Bulletin of the Iranian Mathematical Society Vol. 36 *No. 1* (*2010*), *pp 41-53.*

ON JORDAN GENERALIZED k-DERIVATIONS OF SEMIPRIME Γ_N-RINGS

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Communicated by Fariborz Azarpanah

Abstract. We know that every Jordan generalized k-derivation of a Γ-ring is not a generalized k-derivation of the same, in general. Here, we develop a number of lemmas relating to these derivations of certain Γ-rings and we show that under some conditions every Jordan generalized k-derivation of a 2-torsion free semiprime Γ_N-ring is a generalized k-derivation.

1. **Introduction**

As an extensive generalization of the concept of a classical ring, the notion of a gamma ring was introduced. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of gamma rings have attracted a wider attention as an emerging field of research to the modern algebraists to enrich the world of algebra. A number of prominent mathematicians have worked out on this interesting area of research to determine many basic properties of gamma rings and have extended numerous significant results in this context in the last few decades. There is a large number

MSC(2000): Primary: 16N60; Secondary: 16W25, 16U80.
Keywords: Derivation, generalized k-derivation, Jordan generalized k-derivation, gamma ring, semiprime gamma ring.
Received: 15 April 2008, Accepted: 25 January 2009.
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of researchers throughout the world who are recently engaged to execute more productive and creative results of gamma ring theory.

The notion of a Γ-ring was first introduced by N. Nobusawa [4] (which is presently known as a Γ_N-ring) and afterwards it was generalized by W. E. Barnes [1] in a broad sense (that served us now-a-days to call it as a Γ-ring generally). As an immediate consequence, this generalization states that *every* Γ_N*-ring* is a Γ-ring, but the converse is not necessarily true in general. They obtained many important basic properties of Γ-rings in various ways and determined some more remarkable characteristics of Γ-rings. Later, many mathematicians classified Γ-rings to develop a lot of significant results, one of from which the base of this article has been emanated. We start with the following necessary introductory definitions and examples.

Let M and Γ be two additive abelian groups. If there is a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $M \times \Gamma \times M \to M$ which satisfies the conditions (a) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ and (b) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ*-ring* in the sense of Barnes [1].

For example, if R is an ordinary associative ring, U is any ideal of R, and Z is the ring of integers, then R is a Γ-ring with $\Gamma = U$ or $\Gamma = Z$.

In addition to the definition given above, if there exists another mapping $(\alpha, a, \beta) \mapsto \alpha a\beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the conditions (a*) $(\alpha+\beta)a\gamma = \alpha a\gamma+\beta a\gamma$, $\alpha(a+b)\beta = \alpha a\beta+\alpha b\beta$, $\alpha a(\beta+\gamma) = \alpha a\beta+\alpha a\gamma$, (b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and (c*) $a\alpha b = 0$ implies $\alpha = 0$ for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$, then M is called a Γ-ring in the sense of Nobusawa [4], or simply, a Nobusawa Γ-ring, and then we express it by saying that M is a Γ_N*-ring.*

As an example, if R is an ordinary associative ring with the unity 1, then R is a Γ_N-ring if we consider $\Gamma = R$.

Remark 1.1. M is a Γ_N-ring implies that Γ is an M-ring.

Let M be a Γ-ring. Then, we have the following *definitions*:

(i) M is called *2-torsion free* if $2a = 0$ implies $a = 0$ for all $a \in M$.

(ii) An element $x \in M$ is said to be a *nilpotent element* if $(x\gamma)^n x = 0$, for all $\gamma \in \Gamma$, is satisfied for some positive integer n.

(iii) M is called *semiprime* if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies $a = 0$.
(iv) The set $Z(M) = \{a \in M : a\alpha m = m\alpha a$ for all $\alpha \in \Gamma$ and $m \in M\}$ is called the *centre*.
(v) M is said to be a *commutative* Γ-ring if $a\alpha b = b\alpha a$ holds for all $a, b \in M$ and $\alpha \in \Gamma$.

Note that the notions of derivation and Jordan derivation of a Γ-ring have been introduced by M. Sapanci and A. Nakajima in [5], whereas the concept of the k-derivation of a Γ-ring has been introduced by H. Kandamar [3]. Afterwards, the concept of Jordan generalized derivation of a Γ-ring has been developed by Y. Ceven and M. A. Ozturk in [2], where they have proved that under some constraints every Jordan generalized derivation of a 2-torsion free Γ-ring is a generalized derivation of the same. Here we introduce the concepts of generalized k-derivation and Jordan generalized k-derivation of a Γ-ring to extend the analogous result to the previous one for a 2-torsion free semiprime Γ_N-ring under some suitable conditions. All these concepts are mentioned in the following.

Let M be a Γ-ring. If $d : M \to M$ and $k : \Gamma \to \Gamma$ are two additive mappings such that $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$, then d is said to be a *derivation* of M. And, if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$ for all $a \in M$ and $\alpha \in \Gamma$, then d is called a *Jordan derivation* of M.

Also, for all $a, b \in M$ and $\alpha \in \Gamma$, if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ is satisfied, then d is said to be a *k-derivation* of M. And, if $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$, then d is called a *Jordan k-derivation* of M.

Besides, an additive map $f : M \to M$ is said to be a *generalized derivation* of M if there exists a derivation $d : M \to M$ such that $f(a\alpha b) = f(a)\alpha b + a\alpha d(b)$ is satisfied for all $a, b \in M$ and $\alpha \in \Gamma$. And, f is said to be a *Jordan generalized derivation* of M if there exists a Jordan derivation $d : M \to M$ such that $f(a\alpha a) = f(a)\alpha a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$.

Moreover, an additive map $f : M \to M$ is called a *generalized k-derivation* of M if there exists a k-derivation $d : M \to M$ such that $f(a\alpha b) = f(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$. And, f is called a *Jordan generalized k-derivation* of M if there exists a Jordan

k-derivation $d : M \to M$ such that $f(a\alpha a) = f(a)\alpha a + ak(\alpha)a + a\alpha d(a)$ for all $a \in M$ and $\alpha \in \Gamma$.

Example 1.2. Let M be a Γ_N-ring and f be a generalized k-derivation of M. Then there exists a k-derivation $d : M \to M$ such that $d(x\alpha y) = d(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ and $f(x\alpha y) = f(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Let $M_1 = M \times M$ and $\Gamma_1 = \Gamma \times \Gamma$. Define the operations of addition and multiplication on M_1 and Γ_1 by:

$$(x, y) + (z, w) = (x + z, y + w),\ (x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w), \text{ and}$$
$$(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta),\ (\alpha, \beta)(x, y)(\gamma, \delta) = (\alpha x\gamma, \beta y\delta),$$

for every $x, y, z, w \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$, respectively. Then, M_1 is obviously a Nobusawa Γ_1-ring under these operations.

Let $f_1 : M_1 \to M_1$, $d_1 : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the additive maps defined by $f_1((x, y)) = (f(x), f(y))$, $d_1((x, y)) = (d(x), d(y))$ and $k_1((\alpha, \beta)) = (k(\alpha), k(\beta))$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, respectively. Saying $(x, y) = a \in M_1$, $(z, w) = b \in M_1$ and $(\alpha, \beta) = \gamma \in \Gamma_1$ for any $x, y, z, w \in M$ and $\alpha, \beta \in \Gamma$, we have,

$$\begin{aligned}
d_1(a\gamma b) &= d_1((x, y)(\alpha, \beta)(z, w)) = d_1((x\alpha z, y\beta w)) = (d(x\alpha z), d(y\beta w)) \\
&= (d(x)\alpha z + xk(\alpha)z + x\alpha d(z), d(y)\beta w + yk(\beta)w + y\beta d(w)) \\
&= (d(x)\alpha z, d(y)\beta w) + (xk(\alpha)z, yk(\beta)w) + (x\alpha d(z), y\beta d(w)) \\
&= (d(x), d(y))(\alpha, \beta)(z, w) + (x, y)(k(\alpha), k(\beta))(z, w) \\
&\quad + (x, y)(\alpha, \beta)(d(z), d(w)) \\
&= d_1(a)\gamma b + ak_1(\gamma)b + a\gamma d_1(b),
\end{aligned}$$

which implies that d_1 is a k_1-derivation of M_1. Furthermore, we have,

$$\begin{aligned}
f_1(a\gamma b) &= f_1((x, y)(\alpha, \beta)(z, w)) = f_1((x\alpha z, y\beta w)) = (f(x\alpha z), f(y\beta w)) \\
&= (f(x)\alpha z + xk(\alpha)z + x\alpha d(z), f(y)\beta w + yk(\beta)w + y\beta d(w)) \\
&= (f(x)\alpha z, f(y)\beta w) + (xk(\alpha)z, yk(\beta)w) + (x\alpha d(z), y\beta d(w)) \\
&= (f(x), f(y))(\alpha, \beta)(z, w) + (x, y)(k(\alpha), k(\beta))(z, w) \\
&\quad + (x, y)(\alpha, \beta)(d(z), d(w)) \\
&= f_1(a)\gamma b + ak_1(\gamma)b + a\gamma d_1(b).
\end{aligned}$$

Therefore, it follows that f_1 is a generalized k_1-derivation of M_1 associated with the k_1-derivation d_1 of M_1.

Example 1.3. Let M be a Γ_N-ring and let f be a generalized k-derivation of M. Then, there exists a k-derivation $d : M \to M$ such that $d(x\alpha y) = d(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ and $f(x\alpha y) = f(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Consider $N = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define the operations of addition and multiplication on N and Γ_1 by $(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2)$, $(x_1, x_1)(\alpha, \alpha)(x_2, x_2) = (x_1\alpha x_2, x_1\alpha x_2)$ and $(\alpha_1, \alpha_1) + (\alpha_2, \alpha_2) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$, $(\alpha_1, \alpha_1)(x, x)(\alpha_2, \alpha_2) = (\alpha_1 x\alpha_2, \alpha_1 x\alpha_2)$ for all $x, x_1, x_2 \in M$ and $\alpha, \alpha_1, \alpha_2 \in \Gamma$, respectively. Then, it is clear that N is a Nobusawa Γ_1-ring under these operations.

Now, let $f_1 : N \to N$, $d_1 : N \to N$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the additive maps defined by $f_1((x, x)) = (f(x), f(x))$, $d_1((x, x)) = (d(x), d(x))$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$ for all $x \in M$ and $\alpha \in \Gamma$, respectively. If we say that $(x, x) = a \in N$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$ for any $x \in M$ and $\alpha \in \Gamma$, then we have,

$$\begin{aligned} d_1(a\gamma a) &= d_1((x, x)(\alpha, \alpha)(x, x)) = d_1((x\alpha x, x\alpha x)) = (d(x\alpha x), d(x\alpha x)) \\ &= (d(x)\alpha x + xk(\alpha)x + x\alpha d(x), d(x)\alpha x + xk(\alpha)x + x\alpha d(x)) \\ &= (d(x)\alpha x, d(x)\alpha x) + (xk(\alpha)x, xk(\alpha)x) + (x\alpha d(x), x\alpha d(x)) \\ &= (d(x), d(x))(\alpha, \alpha)(x, x) + (x, x)(k(\alpha), k(\alpha))(x, x) \\ &\quad + (x, x)(\alpha, \alpha)(d(x), d(x)) \\ &= d_1(a)\gamma a + ak_1(\gamma)a + a\gamma d_1(a). \end{aligned}$$

Thus, d_1 is a Jordan k_1-derivation of N. Moreover, we have,

$$\begin{aligned} f_1(a\gamma a) &= f_1((x, x)(\alpha, \alpha)(x, x)) = f_1((x\alpha x, x\alpha x)) = (f(x\alpha x), f(x\alpha x)) \\ &= (f(x)\alpha x + xk(\alpha)x + x\alpha d(x), f(x)\alpha x + xk(\alpha)x + x\alpha d(x)) \\ &= (f(x)\alpha x, f(x)\alpha x) + (xk(\alpha)x, xk(\alpha)x) + (x\alpha d(x), x\alpha d(x)) \\ &= (f(x), f(x))(\alpha, \alpha)(x, x) + (x, x)(k(\alpha), k(\alpha))(x, x) \\ &\quad + (x, x)(\alpha, \alpha)(d(x), d(x)) \\ &= f_1(a)\gamma a + ak_1(\gamma)a + a\gamma d_1(a). \end{aligned}$$

Hence, f_1 is a Jordan generalized k_1-derivation of N associated with the Jordan k_1-derivation d_1 of N. Obviously, f_1 is not a generalized k_1-derivation of N.

From the definitions and examples given above, it is clear that every k-derivation of a Γ-ring M is a Jordan k-derivation of M. But, the converse is not true, in general. Here, we show that under some conditions every Jordan generalized k-derivation of a 2-torsion free semiprime Γ_N-ring M is a generalized k-derivation of M. For this to happen, we develop some useful results as follows.

2. Some preliminary results

Lemma 2.1. *Let M be a Γ_N-ring and let d be a Jordan k-derivation of M. Then, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha d(b) + b\alpha d(a)$;

(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha bk(\beta)a + a\beta bk(\alpha)a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.

In particular, if M is 2-torsion free, then

(iii) $d(a\alpha b\alpha a) = d(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha d(b)\alpha a + a\alpha bk(\alpha)a + a\alpha b\alpha d(a)$;

(iv) $d(a\alpha b\alpha c + c\alpha b\alpha a) = d(a)\alpha b\alpha c + d(c)\alpha b\alpha a + ak(\alpha)b\alpha c + ck(\alpha)b\alpha a + a\alpha d(b)\alpha c + c\alpha d(b)\alpha a + a\alpha bk(\alpha)c + c\alpha bk(\alpha)a + a\alpha b\alpha d(c) + c\alpha b\alpha d(a)$.

Proof. Computing $d((a+b)\alpha(a+b))$ and cancelling the like terms from both sides, we obtain *(i)*. Then replacing $a\beta b + b\beta a$ for b in *(i)*, we get *(ii)*. Since M is 2-torsion free, we obtain *(iii)* by replacing α for β in *(ii)*, and then we obtain *(iv)* by replacing $a + c$ for a in *(iii)*. □

Lemma 2.2. *Let M be a Γ_N-ring and let f be a Jordan generalized k-derivation of M. Then, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:*

(i) $f(a\alpha b + b\alpha a) = f(a)\alpha b + f(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha d(b) + b\alpha d(a)$;

(ii) $f(a\alpha b\beta a + a\beta b\alpha a) = f(a)\alpha b\beta a + f(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha bk(\beta)a + a\beta bk(\alpha)a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.

In particular, if M is 2-torsion free, then

(iii) $f(a\alpha b\alpha a) = f(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha d(b)\alpha a + a\alpha bk(\alpha)a + a\alpha b\alpha d(a)$;

(iv) $f(a\alpha b\alpha c + c\alpha b\alpha a) = f(a)\alpha b\alpha c + f(c)\alpha b\alpha a + ak(\alpha)b\alpha c + ck(\alpha)b\alpha a + a\alpha d(b)\alpha c + c\alpha d(b)\alpha a + a\alpha bk(\alpha)c + c\alpha bk(\alpha)a + a\alpha b\alpha d(c) + c\alpha b\alpha d(a)$.

Proof. Compute $f((a+b)\alpha(a+b))$ and cancel the like terms from both sides to obtain *(i)*. Then replace $a\beta b + b\beta a$ for b in *(i)* and use Lemma 2.1*(i)* to get *(ii)*. Since M is 2-torsion free, *(iii)* is easily obtained by replacing α for β in *(ii)*, and then *(iv)* is obtained by replacing $a+c$ for a in *(iii)*. □

Lemma 2.3. *Let f be a Jordan generalized k-derivation of a 2-torsion free Γ_N-ring M. Then, for all $b \in M$ and $\beta \in \Gamma$,*

$$k(\beta b\beta) = k(\beta)b\beta + \beta d(b)\beta + \beta bk(\beta).$$

Proof. For all $a \in M$ and $\alpha \in \Gamma$, we have $f(a\alpha a) = f(a)\alpha a + ak(\alpha)a + a\alpha d(a)$. Let $b \in M$ and $\beta \in \Gamma$. Then, putting $\beta b\beta$ for α, we get $f(a\beta b\beta a) = f(a)\beta b\beta a + ak(\beta b\beta)a + a\beta b\beta d(a)$. Expanding the LHS by Lemma 2.2*(iii)*, we obtain $a(k(\beta b\beta) - k(\beta)b\beta - \beta d(b)\beta - \beta bk(\beta))a = 0$. Hence, applying the Nobusawa condition (c*) of the definition of Γ_N-ring, we get the proof. □

Lemma 2.4. *If f is both a Jordan generalized k_1-derivation and a Jordan generalized k_2-derivation of a 2-torsion free Γ_N-ring M, then $k_1 = k_2$.*

Proof. Obvious. □

Remark 2.5. If f is a Jordan generalized k-derivation of a 2-torsion free Γ_N-ring M, then k is uniquely determined.

Definition 2.6. Let M be a Γ-ring. Then, for $a, b \in M$ and $\alpha \in \Gamma$, we define $[a, b]_\alpha = a\alpha b - b\alpha a$, known as the commutator of a and b with respect to α.

Lemma 2.7. *If M is a Γ-ring, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*

(i) $[a, b]_\alpha + [b, a]_\alpha = 0$;
(ii) $[a + b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha$;
(iii) $[a, b + c]_\alpha = [a, b]_\alpha + [a, c]_\alpha$;
(iv) $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta$.

Proof. Obvious. □

Remark 2.8. A Γ-ring M is commutative if and only if $[a,b]_\alpha = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.9. Let d be a Jordan k-derivation of a Γ_N-ring M. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define,

$$G_\alpha(a,b) = d(a\alpha b) - d(a)\alpha b - ak(\alpha)b - a\alpha d(b).$$

Lemma 2.10. *If d is a Jordan k-derivation of a Γ_N-ring M, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*

(i) $G_\alpha(a,b) + G_\alpha(b,a) = 0$;
(ii) $G_\alpha(a+b,c) = G_\alpha(a,c) + G_\alpha(b,c)$;
(iii) $G_\alpha(a,b+c) = G_\alpha(a,b) + G_\alpha(a,c)$;
(iv) $G_{\alpha+\beta}(a,b) = G_\alpha(a,b) + G_\beta(a,b)$.

Proof. Obvious.

Remark 2.11. It is clear that d is a k-derivation of a Γ_N-ring M if and only if $G_\alpha(a,b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 2.12. *Let d be a Jordan k-derivation of a 2-torsion free Γ_N-ring M. Then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,*

$$G_\alpha(a,b)\beta m\beta[a,b]_\alpha + [a,b]_\alpha\beta m\beta G_\alpha(a,b) = 0.$$

Proof. First, we compute $d(a\alpha(b\beta m\beta b)\alpha a) + b\alpha(a\beta m\beta a)\alpha b)$ using Lemma 2.1*(iii)* and then compute, $d((a\alpha b)\beta m\beta(b\alpha a) + (b\alpha a)\beta m\beta(a\alpha b))$ using Lemma 2.1*(iv)*. Since these two are equal, cancelling the like terms from both sides of this equality and then rearranging them using Lemma 2.10*(i)*, we get the proof. □

Lemma 2.13. *Let M be a 2-torsion free semiprime Γ_N-ring and suppose that $a, b \in M$. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.*

Proof. Let m and m' be two arbitrary elements of M. Then, from the hypothesis we have, $a\Gamma m\Gamma b = -b\Gamma m\Gamma a$, and consequently we get $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) = -(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)$. So, this implies that $2((a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)) = 0$. Since M is 2-torsion free, we have $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) = 0$. By the semiprimeness of M, we get $a\Gamma m\Gamma b = 0$ for all $m \in M$. Hence, $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$ for all $m \in M$.

Now, applying Lemma 2.13 to the result of Lemma 2.12, we obtain the following result.

Corollary 2.14. *If M is a 2-torsion free semiprime Γ_N-ring, then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,*

(i) $G_\alpha(a, b)\beta m\beta[a, b]_\alpha = 0$;
(ii) $[a, b]_\alpha\beta m\beta G_\alpha(a, b) = 0$.

Definition 2.15. Let f be a Jordan generalized k-derivation of a Γ_N-ring M. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define,

$$F_\alpha(a, b) = f(a\alpha b) - f(a)\alpha b - ak(\alpha)b - a\alpha d(b).$$

Lemma 2.16. *If f is a Jordan generalized k-derivation of a Γ_N-ring M, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*

(i) $F_\alpha(a, b) + F_\alpha(b, a) = 0$;
(ii) $F_\alpha(a + b, c) = F_\alpha(a, c) + F_\alpha(b, c)$;
(iii) $F_\alpha(a, b + c) = F_\alpha(a, b) + F_\alpha(a, c)$;
(iv) $F_{\alpha+\beta}(a, b) = F_\alpha(a, b) + F_\beta(a, b)$.

Proof. Obvious. □

Remark 2.17. *f is a generalized k-derivation of a Γ_N-ring M if and only if $F_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.*

Lemma 2.18. *Let f be a Jordan generalized k-derivation of a 2-torsion free semiprime Γ_N-ring M. If $a, b \in M$ and $\alpha, \beta \in \Gamma$, then for any $m \in M$,*

(i) $F_\alpha(a, b)\beta m\beta[a, b]_\alpha = 0$;
(ii) $[a, b]_\alpha\beta m\beta F_\alpha(a, b) = 0$.

Proof. *(i)* Consider $f(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$ and look what we have here. Compute $f(a\alpha(b\beta m\beta b)\alpha a) + b\alpha(a\beta m\beta a)\alpha b)$ by using Lemma 2.2*(iii)* along with the application of Lemma 2.1*(iii)*. Also, compute

$f((a\alpha b)\beta m\beta(b\alpha a)+(b\alpha a)\beta m\beta(a\alpha b))$ using Lemma 2.2*(iv)*. After equating them, cancel the like terms from both sides to obtain $F_\alpha(a,b)\beta m\beta[a,b]_\alpha + [a,b]_\alpha\beta m\beta G_\alpha(a,b) = 0$ (by rearranging the terms using Lemma 2.16*(i)*). Hence, we get $F_\alpha(a,b)\beta m\beta[a,b]_\alpha = 0$ [by using Corollary 2.14*(ii)*].

(ii) Using *(i)*, we have $[a,b]_\alpha\beta m\beta F_\alpha(a,b)\beta m\beta[a,b]_\alpha\beta m\beta F_\alpha(a,b) = 0$. Since M is semiprime, we obtain $[a,b]_\alpha\beta m\beta F_\alpha(a,b) = 0$. □

Lemma 2.19. *If M is a 2-torsion free semiprime Γ_N-ring, then for all $a,b,u,v,m \in M$ and $\alpha,\beta \in \Gamma$,*

(i) $F_\alpha(a,b)\beta m\beta[u,v]_\alpha = 0$;
(ii) $[u,v]_\alpha\beta m\beta F_\alpha(a,b) = 0$.

Proof. Replacing $a+u$ for a in the result of Lemma 2.18 *(i)*, we obtain that $F_\alpha(a,b)\beta m\beta[u,b]_\alpha + F_\alpha(u,b)\beta m\beta[a,b]_\alpha = 0$. Thus, we get

$$\begin{aligned}&F_\alpha(a,b)\beta m\beta[u,b]_\alpha\beta m\beta F_\alpha(a,b)\beta m\beta[u,b]_\alpha\\ &= -F_\alpha(a,b)\beta m\beta[u,b]_\alpha\beta m\beta F_\alpha(u,b)\beta m\beta[a,b]_\alpha = 0.\end{aligned}$$

Hence, by the semiprimeness of M, we get $F_\alpha(a,b)\beta m\beta[u,b]_\alpha = 0$. Likewise, by replacing $b+v$ for b in this equality, we obtain *(i)*.

Proceeding in the same way as above, by the similar replacements in the result of Lemma 2.18*(ii)*, we get *(ii)*. □

Lemma 2.20. *Let M be a 2-torsion free semiprime Γ_N-ring. Then, for all $a,b,u,v,m \in M$ and $\alpha,\beta,\gamma \in \Gamma$,*

(i) $F_\alpha(a,b)\beta m\beta[u,v]_\gamma = 0$;
(ii) $[u,v]_\gamma\beta m\beta F_\alpha(a,b) = 0$.

Proof. *(i)* Put $\alpha+\gamma$ for α in Lemma 2.19*(i)*, and use Lemma 2.7*(iv)* and Lemma 2.16*(iv)* to get $F_\alpha(a,b)\beta m\beta[u,v]_\gamma + F_\gamma(a,b)\beta m\beta[u,v]_\alpha = 0$. Therefore, we have,

$$\begin{aligned}&F_\alpha(a,b)\beta m\beta[u,v]_\gamma\beta m\beta F_\alpha(a,b)\beta m\beta[u,v]_\gamma\\ &= -F_\alpha(a,b)\beta m\beta[u,v]_\gamma\beta m\beta F_\gamma(a,b)\beta m\beta[u,v]_\alpha = 0.\end{aligned}$$

Since M is semiprime, we get $F_\alpha(a,b)\beta m\beta[u,v]_\gamma = 0$. □

(ii) By the similar replacements in Lemma 2.19*(ii)* as described earlier in the proof of *(i)*, we obtain $[u,v]_\gamma \beta m \beta F_\alpha(a,b) = 0$.

3. **The main result**

Theorem 3.1. *Let f be a Jordan generalized k-derivation of a 2-torsion free semiprime Γ_N-ring M associated with the Jordan k-derivation d of M. If $f(a)\alpha b = f(b)\alpha a$ and $a\alpha d(b) = b\alpha d(a)$ hold for all $a,b \in M$ and $\alpha \in \Gamma$, then f is a generalized k-derivation of M.*

Proof. Let f be a Jordan generalized k-derivation of a 2-torsion free semiprime Γ_N-ring M associated with the Jordan k-derivation d of M. Let $f(a)\alpha b = f(b)\alpha a$ and $a\alpha d(b) = b\alpha d(a)$ for all $a,b \in M$ and $\alpha \in \Gamma$.

Let $a,b,u,v,m \in M$ and $\alpha,\beta,\gamma \in \Gamma$. Then, $F_\alpha(a,b)\beta m \beta [u,v]_\gamma = 0$ [from Lemma 2.20*(i)*]. Using this result, we get

$$\begin{aligned}
&[F_\alpha(a,b),v]_\gamma \beta m \beta [F_\alpha(a,b),v]_\gamma \\
&= (F_\alpha(a,b)\gamma v - v\gamma F_\alpha(a,b))\beta m \beta [F_\alpha(a,b),v]_\gamma \\
&= F_\alpha(a,b)\gamma v \beta m \beta [F_\alpha(a,b),v]_\gamma - v\gamma F_\alpha(a,b)\beta m \beta [F_\alpha(a,b),v]_\gamma = 0,
\end{aligned}$$

since $v\beta m \in M$ and $F_\alpha(a,b) \in M$ for all $a,b,v,m \in M$ and $\alpha,\beta \in \Gamma$. Since M is semiprime, we get $[F_\alpha(a,b),v]_\gamma = 0$, where $v \in M$, $\gamma \in \Gamma$ and $F_\alpha(a,b) \in M$ for all $a,b \in M$, $\alpha \in \Gamma$. Therefore, it follows that $F_\alpha(a,b) \in Z(M)$, the centre of M.

Now, we let $\delta \in \Gamma$. Then, $F_\alpha(a,b)\delta[u,v]_\gamma \beta m \beta F_\alpha(a,b)\delta[u,v]_\gamma = 0$ [by using Lemma 2.20*(ii)*]. By the semiprimeness of M, we get,

$$F_\alpha(a,b)\delta[u,v]_\gamma = 0. \tag{3.1}$$

Similarly, we obtain $[u,v]_\gamma \delta F_\alpha(a,b)\beta m \beta [u,v]_\gamma \delta F_\alpha(a,b) = 0$ [by using Lemma 2.20*(i)*], and then again by the semiprimeness of M, we get,

$$[u,v]_\gamma \delta F_\alpha(a,b) = 0. \tag{3.2}$$

Then, using the hypothesis, we have,

$$\begin{aligned} 2F_\alpha(a,b)\delta F_\alpha(a,b) &= F_\alpha(a,b)\delta(F_\alpha(a,b)+F_\alpha(a,b)) \\ &= F_\alpha(a,b)\delta(F_\alpha(a,b)-F_\alpha(b,a)) \\ &= F_\alpha(a,b)\delta(f(a\alpha b)-f(a)\alpha b-ak(\alpha)b-a\alpha d(b) \\ &\quad -f(b\alpha a)+f(b)\alpha a+bk(\alpha)a+b\alpha d(a)) \\ &= F_\alpha(a,b)\delta f([a,b]_\alpha)-F_\alpha(a,b)\delta[a,b]_{k(\alpha)}. \end{aligned}$$

Here, $k(\alpha)\in\Gamma$ implies, $F_\alpha(a,b)\delta[a,b]_{k(\alpha)}=0$ [by (3.1)]. Thus, we get,

$$2F_\alpha(a,b)\delta F_\alpha(a,b)=F_\alpha(a,b)\delta f([a,b]_\alpha). \tag{3.3}$$

By (3.1) and (3.2), we obtain $F_\alpha(a,b)\delta[u,v]_\gamma+[u,v]_\gamma\delta F_\alpha(a,b)=0$. Then, by Lemma 2.2*(i)* with the applications of (3.1), (3.2) and the hypothesis, we have,

$$\begin{aligned} 0 &= f(F_\alpha(a,b)\delta[u,v]_\gamma+[u,v]_\gamma\delta F_\alpha(a,b)) \\ &= f(F_\alpha(a,b))\delta[u,v]_\gamma+f([u,v]_\gamma)\delta F_\alpha(a,b)+F_\alpha(a,b)k(\delta)[u,v]_\gamma \\ &\quad +[u,v]_\gamma k(\delta)F_\alpha(a,b)+F_\alpha(a,b)\delta d([u,v]_\gamma)+[u,v]_\gamma\delta d(F_\alpha(a,b)) \\ &= f(F_\alpha(a,b))\delta[u,v]_\gamma+f([u,v]_\gamma)\delta F_\alpha(a,b) \\ &\quad +F_\alpha(a,b)\delta d([u,v]_\gamma)+[u,v]_\gamma\delta d(F_\alpha(a,b)) \\ &= 2f([u,v]_\gamma)\delta F_\alpha(a,b)+2[u,v]_\gamma\delta d(F_\alpha(a,b)). \end{aligned}$$

Since M is 2-torsion free, $f([u,v]_\gamma)\delta F_\alpha(a,b)+[u,v]_\gamma\delta d(F_\alpha(a,b))=0$. That is,

$$f([u,v]_\gamma)\delta F_\alpha(a,b)=-[u,v]_\gamma\delta d(F_\alpha(a,b)). \tag{3.4}$$

Next, from (3.3) and (3.4), we obtain,

$$\begin{aligned} 2F_\alpha(a,b)\delta F_\alpha(a,b)\delta F_\alpha(a,b) &= F_\alpha(a,b)\delta f([a,b]_\alpha)\delta F_\alpha(a,b) \\ &= -F_\alpha(a,b)\delta[a,b]_\alpha\delta d(F_\alpha(a,b))=0. \end{aligned}$$

That is, $2(F_\alpha(a,b)\delta)^2F_\alpha(a,b)=0$. Since M is 2-torsion free, we obtain $(F_\alpha(a,b)\delta)^2F_\alpha(a,b)=0$. Hence, it follows that $F_\alpha(a,b)$ is a nilpotent element of the Γ_N-ring M.

But, we know that the center of a semiprime Γ_N-ring does not contain any nonzero nilpotent element. Therefore, we get $F_\alpha(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$. This implies that f is a generalized k-derivation of M. The proof of the theorem is thus complete. □

Acknowledgments

The authors wish to express their sincere thanks to the referee for the valuable comments and important advice to improve the article in the present shape.

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