , $B(\mathcal{X}, \mathcal{K})$ denotes

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the class of all bounded linear operators on \mathcal{X} into \mathcal{K} . For any \mathcal{X} -valued random vector X , we denote the characteristic functional of X by

$$\phi_X(t) = \int_{\Omega} e^{it(X(\omega))} dP(\omega) = Ee^{itX}, \quad t \in \mathcal{X}'.$$

An \mathcal{X} -valued random vector X is said to be α -stable, $0 < \alpha \leq 2$, if for any positive number n there exists a vector x in \mathcal{X} such that

$$[\phi(t)]^n = e^{it(x)} \phi(n^{1/\alpha}t), \quad t \in \mathcal{X}',$$

and X is symmetric if $X \stackrel{d}{=} -X$. For more on Banach valued stable random vectors, see Ledoux and Talagrand (1991).

The Levy-Khinchin Spectral Representation Theorem states that an \mathcal{X} -valued random vector X is α -stable, $0 < \alpha \leq 2$, if and only if there exists a finite measure Γ on S , the unit sphere of \mathcal{X} , and an element $\mu \in \mathcal{X}$ such that the characteristic functional of X can be written as:

$$\phi_X(t) = \exp\left\{-\int_S |t(s)|^\alpha d\Gamma(s) - \varphi_\alpha(\Gamma, t) + it(\mu)\right\}, \quad t \in \mathcal{X}',$$

where,

$$\varphi_\alpha(\Gamma, t) = \begin{cases} \tan(\pi\alpha/2) \int_S |t(s)|^\alpha \operatorname{sign}(t(s)) d\Gamma(s) & \alpha \neq 1, \\ (2/\pi) \int_S t(s) \ln |t(s)| d\Gamma(s) & \alpha = 1. \end{cases}$$

For $0 < \alpha < 2$, this representation is unique and Γ is called the spectral measure of X [Linde (1983), Theorem 6.3.6].

An \mathcal{X} -valued random vector X is symmetric α -stable ($S\alpha S$), if and only if for each $t \in \mathcal{X}'$, $t(X)$ is $S\alpha S$, random variable. If X is $S\alpha S$ then ϕ_X will be real and Γ will be a symmetric measure; moreover,

$$(1.1) \quad \phi_X(t) = \exp\left\{-\int_S |t(s)|^\alpha \Gamma(ds)\right\}, \quad t \in \mathcal{X}'.$$

This characterization, on any real separable Hilbert space, was first obtained by Kulbes (1973). Two \mathcal{X} -valued α -stable random vectors X and Y are jointly $S\alpha S$ if and only if every linear combination $aX + bY$, $a, b \in \mathbb{R}$, is \mathcal{X} -valued $S\alpha S$.

Let $\mathcal{L}_{\mathcal{X}}^0(\Omega)$ denote the set of all \mathcal{X} -valued random vectors on the probability space (Ω, Σ, P) . Also, let (F, \mathcal{F}) be a measurable space. A set function Φ on \mathcal{F} into $\mathcal{L}_{\mathcal{X}}^0(\Omega)$ is a stable random measure if

- (I) $\Phi(\emptyset) = 0$ with probability 1.
- (II) For every choice of $A_1, \dots, A_n \in \mathcal{F}$, $(\Phi(A_1), \dots, \Phi(A_n))$ is jointly α -stable random vector on (Ω, Σ, P) .
- (III) Φ is σ -additive, in the sense that for disjoint \mathcal{F} -sets A_1, A_2, \dots , $\sum_{j=1}^n \Phi(A_j)$ converges to $\Phi\left(\bigcup_{j=1}^{\infty} A_j\right)$ in probability.

The \mathcal{X} -valued α -stable random measure Φ is said to have independent increments if for every disjoint \mathcal{F} -sets A_1, A_2, \dots, A_n , $\Phi(A_1), \dots, \Phi(A_n)$ are independent, and is said to be symmetric if for every $A \in \mathcal{F}$, $\Phi(A)$ is symmetric.

Let Φ be an \mathcal{X} -valued $S\alpha S$ random measure on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with independent increments. It follows from (1.1) that the characteristic functional of $\Phi(A)$ is given by

$$(1.2) \quad \phi_{\Phi(A)}(t) = \exp\left\{-\int_S |t(s)|^\alpha \Gamma_{\Phi(A)}(ds)\right\}, \quad t \in \mathcal{X}'.$$

Note that the spectral measure $\Gamma_{\Phi(A)}$ depends on set A . According to (1.2), the law of Φ is specified by $\{\Gamma_{\Phi(A)}, A \in \mathcal{B}(\mathbb{R}^k)\}$. This will make (1.2) less helpful. As the latter class cannot be identified easily, our aim is to provide a spectral type characterization for Φ . A characterization for multivariate $S\alpha S$ random measures is given in Soltani and Mahmoodi (2004).

A Banach space \mathcal{X} is said to be a Sazonov space provided that there exists a vector topology τ on \mathcal{X}' such that a function ϕ which maps \mathcal{X}' into the set of complex numbers is characteristic functional of a Radon probability measure on \mathcal{X} if and only if (1) $\phi(0) = 1$, (2) ϕ is positive definite and (3) ϕ is continuous in the τ topology (Generalized Bochner Theorem). Such a topology τ is called a Sazonov-topology.

In Section 2, we provide some lemmas and propositions to be used to prove the main result. A complete metric space γ of symmetric finite measures is constructed and employed to characterize the law of Φ . In Section 3, the Radon Nikodym property for the space γ is investigated. Our representation for characteristic functionals of stable random measures is given in Section 4.

2. Symmetric measures on the unit spheres

Let us begin with the following propositions which will be needed through out the article.

Proposition 2.1. *Let \mathcal{X} and \mathcal{K} be two real separable Banach spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{K}}$, respectively. Also, let X be an \mathcal{X} -valued $S\alpha S$ random vector ($0 < \alpha < 2$), and \mathcal{C} be a bounded linear operator from \mathcal{X} into \mathcal{K} ($\mathcal{C} \in B(\mathcal{X}, \mathcal{K})$). Then, $\mathcal{C}X$ is a \mathcal{K} -valued $S\alpha S$ random vector with the spectral measure,*

$$(2.1) \quad \Gamma_{\mathcal{C}X}(A) = \int_{T^{-1}(A)} \|\mathcal{C}s\|_{\mathcal{K}}^{\alpha} \Gamma_X(ds),$$

where, $T(s) = \frac{\mathcal{C}s}{\|\mathcal{C}s\|_{\mathcal{K}}}$ and $A \in \mathcal{B}(\mathcal{X})$.

Proof. The proposition follows by an argument similar to the one given in Mohammadpour and Soltani (2000) and the uniqueness of spectral measures on Banach spaces. \square

The next proposition follows from Proposition 6.6.2 and Proposition 6.6.5 of Linde (1983).

Proposition 2.2. *Let a sequence of \mathcal{X} -valued $S\alpha S$ random vectors $\{X_n\}$ converges weakly to X . Then, X is also an \mathcal{X} -valued $S\alpha S$ random vector. Also, if $\{\Gamma_{X_n}\}$ is the sequence of the spectral measures of $\{X_n\}$, then $\{\Gamma_{X_n}\}$ converges weakly to Γ_X .*

Let Φ be an \mathcal{X} -valued $S\alpha S$ random measure with independent increments on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Also, let

$$(2.2) \quad \mathcal{M} = \overline{\text{sp}}\{\Phi(A); A \in \mathcal{B}(\mathbb{R}^k)\},$$

where the closure is in the sense of convergence in probability. The spectral measure of each $X \in \mathcal{M}$ is denoted by Γ_X . Each spectral measure is symmetric finite measure on the surface of the unit ball S . Let us define,

$$(2.3) \quad \gamma = \{\Gamma_X, X \in \mathcal{M}\}.$$

Equip γ with a vector addition \oplus and a multiplication \otimes defined by

$$\Gamma_X \oplus \Gamma_Y = \Gamma_{X+Y}, \quad a \otimes \Gamma_X = \Gamma_{aX},$$

where a is a scalar. The space γ is a vector space whose scalar field is the set of real numbers. The vector addition \oplus is commutative, associative and has inverse $\ominus \Gamma_X = \Gamma_{-X}$; therefore, $\Gamma_X \ominus \Gamma_X = 0$ and $\Gamma_X \ominus \Gamma_Y = \Gamma_{X-Y}$, $X, Y \in \mathcal{M}$.

For each $\Gamma \in \gamma$, define,

$$\|\Gamma\|_\alpha = (\Gamma(S))^{\min\{1, 1/\alpha\}}.$$

Lemma 2.3. *For $0 < \alpha < 1$, $(\gamma, \|\cdot\|_\alpha)$ is a metric space; and for $1 \leq \alpha < 2$ it is a normed space.*

Proof. Let $\Gamma_X, \Gamma_Y \in \gamma$. Note that X and Y are jointly \mathcal{X} -valued α -stable random vectors. If S' is the unit sphere of a Banach space $\mathcal{X} \times \mathcal{X}$, then by (2.1),

$$(2.4) \quad (\Gamma_{X+Y}(S))^{1/\alpha} = \left(\int_{S'} \|s_1 + s_2\|^\alpha \Gamma_{X,Y}(ds) \right)^{1/\alpha},$$

where $s \in S'$ has the representation $s = s_1 \times s_2$, such that $s_1, s_2 \in \mathcal{X}$. By the Minkowski's inequality, for $1 \leq \alpha < 2$, (2.4) is less than

$$\left(\int_{S'} \|s_1\|^\alpha \Gamma_{X,Y}(ds) \right)^{1/\alpha} + \left(\int_{S'} \|s_2\|^\alpha \Gamma_{X,Y}(ds) \right)^{1/\alpha} = \|\Gamma_X\|_\alpha + \|\Gamma_Y\|_\alpha.$$

For $0 < \alpha < 1$ use the inequality $\|s_1 + s_2\|^\alpha \leq \|s_1\|^\alpha + \|s_2\|^\alpha$. Therefore,

$$\|\Gamma_X \oplus \Gamma_Y\|_\alpha \leq \|\Gamma_X\|_\alpha + \|\Gamma_Y\|_\alpha.$$

Clearly, $d(\Gamma_X, \Gamma_Y) = \|\Gamma_X \ominus \Gamma_Y\|_\alpha = d(\Gamma_Y, \Gamma_X)$ and $d(\Gamma_X, \Gamma_Y) = 0$ imply $X = Y$ with probability 1. Also, we note that $\|c \otimes \Gamma_X\|_\alpha = \left(\int_S |c|^\alpha \|s\|^\alpha \Gamma_X(ds) \right)^{1/\alpha} = |c| \|\Gamma_X\|_\alpha$ for any real number c . The proof is now complete. \square

Proposition 2.4. *Let X_1, X_2, \dots and X be \mathcal{X} -valued $S\alpha S$ random vectors in \mathcal{M} . Then, Γ_{X_n} converges to Γ_X in γ if and only if X_n converges to X in probability.*

Proof. Let Γ_{X_n} converge to Γ_X in γ . Then, Γ_{X_n} is a Cauchy sequence in γ . Therefore, $\Gamma_{X_n - X_m}(S)$ tends to 0 as $m, n \rightarrow \infty$ and then X_n is a Cauchy sequence in probability. For the converse, if $X_n - X$ converges to 0 in probability, then $\Gamma_{X_n - X}$ converges weakly to Γ_0 and then $\Gamma_{X_n - X}(S) \rightarrow 0$. Therefore, $\|\Gamma_{X_n} \ominus \Gamma_X\|_\alpha \rightarrow 0$. \square

Lemma 2.5. *The linear space (γ, ρ) is complete.*

Proof. Let $\{\Gamma_{X_n}\}$ be a Cauchy sequence in γ . So, $\|\Gamma_{X_n} \ominus \Gamma_{X_m}\|_\alpha = \|\Gamma_{X_n - X_m}\|_\alpha \rightarrow 0$ as $n, m \rightarrow \infty$. Then, by Proposition 2.4, X_n is Cauchy in probability and then there exists an \mathcal{X} -valued random vector X such that X_n converges to X in probability. Proposition 2.2 implies that X is $S\alpha S$ random vector in \mathcal{M} . By using Proposition 2.4, Γ_{X_n} converges to Γ_X in γ and $\Gamma_X \in \gamma$. \square

3. The Radon Nikodym property for γ

As observed in Section 2, $(\gamma, \|\cdot\|_\alpha)$, $1 < \alpha < 2$, is a Banach space. Here, we will prove that it is isometrically isomorphic to a certain \mathcal{L}^α space, and consequently apply Radon Nikodym Theorem to certain vector measures with values in γ .

By using Proposition 2.2 and an argument similar to the one given in Soltani (1994) [Theorems 3.1 and 3.2], the following theorem can be proved.

Theorem 3.1. *Let Φ be an \mathcal{X} -valued $S\alpha S$ random measure with independent increments on measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and the class \mathcal{M} be as in (2.2). Then, there is a unique bimeasure π on $\mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(S)$ such that*

$$(3.1) \quad \pi(A, \cdot) = \Gamma_{\Phi(A)}(\cdot), \quad \text{for every } A \in \mathcal{B}(\mathbb{R}^k),$$

where $\Gamma_{\Phi(A)}$ is the spectral measure of $\Phi(A)$. Moreover,

(i) *For every $Y \in \mathcal{M}$, there is a real valued function $g \in \mathcal{L}^\alpha(\pi(\cdot, S))$ such that the spectral measure of Y is given by*

$$(3.2) \quad \Gamma_Y(B) = \int_{\mathbb{R}^k} |g(t)|^\alpha \pi(dt, B),$$

for every $B \in \mathcal{B}(S)$.

(ii) *If g is a real valued Borel function in $\mathcal{L}^\alpha(\pi(\cdot, S))$, then there is a unique \mathcal{X} -valued $S\alpha S$ random vector Y in \mathcal{M} for which its spectral measure is given by (3.2).*

Let us apply Theorem 3.1 to establish an isomorphism between the space $(\gamma, \|\cdot\|_\alpha)$ and $\mathcal{L}^\alpha(\pi(\cdot, S))$. According to parts 1 and 2 of this theorem, for every $g \in \mathcal{L}^\alpha(\pi(\cdot, S))$, the stochastic integral $Y = \int_{\mathbb{R}^k} g(x) d\Phi(x)$ is well defined, in the weak sense, and defines an \mathcal{X} -valued $S\alpha S$ random vector. Clearly, $Y \in \mathcal{M}$ and then $\Gamma_Y \in \gamma$. Now, let us set $T(\Gamma_Y) = g$. We have,

$$\begin{aligned} \left(\int_{\mathbb{R}^k} |g(t)|^\alpha \pi(dt, S) \right)^{1/\alpha} &= (\Gamma_Y(S))^{1/\alpha} \\ &= \|\Gamma_Y\|_\alpha. \end{aligned}$$

Hence, T is an isometric isomorphism of γ into $\mathcal{L}^\alpha(\pi(\cdot, S))$.

Theorem 3.2. *For $1 < \alpha < 2$, let Φ be an \mathcal{X} -valued $S\alpha S$ random measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with independent increments and also let γ be the space as in (2.3). Then, γ has the Radon Nikodym property.*

Proof. If $\pi(\cdot, \cdot)$ is defined as in (3.1), then $\pi(\mathbb{R}^k, S) = \Gamma_{\Phi(\mathbb{R}^k)}(S) < \infty$. Now, since for $1 < \alpha$, $\mathcal{L}^\alpha(\pi(\cdot, S))$ has the Radon Nikodym property [Diestel and Uhl(1977), page 140, Theorem 1], and $\mathcal{L}^\alpha(\pi(\cdot, S))$ and γ are isometrically isomorphic, then γ has the Radon Nikodym property.

4. The main result

Let $\psi(A) = \Gamma_{\Phi(A)}$, $A \in \mathcal{B}(\mathbb{R}^k)$. Since $(\Phi(\bigcup_{j=1}^\infty A_j) - \sum_{j=1}^n \Phi(A_j)) \rightarrow 0$ in probability for any given sequence of disjoint sets A_1, A_2, \dots ,

$$\left\| \Gamma_{\Phi(\bigcup_{j=1}^\infty A_j) - \sum_{j=1}^n \Phi(A_j)} \right\|_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

giving that

$$\psi(\bigcup_{j=1}^\infty A_j) = \Gamma_{\Phi(\bigcup_{j=1}^\infty A_j)} = \Gamma_{\sum_{j=1}^\infty \Phi(A_j)} = \bigoplus_j \psi(A_j),$$

in $(\gamma, \|\cdot\|_\alpha)$. Therefore, ψ is a vector measure on $\mathcal{B}(\mathbb{R}^k)$ with values in γ .

Lemma 4.1. *The vector measure ψ possesses the following properties:*

(I) *There is a finite positive measure μ on $\mathcal{B}(\mathbb{R}^k)$ such that ψ is μ -continuous (i.e., $\|\psi(A_n)\|_\alpha \rightarrow 0$ as $\mu(A_n) \rightarrow 0$).*

(II) ψ is of bounded variation.

Proof. For (I), when $1 \leq \alpha < 2$, let $\mu(A) = \|\psi(A)\|_\alpha^\alpha$, ($A \in \mathcal{B}(\mathbb{R}^k)$). Note that if X and Y are independent, then $\Gamma_{X+Y} = \Gamma_X + \Gamma_Y$; therefore, since Φ is independently scattered, it follows that for disjoint sets A_1, A_2 ,

$$\begin{aligned} \mu(A_1 \cup A_2) &= \|\Gamma_{\Phi(A_1 \cup A_2)}\|_\alpha^\alpha = \|\Gamma_{\Phi(A_1)} + \Gamma_{\Phi(A_2)}\|_\alpha^\alpha \\ &= \|\Gamma_{\Phi(A_1)}\|_\alpha^\alpha + \|\Gamma_{\Phi(A_2)}\|_\alpha^\alpha, \end{aligned}$$

and thus μ is finitely additive and $\mu(\cup_{i=n+1}^\infty A_i) = \|\Gamma_{\Phi(\cup_{i=n+1}^\infty A_i)}\|_\alpha \rightarrow 0$

as $n \rightarrow \infty$. Therefore, $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^n \mu(A_i) + \mu(\cup_{i=n+1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

The same reasoning also applies to $0 < \alpha < 1$, with $\mu(A) = \|\psi(A)\|_\alpha$. It also easily follows that Ψ is μ -continuous. Part (II) follows from the fact that μ is a finite measure, [Proposition 11 in Diestel and Uhl(1977)]. \square

Our main result is the following theorem.

Theorem 4.2. *Let \mathcal{X} be a real separable Sazonov space with Sazonov topology τ and Φ be an \mathcal{X} -valued S α S random measure, $1 < \alpha < 2$, with independent increments on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Then, the law of Φ is uniquely specified by a control pair (μ, k) , through*

$$(4.1) \quad -\log \phi_{\sum_{i=1}^n a_i \Phi(A_i)}(t) = \sum_{i=1}^n |a_i|^\alpha \int_{A_i} k(t, y) \mu(dy),$$

$t \in \mathcal{X}'$, $y \in \mathbb{R}^k$, $A_i \in \mathcal{B}(\mathbb{R}^k)$, $a_i \in \mathbb{R}$, where μ is a positive measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and $k : \mathcal{X}' \times \mathbb{R}^k \mapsto \mathbb{R}^+$ is a measurable mapping with the following properties:

(I) For every $t \in \mathcal{X}'$, $k(t, \cdot)$ is integrable with respect to μ .

(II) For y , μ a.e., $k(\cdot, y)$ is of negative type and homogeneous, that is,

$$\sum_{i,j=1}^N c_i c_j k(t_i - t_j, y) \leq 0,$$

for every integer N and every choice of real numbers c_1, \dots, c_N subject to $\sum_{j=1}^N c_j = 0$, and $t_1, \dots, t_N \in \mathcal{X}'$. Moreover,

$$k(ct, y) = |c|^\alpha k(t, y),$$

for every scalar c and every $t \in \mathcal{X}'$.

(III) For y, μ a.e., $k(\cdot, y)$ is τ -continuous.

Conversely for a measurable mapping $k : \mathcal{X}' \times \mathbb{R}^k \mapsto \mathbb{R}^+$ having the properties (I), (II) and (III), there is an \mathcal{X} -valued $S\alpha S$ random measure Φ with independent increments on a measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ which satisfies (4.1).

Proof. The Radon Nikodym Theorem for the vector measure ψ with respect to μ implies that there is a unique γ -valued μ Bochner integrable function $p(y) =: p(y, ds)$ on \mathbb{R}^k , for which $\psi(dy) = p(y)\mu(dy)$, [Diestel (1977), pages 47 and 59]. This will allow approximating $\psi(A)$ by a sequence $\psi_N = \sum_{j=1}^N 1_{E_j}(y)p(y_j)\mu(E_j)$ in $(\gamma, \|\cdot\|_\alpha)$, where E_1, \dots, E_N is a finite partition of $\mathcal{B}(\mathbb{R}^k)$ -sets for A . But if a sequence $\{\Gamma_{X_n}\}$ converges to Γ in $(\gamma, \|\cdot\|_\alpha)$, then X_n will converge weakly to X . Consequently, for each bounded function q ,

$$\begin{aligned} \int_S q(s)\psi(A)(ds) &= \lim_{N \rightarrow \infty} \int_S q(s)\psi_N(ds) \\ &= \lim_{N \rightarrow \infty} \sum_{j=1}^N \left\{ \int_S q(s)p(y_j, ds) \right\} 1_{E_j}(y)\mu(E_j) \\ &= \int_A \left\{ \int_S q(s)p(y, ds) \right\} \mu(dy). \end{aligned}$$

Now, since

$$\begin{aligned} -\log \phi_{\Phi(A)}(t) &= \int_S |t(s)|^\alpha \Gamma_{\Phi(A)}(ds) \\ &= \int_S |t(s)|^\alpha \psi(A)(ds), \end{aligned}$$

we obtain:

$$-\log \phi_{\Phi(A)}(t) = \int_A \left\{ \int_S |t(s)|^\alpha p(y, ds) \right\} \mu(dy).$$

Let

$$k(t, y) = \int_S |t(s)|^\alpha p(y, ds).$$

Then, (4.1) will evidently be satisfied. What remains to prove is that $k(., .)$ possesses the properties (I), (II) and (III). The property (I) follows from the fact that Φ is defined on $\mathcal{B}(\mathbb{R}^k)$. Indeed, $\Phi(\mathbb{R}^k)$ is an \mathcal{X} -valued $S\alpha S$ random vector. For the property (II), the function $f(x) = |x|^\alpha$ is of negative type on $(-\infty, +\infty)$ [Schoenberg (1938)]. Therefore, for $t_1, t_2, \dots, t_N \in \mathcal{X}'$ and c_1, c_2, \dots, c_N , given real numbers, such that $\sum_{j=1}^N c_j = 0$, and every $s \in S$, giving that

$$\sum_{i,j=1}^N c_i c_j |t_i(s) - t_j(s)|^\alpha \leq 0,$$

it follows that,

$$\sum_{i,j=1}^N c_i c_j k(t_i - t_j, y) \leq 0.$$

Also,

$$k(ct, y) = \int |ct(s)|^\alpha p(y, ds) = |c|^\alpha k(t, y).$$

For (III), we note that it will be sufficient to show $k(., y)$ is τ -continuous at zero. But since $p(y, S) < \infty$, for every $y \in \mathbb{R}^k$, it follows from the Lyapounov's inequality that

$$\begin{aligned} k(t, y) &= (p(y, S) \int_S |t(s)|^\alpha \frac{p(y, ds)}{p(y, S)}) \\ &\leq (p(y, S) [\int_S (t(s))^2 \frac{p(y, ds)}{p(y, S)}]^\alpha)^{1/2}. \end{aligned}$$

According to the Levy-Khinchin Spectral Representation Theorem, $\exp\{\int_S (t(s))^2 p(y, ds)\}$ is a Gaussian characteristic functional. Therefore, it is τ -continuous and consequently $k(., y)$ is τ -continuous everywhere on \mathcal{X}' .

For the converse, assume (k, μ) is given and k satisfies properties (I), (II) and (III). Since $k(0, y) = 0$, and $k(., y)$ is τ -continuous on \mathcal{X}' , it follows that $\int_A k(0, y) \mu(dy) = 0$ and $\int_A k(., y) \mu(dy)$ is τ -continuous on \mathcal{X}' for every $A \in \mathcal{B}(\mathbb{R}^k)$. However, the fact that $\int_A k(., y) \mu(dy)$ is positive definite on \mathcal{X}' follows immediately from [Gelefan, page 279, Theorem 4]. Therefore, $\phi_A(\cdot) = \exp\{-\int_A k(., y) \mu(dy)\}$ is a characteristic functional. Let $\Phi(A)$ be an \mathcal{X} -valued random vector with characteristic functional

$\phi_A(\cdot)$. It follows by a classical argument that $\{\Phi(A), A \in \mathcal{B}(\mathbb{R}^k)\}$ induces an \mathcal{X} -valued random measure on $\mathcal{B}(\mathbb{R}^k)$. It is plain to verify that $\Phi(A)$ is $S\alpha S$. Indeed, $\phi_A(t)$ is real and for $n > 0$,

$$[\phi_A(t)]^n = \exp\{-n \int_A k(t, y) \mu(dy)\} = \phi(n^{1/\alpha} t).$$

The proof is now complete. \square

Remark 4.3. Each Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ is a Sazonov space and the Sazonov topology on $\mathcal{H}' (= \mathcal{H})$ is the locally convex topology generated by the semi-norms p with $p(x) = \langle Sx, x \rangle^{1/2}$, where $S : \mathcal{H} \rightarrow \mathcal{H}$ varies over the symmetric positive trace class operators on \mathcal{H} . Therefore, the theorem is true in this case.

Example 1. For real separable Banach space \mathcal{X} , let $x_i \in \mathcal{X}, i = 1, 2, \dots$ and $\sum_{i=1}^{\infty} \|x_i\|^\alpha < \infty$. For $0 < \alpha < 2$, let $\{\theta_i^{(\alpha)}\}$ denote a sequence of independent $S\alpha S$ random variables such that for every $x \in \mathbb{R}$,

$$\phi_{\theta_i^{(\alpha)}}(x) = e^{-|x|^\alpha}.$$

Assume $\sum_{i=1}^{\infty} \theta_i^{(\alpha)} x_i$ exists almost everywhere. Define $\Phi(A) := \sum_{i \in A} \theta_i^{(\alpha)} x_i$. Then, Φ is an $S\alpha S$ random measure and

$$\phi_{\Phi(A)}(t) = e^{-\sum_{i \in A} |t(x_i)|^\alpha} \quad t \in \mathcal{X}',$$

and

$$\Gamma_{\Phi(A)}(ds) = 1/2 \sum_{i \in A} \|x_i\|^\alpha (\delta_{x_i/\|x_i\|}(ds) + \delta_{-x_i/\|x_i\|}(ds)).$$

It follows that $\mu(A) = \sum_{i \in A} \|x_i\|^\alpha$ and $p(i, ds) = 1/2(\delta_{x_i/\|x_i\|}(ds) + \delta_{-x_i/\|x_i\|}(ds))$, where δ_a is the direct measure concentrated on a . Therefore,

$$k(t, i) = \int_S |t(s)|^\alpha p(i, ds) = |t(x_i)/\|x_i\|^\alpha, \quad t \in \mathcal{X}'.$$

With the assumption on this Banach space, it will become a Sazonov space and its topology is a topology by the following neighborhood basis

of zero:

$\{\{t \in \mathcal{X}' ; -\log \phi_X(t) \leq 1\}$; for each \mathcal{X} -valued α stable random vector $X\}$.

[Linde (1983), page 176].

Example 2. Let Φ be a Levy $S\alpha S$ random measure. Then, $\Gamma_{\Phi(A)}(ds) = \lambda(A)v(ds)$, where λ is the Lebesgue measure and v is a symmetric probability measure on S , which is not supported by any subspace, $v\{s \in S; \beta(s) = 0\} < 1$ for all $\beta \in S'$. Therefore,

$$\phi_{\Phi(A)}(t) = \exp\{-\lambda(A) \int_S |t(s)|^\alpha dv(s)\}, \quad t \in \mathcal{X}', A \in \mathcal{B}(\mathbb{R}).$$

Thus, in Theorem 4.2, μ is the Lebesgue measure and

$$k(t, y) = \int_S |t(s)|^\alpha dv(s) \quad t \in \mathcal{X}', y \in \mathbb{R}.$$

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