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A REPRESENTATION FOR CHARACTERISTIC FUNCTIONALS OF STABLE RANDOM MEASURES WITH VALUES IN SAZONOV SPACES

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Abstract. We deal with a Sazonov space ($\mathcal{X}$: real separable) valued symmetric α stable random measure Φ with independent increments on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. A pair (k, μ), called here a control pair, for which $k : \mathcal{X} \times \mathbb{R}^k \to \mathbb{R}^+$, μ a positive measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, is introduced. It is proved that the law of Φ is governed by a control pair; and every control pair will induce such Φ. Moreover, k is unique for a given μ. Our derivations are based on the Generalized Bochner Theorem and the Radon- Nikodym Theorem for vector measures.

1. Introduction

Let $\mathcal{X}$ be a real separable Banach space equipped with the norm $\|.\|_{\mathcal{X}}$, on occasion $\|.\|$, whenever there is no ambiguity. Also, let (Ω, Σ, P) be a probability space. An $\mathcal{X}$-valued random vector X is a measurable mapping from the probability space (Ω, Σ, P) into the Banach space $\mathcal{X}$ equipped with its Borel σ-field $\mathcal{B}(\mathcal{X})$ generated by the open subsets in $\mathcal{X}$. Let $\mathcal{X}'$ be the topological dual of $\mathcal{X}$; i.e., the space of all bounded linear functionals on $\mathcal{X}$. For two Banach spaces $\mathcal{X}$ and $\mathcal{K}$, $B(\mathcal{X}, \mathcal{K})$ denotes

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the class of all bounded linear operators on $\mathcal{X}$ into $\mathcal{K}$. For any $\mathcal{X}$-valued random vector X, we denote the characteristic functional of X by

$$\phi_X(t) = \int\limits_{\Omega} e^{it(X(\omega))} dP(\omega) = Ee^{itX}, \qquad t \in \mathcal{X}'.$$

An $\mathcal{X}$-valued random vector X is said to be α-stable, $0 < \alpha \leq 2$, if for any positive number n there exists a vector x in $\mathcal{X}$ such that

$$[\phi(t)]^n = e^{it(x)}\phi(n^{1/\alpha}t), \qquad t \in \mathcal{X}',$$

and X is symmetric if $X \stackrel{d}{=} -X$. For more on Banach valued stable random vectors, see Ledoux and Talagrand (1991).

The Levy-Khinchin Spectral Representation Theorem states that an $\mathcal{X}$-valued random vector X is α-stable, $0 < \alpha \leq 2$, if and only if there exists a finite measure Γ on S, the unit sphere of $\mathcal{X}$, and an element $\mu \in \mathcal{X}$ such that the characteristic functional of X can be written as:

$$\phi_X(t) = \exp\{-\int\limits_S |t(s)|^\alpha \, d\Gamma(s) - \varphi_\alpha(\Gamma, t) + it(\mu)\}, \qquad t \in \mathcal{X}',$$

where,

$$\varphi_\alpha(\Gamma, t) = \begin{cases} \tan(\pi\alpha/2) \int\limits_S |t(s)|^\alpha \, sign(t(s)) d\Gamma(s) & \alpha \neq 1, \\ (2/\pi) \int\limits_S t(s) \ln |t(s)| \, d\Gamma(s) & \alpha = 1. \end{cases}$$

For $0 < \alpha < 2$, this representation is unique and Γ is called the spectral measure of X [Linde (1983), Theorem 6.3.6].

An $\mathcal{X}$-valued random vector X is symmetric α-stable ($S\alpha S$), if and only if for each $t \in \mathcal{X}'$, $t(X)$ is $S\alpha S$, random variable. If X is $S\alpha S$ then ϕ_X will be real and Γ will be a symmetric measure; moreover,

$$\phi_X(t) = \exp\{-\int\limits_S |t(s)|^\alpha \, \Gamma(ds)\}, \qquad t \in \mathcal{X}'. \tag{1.1}$$

This characterization, on any real separable Hilbert space, was first obtained by Kulbes (1973). Two $\mathcal{X}$-valued α-stable random vectors X and Y are jointly $S\alpha S$ if and only if every linear combination $aX + bY$, $a, b \in \mathbb{R}$, is $\mathcal{X}$-valued $S\alpha S$.

Let $\mathcal{L}^0_{\mathcal{X}}(\Omega)$ denote the set of all $\mathcal{X}$-valued random vectors on the probability space (Ω, Σ, P). Also, let $(F, \mathcal{F})$ be a measurable space. A set function Φ on $\mathcal{F}$ into $\mathcal{L}^0_{\mathcal{X}}(\Omega)$ is a stable random measure if

(I) $\Phi(\emptyset) = 0$ with probability 1.

(II) For every choice of $A_1, \cdots, A_n \in \mathcal{F}$, $(\Phi(A_1), ..., \Phi(A_n))$ is jointly α-stable random vector on (Ω, Σ, P).

(III) Φ is σ-additive, in the sense that for disjoint $\mathcal{F}$-sets $A_1, A_2, \cdots$, $\sum_{j=1}^{n} \Phi(A_j)$ converges to $\Phi\left(\bigcup_{j=1}^{\infty} A_j\right)$ in probability.

The $\mathcal{X}$-valued α-stable random measure Φ is said to have independent increments if for every disjoint $\mathcal{F}$-sets $A_1, A_2, ..., A_n$, $\Phi(A_1), ..., \Phi(A_n)$ are independent, and is said to be symmetric if for every $A \in \mathcal{F}$, $\Phi(A)$ is symmetric.

Let Φ be an $\mathcal{X}$-valued $S\alpha S$ random measure on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with independent increments. It follows from (1.1) that the characteristic functional of $\Phi(A)$ is given by

$$(1.2) \qquad \phi_{\Phi(A)}(t) = \exp\{-\int_S |t(s)|^\alpha \Gamma_{\Phi(A)}(ds)\}, \qquad t \in \mathcal{X}'.$$

Note that the spectral measure $\Gamma_{\Phi(A)}$ depends on set A. According to (1.2), the law of Φ is specified by $\{\Gamma_{\Phi(A)}, A \in \mathcal{B}(\mathbb{R}^k)\}$. This will make (1.2) less helpful. As the latter class cannot be identified easily, our aim is to provide a spectral type characterization for Φ. A characterization for multivariate $S\alpha S$ random measures is given in Soltani and Mahmoodi (2004).

A Banach space $\mathcal{X}$ is said to be a Sazonov space provided that there exists a vector topology τ on $\mathcal{X}'$ such that a function ϕ which maps $\mathcal{X}'$ into the set of complex numbers is characteristic functional of a Radon probability measure on $\mathcal{X}$ if and only if (1) $\phi(0) = 1$, (2) ϕ is positive definite and (3) ϕ is continuous in the τ topology (Generalized Bochner Theorem). Such a topology τ is called a Sazonov-topology.

In Section 2, we provide some lemmas and propositions to be used to prove the main result. A complete metric space γ of symmetric finite measures is constructed and employed to characterize the law of Φ. In Section 3, the Radon Nikodym property for the space γ is investigated. Our representation for characteristic functionals of stable random measures is given in Section 4.

2. **Symmetric measures on the unit spheres**

Let us begin with the following propositions which will be needed through out the article.

Proposition 2.1. *Let $\mathcal{X}$ and $\mathcal{K}$ be two real separable Banach spaces with norms $\|.\|_{\mathcal{X}}$ and $\|.\|_{\mathcal{K}}$, respectively. Also, let X be an $\mathcal{X}$-valued $S\alpha S$ random vector ($0 < \alpha < 2$), and $\mathcal{C}$ be a bounded linear operator from $\mathcal{X}$ into $\mathcal{K}$ ($\mathcal{C} \in B(\mathcal{X}, \mathcal{K})$). Then, $\mathcal{C}X$ is a $\mathcal{K}$-valued $S\alpha S$ random vector with the spectral measure,*

$$\Gamma_{\mathcal{C}X}(A) = \int_{T^{-1}(A)} \|\mathcal{C}s\|_{\mathcal{K}}^{\alpha} \, \Gamma_X(ds), \tag{2.1}$$

where, $T(s) = \dfrac{\mathcal{C}s}{\|\mathcal{C}s\|_{\mathcal{K}}}$ *and* $A \in \mathcal{B}(\mathcal{X})$.

Proof. The proposition follows by an argument similar to the one given in Mohammadpour and Soltani (2000) and the uniqueness of spectral measures on Banach spaces. □

The next proposition follows from Proposition 6.6.2 and Proposition 6.6.5 of Linde (1983).

Proposition 2.2. *Let a sequence of $\mathcal{X}$-valued $S\alpha S$ random vectors $\{X_n\}$ converges weakly to X. Then, X is also an $\mathcal{X}$-valued $S\alpha S$ random vector. Also, if $\{\Gamma_{X_n}\}$ is the sequence of the spectral measures of $\{X_n\}$, then $\{\Gamma_{X_n}\}$ converges weakly to Γ_X.*

Let Φ be an $\mathcal{X}$-valued $S\alpha S$ random measure with independent increments on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Also, let

$$\mathcal{M} = \overline{\text{sp}}\{\Phi(A); A \in \mathcal{B}(\mathbb{R}^k)\}, \tag{2.2}$$

where the closure is in the sense of convergence in probability. The spectral measure of each $X \in \mathcal{M}$ is denoted by Γ_X. Each spectral measure is symmetric finite measure on the surface of the unit ball S. Let us define,

$$\gamma = \{\Gamma_X, \, X \in \mathcal{M}\}. \tag{2.3}$$

Equip γ with a vector addition $\oplus$ and a multiplication $\otimes$ defined by

$$\Gamma_X \oplus \Gamma_Y = \Gamma_{X+Y}, \;\; a \otimes \Gamma_X = \Gamma_{aX},$$

where a is a scalar. The space γ is a vector space whose scalar field is the set of real numbers. The vector addition $\oplus$ is commutative, associative and has inverse $\ominus\Gamma_X = \Gamma_{-X}$; therefore, $\Gamma_X \ominus \Gamma_X = 0$ and $\Gamma_X \ominus \Gamma_Y = \Gamma_{X-Y}$, $X, Y \in \mathcal{M}$.

For each $\Gamma \in \gamma$, define,

$$\|\Gamma\|_{\alpha} = (\Gamma(S))^{\min\{1,1/\alpha\}}.$$

Lemma 2.3. *For $0 < \alpha < 1$, $(\gamma, \|.\|_{\alpha})$ is a metric space; and for $1 \leq \alpha < 2$ it is a normed space.*

Proof. Let $\Gamma_X, \Gamma_Y \in \gamma$. Note that X and Y are jointly $\mathcal{X}$-valued α-stable random vectors. If S' is the unit sphere of a Banach space $\mathcal{X} \times \mathcal{X}$, then by (2.1),

$$(\Gamma_{X+Y}(S))^{1/\alpha} = (\int_{S'} \|s_1 + s_2\|^{\alpha} \Gamma_{X,Y}(ds))^{1/\alpha}, \tag{2.4}$$

where $s \in S'$ has the representation $s = s_1 \times s_2$, such that $s_1, s_2 \in \mathcal{X}$. By the Minkowski's inequality, for $1 \leq \alpha < 2$, (2.4) is less than

$$(\int_{S'} \|s_1\|^{\alpha} \Gamma_{X,Y}(ds))^{1/\alpha} + (\int_{S'} \|s_2\|^{\alpha} \Gamma_{X,Y}(ds))^{1/\alpha} = \|\Gamma_X\|_{\alpha} + \|\Gamma_Y\|_{\alpha}.$$

For $0 < \alpha < 1$ use the inequality $\|s_1 + s_2\|^{\alpha} \leq \|s_1\|^{\alpha} + \|s_2\|^{\alpha}$. Therefore,

$$\|\Gamma_X \oplus \Gamma_Y\|_{\alpha} \leq \|\Gamma_X\|_{\alpha} + \|\Gamma_Y\|_{\alpha}.$$

Clearly, $d(\Gamma_X, \Gamma_Y) = \|\Gamma_X \ominus \Gamma_Y\|_{\alpha} = d(\Gamma_Y, \Gamma_X)$ and $d(\Gamma_X, \Gamma_Y) = 0$ imply $X = Y$ with probability 1. Also, we note that $\|c \otimes \Gamma_X\|_{\alpha} = (\int_S |c|^{\alpha} \|s\|^{\alpha} \Gamma_X(ds))^{1/\alpha} = |c| \|\Gamma_X\|_{\alpha}$ for any real number c. The proof is now complete. □

Proposition 2.4. *Let $X_1, X_2, ...$ and X be $\mathcal{X}$-valued $S\alpha S$ random vectors in $\mathcal{M}$. Then, Γ_{X_n} converges to Γ_X in γ if and only if X_n converges to X in probability.*

Proof. Let Γ_{X_n} converge to Γ_X in γ. Then, Γ_{X_n} is a Cauchy sequence in γ. Therefore, $\Gamma_{X_n - X_m}(S)$ tends to 0 as $m, n \to \infty$ and then X_n is a Cauchy sequence in probability. For the converse, if $X_n - X$ converges to 0 in probability, then $\Gamma_{X_n - X}$ converges weakly to Γ_0 and then $\Gamma_{X_n - X}(S) \to 0$. Therefore, $\|\Gamma_{X_n} \ominus \Gamma_X\|_{\alpha} \to 0$. □

Lemma 2.5. *The linear space (γ, ρ) is complete.*

Proof. Let $\{\Gamma_{X_n}\}$ be a Cauchy sequence in γ. So, $\|\Gamma_{X_n} \ominus \Gamma_{X_m}\|_\alpha = \|\Gamma_{X_n - X_m}\|_\alpha \to 0$ as $n, m \to \infty$. Then, by Proposition 2.4, X_n is Cauchy in probability and then there exists an $\mathcal{X}$-valued random vector X such that X_n converges to X in probability. Proposition 2.2 implies that X is $S\alpha S$ random vector in $\mathcal{M}$. By using Proposition 2.4, Γ_{X_n} converges to Γ_X in γ and $\Gamma_X \in \gamma$. □

3. **The Radon Nikodym property for γ**

As observed in Section 2, $(\gamma, \|.\|_\alpha)$, $1 < \alpha < 2$, is a Banach space. Here, we will prove that it is isometrically isomorphic to a certain $\mathcal{L}^\alpha$ space, and consequently apply Radon Nikodym Theorem to certain vector measures with values in γ.

By using Proposition 2.2 and an argument similar to the one given in Soltani (1994) [Theorems 3.1 and 3.2], the following theorem can be proved.

Theorem 3.1. *Let Φ be an $\mathcal{X}$-valued $S\alpha S$ random measure with independent increments on measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and the class $\mathcal{M}$ be as in (2.2). Then, there is a unique bimeasure π on $\mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(S)$ such that*

$$\pi(A, .) = \Gamma_{\Phi(A)}(.), \qquad \text{for every } A \in \mathcal{B}(\mathbb{R}^k), \tag{3.1}$$

where $\Gamma_{\Phi(A)}$ is the spectral measure of $\Phi(A)$. Moreover,

(i) For every $Y \in \mathcal{M}$, there is a real valued function $g \in \mathcal{L}^\alpha(\pi(., S))$ such that the spectral measure of Y is given by

$$\Gamma_Y(B) = \int_{\mathbb{R}^k} |g(t)|^\alpha \, \pi(dt, B), \tag{3.2}$$

for every $B \in \mathcal{B}(S)$.

(ii) If g is a real valued Borel function in $\mathcal{L}^\alpha(\pi(., S))$, then there is a unique $\mathcal{X}$-valued $S\alpha S$ random vector Y in $\mathcal{M}$ for which its spectral measure is given by (3.2).

Let us apply Theorem 3.1 to establish an isomorphism between the space $(\gamma, \|.\|_\alpha)$ and $\mathcal{L}^\alpha(\pi(., S))$. According to parts 1 and 2 of this theorem, for every $g \in \mathcal{L}^\alpha(\pi(., S))$, the stochastic integral $Y = \int_{\mathbb{R}^k} g(x) d\Phi(x)$ is well defined, in the weak sense, and defines an $\mathcal{X}$-valued $S\alpha S$ random vector. Clearly, $Y \in \mathcal{M}$ and then $\Gamma_Y \in \gamma$. Now, let us set $T(\Gamma_Y) = g$. We have,

$$\begin{aligned}(\int_{\mathbb{R}^k} |g(t)|^\alpha \, \pi(dt, S))^{1/\alpha} &= (\Gamma_Y(S))^{1/\alpha} \\ &= \|\Gamma_Y\|_\alpha \,.\end{aligned}$$

Hence, T is an isometric isomorphism of γ into $\mathcal{L}^\alpha(\pi(., S))$.

Theorem 3.2. *For $1 < \alpha < 2$, let Φ be an $\mathcal{X}-$valued $S\alpha S$ random measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with independent increments and also let γ be the space as in (2.3). Then, γ has the Radon Nikodym property.*

Proof. If $\pi(., .)$ is defined as in (3.1), then $\pi(\mathbb{R}^k, S) = \Gamma_{\Phi(\mathbb{R}^k)}(S) < \infty$. Now, since for $1 < \alpha$, $\mathcal{L}^\alpha(\pi(., S))$ has the Radon Nikodym property [Diestel and Uhl(1977), page 140, Theorem 1], and $\mathcal{L}^\alpha(\pi(., S))$ and γ are isometrically isomorphic, then γ has the Radon Nikodym property.

4. **The main result**

Let $\psi(A) = \Gamma_{\Phi(A)}$, $A \in \mathcal{B}(\mathbb{R}^k)$. Since $(\Phi(\bigcup_{j=1}^{\infty} A_j) - \sum_{j=1}^{n} \Phi(A_j)) \to 0$ in probability for any given sequence of disjoint sets $A_1, A_2, \ldots$,

$$\left\| \Gamma_{\Phi(\cup_{j=1}^\infty A_j) - \sum_{j=1}^n \Phi(A_j)} \right\|_\alpha \to 0 \qquad \text{as } n \to \infty,$$

giving that

$$\psi(\cup_{j=1}^\infty A_j) = \Gamma_{\Phi(\cup_{j=1}^\infty A_j)} = \Gamma_{\sum_{j=1}^\infty \Phi(A_j)} = \oplus_j \psi(A_j),$$

in $(\gamma, \|.\|_\alpha)$. Therefore, ψ is a vector measure on $\mathcal{B}(\mathbb{R}^k)$ with values in γ.

Lemma 4.1. *The vector measure ψ possesses the following properties:*

(I) There is a finite positive measure μ on $\mathcal{B}(\mathbb{R}^k)$ such that ψ is μ-continuous (i.e., $\|\psi(A_n)\|_\alpha \to 0$ as $\mu(A_n) \to 0$).

(II) ψ is of bounded variation.

Proof. For (I), when $1 \leq \alpha < 2$, let $\mu(A) = \|\psi(A)\|_\alpha^\alpha$, $(A \in \mathcal{B}(\mathbb{R}^k))$. Note that if X and Y are independent, then $\Gamma_{X+Y} = \Gamma_X + \Gamma_Y$; therefore, since Φ is independently scattered, it follows that for disjoint sets A_1, A_2,

$$\begin{aligned} \mu(A_1 \cup A_2) &= \left\|\Gamma_{\Phi(A_1 \cup A_2)}\right\|_\alpha^\alpha = \left\|\Gamma_{\Phi(A_1)} + \Gamma_{\Phi(A_2)}\right\|_\alpha^\alpha \\ &= \left\|\Gamma_{\Phi(A_1)}\right\|_\alpha^\alpha + \left\|\Gamma_{\Phi(A_2)}\right\|_\alpha^\alpha, \end{aligned}$$

and thus μ is finitely additive and $\mu(\cup_{i=n+1}^\infty A_i) = \left\|\Gamma_{\Phi(\cup_{i=n+1}^\infty A_i)}\right\|_\alpha \to 0$ as $n \to \infty$. Therefore, $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^n \mu(A_i) + \mu(\cup_{i=n+1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$. The same reasoning also applies to $0 < \alpha < 1$, with $\mu(A) = \|\psi(A)\|_\alpha$. It also easily follows that Ψ is μ-continuous. Part (II) follows from the fact that μ is a finite measure, [Proposition 11 in Diestel and Uhl(1977)]. $\square$

Our main result is the following theorem.

Theorem 4.2. *Let $\mathcal{X}$ be a real separable Sazonov space with Sazonov topology τ and Φ be an $\mathcal{X}$-valued $S\alpha S$ random measure, $1 < \alpha < 2$, with independent increments on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Then, the law of Φ is uniquely specified by a control pair (μ, k), through*

$$-\log \phi_{\sum_{i=1}^n a_i \Phi(A_i)}(t) = \sum_{i=1}^n |a_i|^\alpha \int_{A_i} k(t,y)\mu(dy), \tag{4.1}$$

$t \in \mathcal{X}', y \in \mathbb{R}^k$, $A_i \in \mathcal{B}(\mathbb{R}^k), a_i \in \mathbb{R}$, where μ is a positive measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and $k : \mathcal{X}' \times \mathbb{R}^k \longmapsto \mathbb{R}^+$ is a measurable mapping with the following properties:

(I) For every $t \in \mathcal{X}'$, $k(t,.)$ is integrable with respect to μ.

(II) For y, μ a.e., $k(.,y)$ is of negative type and homogeneous, that is,

$$\sum_{i,j=1}^N c_i c_j k(t_i - t_j, y) \leq 0,$$

for every integer N and every choice of real numbers $c_1, \ldots, c_N$ subject to $\sum_{j=1}^N c_j = 0$, and $t_1, \ldots, t_N \in \mathcal{X}'$. Moreover,

$$k(ct, y) = |c|^\alpha k(t,y),$$

for every scalar c and every $t \in \mathcal{X}'$.

(III) For y, μ a.e. , $k(.,y)$ is τ-continuous.
Conversely for a measurable mapping $k : \mathcal{X}' \times \mathbb{R}^k \longmapsto \mathbb{R}^+$ having the properties (I), (II) and (III), there is an $\mathcal{X}$-valued $S\alpha S$ random measure Φ with independent increments on a measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ which satisfies (4.1).

Proof. The Radon Nikodym Theorem for the vector measure ψ with respect to μ implies that there is a unique γ-valued μ Bochner integrable function $p(y) =: p(y, ds)$ on $\mathbb{R}^k$, for which $\psi(dy) = p(y)\mu(dy)$, [Diestel (1977), pages 47 and 59]. This will allow approximating $\psi(A)$ by a sequence $\psi_N = \sum_{j=1}^{N} 1_{E_j}(y)p(y_j)\mu(E_j)$ in $(\gamma, \|.\|_\alpha)$, where $E_1, ..., E_N$ is a finite partition of $\mathcal{B}(\mathbb{R}^k)$-sets for A. But if a sequence $\{\Gamma_{X_n}\}$ converges to Γ in $(\gamma, \|.\|_\alpha)$, then X_n will converge weakly to X. Consequently, for each bounded function q,

$$\begin{aligned}\int_S q(s)\psi(A)(ds) &= \lim_{N\to\infty} \int_S q(s)\psi_N(ds) \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} \{\int_S q(s)p(y_j, ds)\}1_{E_j}(y)\mu(E_j) \\ &= \int_A \{\int_S q(s)p(y, ds)\}\mu(dy).\end{aligned}$$

Now, since

$$\begin{aligned}-\log \phi_{\Phi(A)}(t) &= \int_S |t(s)|^\alpha \Gamma_{\Phi(A)}(ds) \\ &= \int_S |t(s)|^\alpha \psi(A)(ds),\end{aligned}$$

we obtain:

$$-\log \phi_{\Phi(A)}(t) = \int_A \{\int_S |t(s)|^\alpha p(y, ds)\}\mu(dy).$$

Let

$$k(t, y) = \int_S |t(s)|^\alpha p(y, ds).$$

Then, (4.1) will evidently be satisfied. What remains to prove is that $k(.,.)$ possesses the properties (I), (II) and (III). The property (I) follows from the fact that Φ is defined on $\mathcal{B}(\mathbb{R}^k)$. Indeed, $\Phi(\mathbb{R}^k)$ is an $\mathcal{X}$-valued $S\alpha S$ random vector. For the property (II), the function $f(x) = |x|^\alpha$ is of negative type on $(-\infty, +\infty)$ [Schoenberg (1938)]. Therefore, for $t_1, t_2, ..., t_N \in \mathcal{X}'$ and $c_1, c_2, ..., c_N$, given real numbers, such that $\sum\limits_{j=1}^{N} c_j = 0$, and every $s \in S$, giving that

$$\sum_{i,j=1}^{N} c_i c_j \left|t_i(s) - t_j(s)\right|^\alpha \leq 0,$$

it follows that,

$$\sum_{i,j=1}^{N} c_i c_j k(t_i - t_j, y) \leq 0.$$

Also,

$$k(ct, y) = \int \left|ct(s)\right|^\alpha p(y, ds) = |c|^\alpha k(t, y).$$

For (III), we note that it will be sufficient to show $k(., y)$ is τ-continuous at zero. But since $p(y, S) < \infty$, for every $y \in \mathbb{R}^k$, it follows from the Lyapounov′s inequality that

$$\begin{aligned} k(t, y) &= (p(y, S) \int\limits_S |t(s)|^\alpha \frac{p(y, ds)}{p(y, S)} \\ &\leq (p(y, S)[\int\limits_S (t(s))^2 \frac{p(y, ds)}{p(y, S)}]^{\alpha/2}. \end{aligned}$$

According to the Levy-Khinchin Spectral Representation Theorem, $\exp\{\int\limits_S (t(s))^2 p(y, ds)\}$ is a Gaussian characteristic functional. Therefore, it is $\tau-$continuous and consequently $k(., y)$ is τ-continuous everywhere on $\mathcal{X}'$.

For the converse, assume (k, μ) is given and k satisfies properties (I), (II) and (III). Since $k(0, y) = 0$, and $k(., y)$ is τ-continuous on $\mathcal{X}'$, it follows that $\int_A k(0, y)\mu(dy) = 0$ and $\int_A k(., y)\mu(dy)$ is τ-continuous on $\mathcal{X}'$ for every $A \in \mathcal{B}(\mathbb{R}^k)$. However, the fact that $\int_A k(., y)\mu(dy)$ is positive definite on $\mathcal{X}'$ follows immediately from [Gelefand, page 279, Theorem 4]. Therefore, $\phi_A(.) = \exp\{-\int_A k(., y)\mu(dy)\}$ is a characteristic functional. Let $\Phi(A)$ be an $\mathcal{X}$-valued random vector with characteristic functional

$\phi_A(.)$. It follows by a classical argument that $\{\Phi(A), A \in \mathcal{B}(\mathbb{R}^k)\}$ induces an $\mathcal{X}$- valued random measure on $\mathcal{B}(\mathbb{R}^k)$. It is plain to verify that $\Phi(A)$ is $S\alpha S$. Indeed, $\phi_A(t)$ is real and for $n > 0$,

$$[\phi_A(t)]^n = \exp\{-n \int_A k(t,y)\mu(dy)\} = \phi(n^{1/\alpha}t).$$

The proof is now complete. □

Remark 4.3. Each Hilbert space with inner-product $< .,. >$ is a Sazonov space and the Sazonov topology on $\mathcal{H}'(=\mathcal{H})$ is the locally convex topology generated by the semi-norms p with $p(x) =< Sx, x >^{1/2}$, where $S : \mathcal{H} \to \mathcal{H}$ varies over the symmetric positive trace class operators on $\mathcal{H}$. Therefore, the theorem is true in this case.

Example 1. For real separable Banach space $\mathcal{X}$, let $x_i \in \mathcal{X}$, $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} \|x_i\|^\alpha < \infty$. For $0 < \alpha < 2$, let $\{\theta_i^{(\alpha)}\}$ denote a sequence of independent $S\alpha S$ random variables such that for every $x \in \mathbb{R}$,

$$\phi_{\theta_i^{(\alpha)}}(x) = e^{-|x|^\alpha}.$$

Assume $\sum_{i=1}^{\infty} \theta_i^{(\alpha)} x_i$ exists almost everywhere. Define $\Phi(A) := \sum_{i\in A} \theta_i^{(\alpha)} x_i$. Then, Φ is an $S\alpha S$ random measure and

$$\phi_{\Phi(A)}(t) = e^{-\sum_{i\in A}|t(x_i)|^\alpha} \qquad t \in \mathcal{X}',$$

and

$$\Gamma_{\Phi(A)}(ds) = 1/2 \sum_{i\in A} \|x_i\|^\alpha \left(\delta_{x_i/\|x_i\|}(ds) + \delta_{-x_i/\|x_i\|}(ds)\right).$$

It follows that $\mu(A) = \sum_{i\in A} \|x_i\|^\alpha$ and $p(i, ds) = 1/2(\delta_{x_i/\|x_i\|}(ds)$ $+\delta_{-x_i/\|x_i\|}(ds))$, where δ_a is the direct measure concentrated on a. Therefore,

$$k(t,i) = \int_S |t(s)|^\alpha p(i,ds) = |t(x_i)/\|x_i\||^\alpha, \qquad t \in \mathcal{X}'.$$

With the assumption an this Banach space, it will become a Sazanov space and its topology is a topology by the following neighborhood basis of zero:

$\{\{t \in \mathcal{X}'; -\log \phi_X(t) \le 1\};$ for each $\mathcal{X}$-valued α stable random vector $X\}$.

[Linde (1983), page 176].

Example 2. Let Φ be a Levy $S\alpha S$ random measure. Then, $\Gamma_{\Phi(A)}(ds) = \lambda(A)\upsilon(ds)$, where λ is the Lebesgue measure and υ is a symmetric probability measure on S, which is not supported by any subspace, $\upsilon\{s \in S; \beta(s) = 0\} < 1$ for all $\beta \in S'$. Therefore,

$$\phi_{\Phi(A)}(t) = \exp\{-\lambda(A) \int_S |t(s)|^\alpha \, d\upsilon(s)\}, \qquad t \in \mathcal{X}',\ A \in \mathcal{B}(\mathbb{R}).$$

Thus, in Theorem 4.2, μ is the Lebesgue measure and

$$k(t,y) = \int_S |t(s)|^\alpha \, d\upsilon(s) \quad t \in \mathcal{X}',\ y \in \mathbb{R}.$$

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