ON JORDAN GENERALIZED $k$-DERIVATIONS OF SEMIPRIME $\Gamma_N$-RINGS

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Abstract. We know that every Jordan generalized $k$-derivation of a $\Gamma$-ring is not a generalized $k$-derivation of the same, in general. Here, we develop a number of lemmas relating to these derivations of certain $\Gamma$-rings and we show that under some conditions every Jordan generalized $k$-derivation of a 2-torsion free semiprime $\Gamma_N$-ring is a generalized $k$-derivation.

1. Introduction

As an extensive generalization of the concept of a classical ring, the notion of a gamma ring was introduced. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of gamma rings have attracted a wider attention as an emerging field of research to the modern algebraists to enrich the world of algebra. A number of prominent mathematicians have worked out on this interesting area of research to determine many basic properties of gamma rings and have extended numerous significant results in this context in the last few decades. There is a large number

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of researchers throughout the world who are recently engaged to execute more productive and creative results of gamma ring theory.

The notion of a $\Gamma$-ring was first introduced by N. Nobusawa [4] (which is presently known as a $\Gamma_N$-ring) and afterwards it was generalized by W. E. Barnes [1] in a broad sense (that served us now-a-days to call it as a $\Gamma$-ring generally). As an immediate consequence, this generalization states that every $\Gamma_N$-ring is a $\Gamma$-ring, but the converse is not necessarily true in general. They obtained many important basic properties of $\Gamma$-rings in various ways and determined some more remarkable characteristics of $\Gamma$-rings. Later, many mathematicians classified $\Gamma$-rings to develop a lot of significant results, one of from which the base of this article has been emanated. We start with the following necessary introductory definitions and examples.

Let $M$ and $\Gamma$ be two additive abelian groups. If there is a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $M \times \Gamma \times M \to M$ which satisfies the conditions (a) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ and (b) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is called a $\Gamma$-ring in the sense of Barnes [1].

For example, if $R$ is an ordinary associative ring, $U$ is any ideal of $R$, and $Z$ is the ring of integers, then $R$ is a $\Gamma$-ring with $\Gamma = U$ or $\Gamma = Z$.

In addition to the definition given above, if there exists another mapping $(\alpha, a, \beta) \mapsto a\alpha \beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the conditions (a*) $(\alpha + \beta)a\gamma = a\alpha \gamma + a\beta \gamma$, $\alpha(a + b)\beta = a\alpha \beta + a\beta \beta$, $a\alpha(\beta + \gamma) = a\alpha \beta + a\alpha \gamma$, (b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and (c*) $a\alpha b = 0$ implies $\alpha = 0$ for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$, then $M$ is called a $\Gamma$-ring in the sense of Nobusawa [4], or simply, a Nobusawa $\Gamma$-ring, and then we express it by saying that $M$ is a $\Gamma_N$-ring.

As an example, if $R$ is an ordinary associative ring with the unity $1$, then $R$ is a $\Gamma_N$-ring if we consider $\Gamma = U$ or $\Gamma = Z$.

**Remark 1.1.** $M$ is a $\Gamma_N$-ring implies that $\Gamma$ is an $M$-ring.

Let $M$ be a $\Gamma$-ring. Then, we have the following definitions:

(i) $M$ is called 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$.

(ii) An element $x \in M$ is said to be a nilpotent element if $(x\gamma)^n x = 0$, for all $\gamma \in \Gamma$, is satisfied for some positive integer $n$. 

(iii) \( M \) is called semiprime if \( a \Gamma M \Gamma a = 0 \) with \( a \in M \) implies \( a = 0 \).

(iv) The set \( Z(M) = \{ a \in M : a \alpha m = m \alpha a \mbox{ for all } \alpha \in \Gamma \mbox{ and } m \in M \} \) is called the centre.

(v) \( M \) is said to be a commutative \( \Gamma \)-ring if \( a \alpha b = b \alpha a \) holds for all \( a, b \in M \) and \( \alpha \in \Gamma \).

Note that the notions of derivation and Jordan derivation of a \( \Gamma \)-ring have been introduced by M. Sapanci and A. Nakajima in [5], whereas the concept of the \( k \)-derivation of a \( \Gamma \)-ring has been introduced by H. Kandamar [3]. Afterwards, the concept of Jordan generalized derivation of a \( \Gamma \)-ring has been developed by Y. Ceven and M. A. Ozturk in [2], where they have proved that under some constraints every Jordan generalized derivation of a 2-torsion free \( \Gamma \)-ring is a generalized derivation of the same. Here we introduce the concepts of generalized \( k \)-derivation and Jordan generalized \( k \)-derivation of a \( \Gamma \)-ring to extend the analogous result to the previous one for a 2-torsion free semiprime \( \Gamma_N \)-ring under some suitable conditions. All these concepts are mentioned in the following.

Let \( M \) be a \( \Gamma \)-ring. If \( d : M \rightarrow M \) and \( k : \Gamma \rightarrow \Gamma \) are two additive mappings such that \( d(a \alpha b) = d(a) \alpha b + a \alpha d(b) \) for all \( a, b \in M \) and \( \alpha \in \Gamma \), then \( d \) is said to be a derivation of \( M \). And, if \( d(a \alpha a) = d(a) \alpha a + a \alpha d(a) \) for all \( a \in M \) and \( \alpha \in \Gamma \), then \( d \) is called a Jordan derivation of \( M \).

Also, for all \( a, b \in M \) and \( \alpha \in \Gamma \), if \( d(a \alpha b) = d(a) \alpha b + ak(\alpha) b + a \alpha d(b) \) is satisfied, then \( d \) is said to be a \( k \)-derivation of \( M \). And, if \( d(a \alpha a) = d(a) \alpha a + ak(\alpha) a + a \alpha d(a) \) holds for all \( a \in M \) and \( \alpha \in \Gamma \), then \( d \) is called a Jordan \( k \)-derivation of \( M \).

Besides, an additive map \( f : M \rightarrow M \) is said to be a generalized derivation of \( M \) if there exists a derivation \( d : M \rightarrow M \) such that \( f(a \alpha b) = f(a) \alpha b + a \alpha d(b) \) is satisfied for all \( a, b \in M \) and \( \alpha \in \Gamma \). And, \( f \) is said to be a Jordan generalized derivation of \( M \) if there exists a Jordan derivation \( d : M \rightarrow M \) such that \( f(a \alpha a) = f(a) \alpha a + a \alpha d(a) \) holds for all \( a \in M \) and \( \alpha \in \Gamma \).

Moreover, an additive map \( f : M \rightarrow M \) is called a generalized \( k \)-derivation of \( M \) if there exists a \( k \)-derivation \( d : M \rightarrow M \) such that \( f(a \alpha b) = f(a) \alpha b + ak(\alpha) b + a \alpha d(b) \) for all \( a, b \in M \) and \( \alpha \in \Gamma \). And, \( f \) is called a Jordan generalized \( k \)-derivation of \( M \) if there exists a Jordan
The document contains a mathematical proof and an example related to \( k \)-derivation in the context of \( \Gamma \)-ring theory.

### Proof

Let \( d : M \to M \) be a \( k \)-derivation, meaning

\[
f(aa) = f(a)aa + k(a)a + aod(a)
\]

for all \( a \in M \) and \( \alpha \in \Gamma \).

### Example 1.2

Let \( M \) be a \( \Gamma \)-ring and \( f \) be a generalized \( k \)-derivation of \( M \). Then there exists a \( k \)-derivation \( d : M \to M \) such that

\[
d(ax) = d(x)ay + k(ax) + aod(y)
\]

and

\[
f(ax) = f(x)ay + k(ax) + aod(y)
\]

for all \( x, y \in M \) and \( \alpha \in \Gamma \). Let \( M_1 = M \times M \) and \( \Gamma_1 = \Gamma \times \Gamma \). Define the operations of addition and multiplication on \( M_1 \) and \( \Gamma_1 \) by:

\[
(x, y) + (z, w) = (x + z, y + w), \quad (x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w),
\]

\[
(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta), \quad (\alpha, \beta)(x, y)(\gamma, \delta) = (\alpha x\gamma, \beta y\delta),
\]

for every \( x, y, z, w \in M \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \), respectively. Then, \( M_1 \) is obviously a Noibusawa \( \Gamma_1 \)-ring under these operations.

Let \( f_1 : M_1 \to M_1, d_1 : M_1 \to M_1 \) and \( k_1 : \Gamma_1 \to \Gamma_1 \) be the additive maps defined by \( f_1((x, y)) = (f(x), f(y)) \), \( d_1((x, y)) = (d(x), d(y)) \) and \( k_1((\alpha, \beta)) = (k(\alpha), k(\beta)) \) for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \), respectively. Saying \( (x, y) = a \in M_1, (z, w) = b \in M_1 \) and \( (\alpha, \beta) = \gamma \in \Gamma_1 \) for any \( x, y, z, w \in M \) and \( \alpha, \beta \in \Gamma \), we have,

\[
d_1(a\gamma b) = d_1((x, y)(\alpha, \beta)(z, w)) = d_1((x\alpha z, y\beta w)) = d(x\alpha z, d(y\beta w))
\]

\[
= (d(x)\alpha z + k(\alpha)z + x\alpha z, d(y)\beta w + y\beta d(w))
\]

\[
= (d(x)\alpha z, d(y)\beta w) + (k(\alpha)z, x\alpha z, y\beta d(w))
\]

\[
= (d(x), d(y))(\alpha, \beta)(z, w) + (x, y)(k(\alpha), k(\beta))(z, w)
\]

\[
+ (x, y)(\alpha, \beta)(d(z), d(w))
\]

\[
= d_1(a\gamma b) + ak_1(\gamma) + a\gamma d_1(b),
\]

which implies that \( d_1 \) is a \( k \)-derivation of \( M_1 \). Furthermore, we have,

\[
f_1(a\gamma b) = f_1((x, y)(\alpha, \beta)(z, w)) = f_1((x\alpha z, y\beta w)) = f(x\alpha z, f(y\beta w))
\]

\[
= f(x)\alpha z + k(x)z + x\alpha z, f(y)\beta w + y\beta d(w)
\]

\[
= (f(x)\alpha z, f(y)\beta w) + (k(x)z, x\alpha z, y\beta d(w))
\]

\[
= (f(x), f(y))(\alpha, \beta)(z, w) + (x, y)(k(\alpha), k(\beta))(z, w)
\]

\[
+ (x, y)(\alpha, \beta)(d(z), d(w))
\]

\[
= f_1(a\gamma b) + ak_1(\gamma) + a\gamma d_1(b).
\]

Therefore, it follows that \( f_1 \) is a generalized \( k \)-derivation of \( M_1 \) associated with the \( k_1 \)-derivation \( d_1 \) of \( M_1 \).
Example 1.3. Let $M$ be a $\Gamma_N$-ring and let $f$ be a generalized $k$-derivation of $M$. Then, there exists a $k$-derivation $d : M \to M$ such that $d(xy) = d(x)yx + xkd(y)$ and $f(xy) = f(y)yx + xkd(y)$ for all $x, y \in M$ and $k \in \Gamma$.

Consider $N = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define the operations of addition and multiplication on $N$ and $\Gamma_1$ by $(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2)$, $(x_1, x_1)(\alpha, \alpha)(x_2, x_2) = (x_1\alpha x_2, x_1\alpha x_2)$ and $(\alpha_1, \alpha_1) + (\alpha_2, \alpha_2) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$, $(\alpha_1, \alpha_1)(x, x)(\alpha_2, \alpha_2) = (\alpha_1 x \alpha_2, \alpha_1 x \alpha_2)$ for all $x, x_1, x_2 \in M$ and $\alpha, \alpha_1, \alpha_2 \in \Gamma$, respectively. Then, it is clear that $N$ is a Nobusawa $\Gamma_1$-ring under these operations.

Now, let $f_1 : N \to N$, $d_1 : N \to N$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the additive maps defined by $f_1((x, x)) = (f(x), f(x))$, $d_1((x, x)) = (d(x), d(x))$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$ for all $x \in M$ and $\alpha \in \Gamma$, respectively. If we say that $(x, x) = a \in N$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$ for any $x \in M$ and $\alpha \in \Gamma$, then we have,

$$d_1(a \gamma a) = d_1((x, x)(\alpha, \alpha)(x, x)) = d_1((x ox, xo x)) = (d(x ox), d(x o x))$$
$$= (d(x o x) + x k(\alpha) x + xo d(x), d(x o x) + x k(\alpha) x + xo d(x))$$
$$= (d(x o x), d(x o x)) + (x k(\alpha) x, x k(\alpha) x) + (xo d(x), xo d(x))$$
$$= (d(x), d(x))(\alpha, \alpha)(x, x) + (x, x)(k(\alpha), k(\alpha))(x, x)$$
$$+ (x, x)(k(\alpha), k(\alpha))(d(x), d(x))$$
$$= d_1(a)(\gamma a) + ak_1(\gamma) a + a \gamma d_1(a).$$

Thus, $d_1$ is a Jordan $k_1$-derivation of $N$. Moreover, we have,

$$f_1(a \gamma a) = f_1((x, x)(\alpha, \alpha)(x, x)) = f_1((x ox, xo x)) = (f(x ox), f(x o x))$$
$$= (f(x o x) + x k(\alpha) x + xo d(x), f(x o x) + x k(\alpha) x + xo d(x))$$
$$= (f(x o x), f(x o x)) + (x k(\alpha) x, x k(\alpha) x) + (xo d(x), xo d(x))$$
$$= (f(x), f(x))(\alpha, \alpha)(x, x) + (x, x)(k(\alpha), k(\alpha))(x, x)$$
$$+ (x, x)(k(\alpha), k(\alpha))(d(x), d(x))$$
$$= f_1(a)(\gamma a) + ak_1(\gamma) a + a \gamma d_1(a).$$

Hence, $f_1$ is a Jordan generalized $k_1$-derivation of $N$ associated with the Jordan $k_1$-derivation $d_1$ of $N$. Obviously, $f_1$ is not a generalized $k_1$-derivation of $N$. 
From the definitions and examples given above, it is clear that every $k$-derivation of a $\Gamma$-ring $M$ is a Jordan $k$-derivation of $M$. But, the converse is not true, in general. Here, we show that under some conditions every Jordan generalized $k$-derivation of a 2-torsion free semiprime $\Gamma$-ring $M$ is a generalized $k$-derivation of $M$. For this to happen, we develop some useful results as follows.

2. Some preliminary results

**Lemma 2.1.** Let $M$ be a $\Gamma_N$-ring and let $d$ be a Jordan $k$-derivation of $M$. Then, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha b + b\alpha a$;

(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a + a\alpha b\beta d(a) + a\beta b\alpha d(a) + a\beta b\alpha a + a\alpha b\beta a$.

In particular, if $M$ is 2-torsion free, then

(iii) $d(a\alpha b\alpha a) = d(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha b\alpha d(a)$;

(iv) $d(a\alpha b\alpha c + c\alpha b\alpha a) = d(a)\alpha b\alpha c + d(c)\alpha b\alpha a + ak(\alpha)b\alpha c + c\alpha b\alpha a + a\alpha b\alpha c + c\alpha b\alpha a$.

**Proof.** Computing $d((a + b)(\alpha + b))$ and cancelling the like terms from both sides, we obtain (i). Then replacing $a\beta b + b\beta a$ for $b$ in (i), we get (ii). Since $M$ is 2-torsion free, we obtain (iii) by replacing $\alpha$ for $\beta$ in (ii), and then we obtain (iv) by replacing $a + c$ for $a$ in (iii). \qed

**Lemma 2.2.** Let $M$ be a $\Gamma_N$-ring and let $f$ be a Jordan generalized $k$-derivation of $M$. Then, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:

(i) $f(a\alpha b + b\alpha a) = f(a)\alpha b + f(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha b + b\alpha a$;

(ii) $f(a\alpha b\beta a + a\beta b\alpha a) = f(a)\alpha b\beta a + f(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.

In particular, if $M$ is 2-torsion free, then

(iii) $f(a\alpha b\alpha a) = f(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha b\alpha d(a)$;

(iv) $f(a\alpha b\alpha c + c\alpha b\alpha a) = f(a)\alpha b\alpha c + f(c)\alpha b\alpha a + ak(\alpha)b\alpha c + c\alpha b\alpha a + a\alpha b\alpha c + c\alpha b\alpha a$.
Proof. Compute \( f((a + b)\alpha(a + b)) \) and cancel the like terms from both sides to obtain \((i)\). Then replace \(a\beta b + b\alpha a\) for \(b\) in \((i)\) and use Lemma 2.1\((i)\) to get \((ii)\). Since \(M\) is 2-torsion free, \((iii)\) is easily obtained by replacing \(\alpha\) for \(\beta\) in \((ii)\), and then \((iv)\) is obtained by replacing \(a + c\) for \(a\) in \((iii)\).

\[
\text{Lemma 2.3. Let } f \text{ be a Jordan generalized } k\text{-derivation of a 2-torsion free } \Gamma_N\text{-ring } M. \text{ Then, for all } b \in M \text{ and } \beta \in \Gamma,
\]

\[
k(\beta b\beta) = k(\beta)b\beta + \beta d(b)\beta + \beta bk(\beta).
\]

Proof. For all \(a \in M\) and \(\alpha \in \Gamma\), we have
\[
f(aa\alpha) = f(a)\alpha a + ak(\alpha)a + aod(a).
\]
Let \(b \in M\) and \(\beta \in \Gamma\). Then, putting \(\beta b\beta\) for \(\alpha\), we get
\[
f(a\beta b\beta a) = f(a)\beta b\beta a + ak(\beta)\beta a + a\beta b\beta d(a).
\]
Expanding the LHS by Lemma 2.2\((iii)\), we obtain
\[
a(k(\beta b\beta) - k(\beta)b\beta - \beta d(b)\beta - \beta bk(\beta))a = 0.
\]
Hence, applying the Nobusawa condition \((c^*)\) of the definition of \(\Gamma_N\)-ring, we get the proof.

\[
\text{Lemma 2.4. If } f \text{ is both a Jordan generalized } k_1\text{-derivation and a Jordan generalized } k_2\text{-derivation of a 2-torsion free } \Gamma_N\text{-ring } M, \text{ then } k_1 = k_2.
\]

Proof. Obvious.

\[
\text{Remark 2.5. If } f \text{ is a Jordan generalized } k\text{-derivation of a 2-torsion free } \Gamma_N\text{-ring } M, \text{ then } k \text{ is uniquely determined.}
\]

\[
\text{Definition 2.6. Let } M \text{ be a } \Gamma\text{-ring. Then, for } a, b \in M \text{ and } \alpha \in \Gamma, \text{ we define } [a, b]_{\alpha} = aab - boa, \text{ known as the commutator of } a \text{ and } b \text{ with respect to } \alpha.
\]

\[
\text{Lemma 2.7. If } M \text{ is a } \Gamma\text{-ring, then for all } a, b, c \in M \text{ and } \alpha, \beta \in \Gamma,
\]

\[
\begin{align*}
(i) \quad [a, b]_{\alpha} + [b, a]_{\alpha} & = 0; \\
(ii) \quad [a + b, c]_{\alpha} & = [a, c]_{\alpha} + [b, c]_{\alpha}; \\
(iii) \quad [a, b + c]_{\alpha} & = [a, b]_{\alpha} + [a, c]_{\alpha}; \\
(iv) \quad [a, b]_{\alpha + \beta} & = [a, b]_{\alpha} + [a, b]_{\beta}.
\end{align*}
\]

Proof. Obvious.
Remark 2.8. A \( \Gamma \)-ring \( M \) is commutative if and only if \( [a, b]_\alpha = 0 \) for all \( a, b \in M \) and \( \alpha \in \Gamma \).

Definition 2.9. Let \( d \) be a Jordan \( k \)-derivation of a \( \Gamma_N \)-ring \( M \). Then for \( a, b \in M \) and \( \alpha \in \Gamma \), we define,

\[
G_\alpha(a, b) = d(aab) - d(a)ab - ak(\alpha)b - a\alpha d(b).
\]

Lemma 2.10. If \( d \) is a Jordan \( k \)-derivation of a \( \Gamma_N \)-ring \( M \), then for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \),

\begin{enumerate}
  \item \( G_\alpha(a, b) + G_\alpha(b, a) = 0; \)
  \item \( G_\alpha(a + b, c) = G_\alpha(a, c) + G_\alpha(b, c); \)
  \item \( G_\alpha(a, b + c) = G_\alpha(a, b) + G_\alpha(a, c); \)
  \item \( G_{\alpha + \beta}(a, b) = G_\alpha(a, b) + G_\beta(a, b). \)
\end{enumerate}

Proof. Obvious.

Remark 2.11. It is clear that \( d \) is a \( k \)-derivation of a \( \Gamma_N \)-ring \( M \) if and only if \( G_\alpha(a, b) = 0 \) for all \( a, b \in M \) and \( \alpha \in \Gamma \).

Lemma 2.12. Let \( d \) be a Jordan \( k \)-derivation of a 2-torsion free \( \Gamma_N \)-ring \( M \). Then for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \),

\[
G_\alpha(a, b) = d(aab) - d(a)ab - ak(\alpha)b - a\alpha d(b).
\]

Proof. First, we compute \( d(a\alpha(b\beta m b)\alpha a) + b\alpha(b\beta m b)\alpha a \) using Lemma 2.1(iii) and then compute, \( d((a\alpha b)\beta m b(boa) + (boa)\beta m b(aab)) \) using Lemma 2.1(iv). Since these two are equal, cancelling the like terms from both sides of this equality and then rearranging them using Lemma 2.10(i), we get the proof. \( \square \)

Lemma 2.13. Let \( M \) be a 2-torsion free semiprime \( \Gamma_N \)-ring and suppose that \( a, b \in M \). If \( a\Gamma m \Gamma b + b\Gamma m \Gamma a = 0 \) for all \( m \in M \), then \( a\Gamma m \Gamma b = b\Gamma m \Gamma a = 0 \).

Proof. Let \( m \) and \( m' \) be two arbitrary elements of \( M \). Then, from the hypothesis we have, \( a\Gamma m \Gamma b = -b\Gamma m \Gamma a \), and consequently we get \( (a\Gamma m \Gamma b)\Gamma m \Gamma (a\Gamma m \Gamma b) = -(a\Gamma m \Gamma b)\Gamma m \Gamma (a\Gamma m \Gamma b) \). So, this implies
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that $2((a\Gamma m\Gamma b)\Gamma m'(\Gamma(a\Gamma m\Gamma b))) = 0$. Since $M$ is 2-torsion free, we have $(a\Gamma m\Gamma b)\Gamma m'(\Gamma(a\Gamma m\Gamma b)) = 0$. By the semiprimeness of $M$, we get $a\Gamma m\Gamma b = 0$ for all $m \in M$. Hence, $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$ for all $m \in M$. 

Now, applying Lemma 2.13 to the result of Lemma 2.12, we obtain the following result.

**Corollary 2.14.** If $M$ is a 2-torsion free semiprime $\Gamma_N$-ring, then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,

(i) $G_\alpha(a, b)\beta m\beta[a, b]\alpha = 0$;
(ii) $[a, b]_\alpha\beta m\beta G_\alpha(a, b) = 0$.

**Definition 2.15.** Let $f$ be a Jordan generalized $k$-derivation of a $\Gamma_N$-ring $M$. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define,

$$F_\alpha(a, b) = f(a\alpha b) - f(a)\alpha b - ak(\alpha)b - a\alpha d(b).$$

**Lemma 2.16.** If $f$ is a Jordan generalized $k$-derivation of a $\Gamma_N$-ring $M$, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,

(i) $F_\alpha(a, b) + F_\alpha(b, a) = 0$;
(ii) $F_\alpha(a + b, c) = F_\alpha(a, c) + F_\alpha(b, c)$;
(iii) $F_\alpha(a, b + c) = F_\alpha(a, b) + F_\alpha(a, c)$;
(iv) $F_{\alpha + \beta}(a, b) = F_\alpha(a, b) + F_\beta(a, b)$.

**Proof.** Obvious. □

**Remark 2.17.** $f$ is a generalized $k$-derivation of a $\Gamma_N$-ring $M$ if and only if $F_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

**Lemma 2.18.** Let $f$ be a Jordan generalized $k$-derivation of a 2-torsion free semiprime $\Gamma_N$-ring $M$. If $a, b \in M$ and $\alpha, \beta \in \Gamma$, then for any $m \in M$,

(i) $F_{\alpha}(a, b)\beta m\beta[a, b]\alpha = 0$;
(ii) $[a, b]_\alpha\beta m\beta F_{\alpha}(a, b) = 0$.

**Proof.** (i) Consider $f(a\alpha b\beta m\beta ba + b\alpha a\beta m\beta ab)$ and look what we have here. Compute $f(a\alpha b\beta m\beta ba + b\alpha a\beta m\beta ab)$ by using Lemma 2.2(iii) along with the application of Lemma 2.1(iii). Also, compute
\[ f((aob)\beta m\beta(boa)+(boa)\beta m\beta(aob)) \text{ using Lemma 2.2(iv)}. \] After equating them, cancel the like terms from both sides to obtain \( F_\alpha(a, b)\beta m\beta[a, b]_\alpha + [a, b]_\alpha \beta m\beta F_\alpha(a, b) = 0 \) (by rearranging the terms using Lemma 2.16(i)). Hence, we get \( F_\alpha(a, b)\beta m\beta[a, b]_\alpha = 0 \) [by using Corollary 2.14(ii)].

(ii) Using (i), we have \([a, b]_\alpha \beta m\beta F_\alpha(a, b)\beta m\beta[a, b]_\alpha \beta m\beta F_\alpha(a, b) = 0\). Since \( M \) is semiprime, we obtain \([a, b]_\alpha \beta m\beta F_\alpha(a, b) = 0\). \(\square\)

**Lemma 2.19.** If \( M \) is a 2-torsion free semiprime \( \Gamma_N \)-ring, then for all \( a, b, u, v, m \in M \) and \( \alpha, \beta \in \Gamma \),

(i) \( F_\alpha(a, b)\beta m\beta[u, v]_\alpha = 0 \);

(ii) \([u, v]_\alpha \beta m\beta F_\alpha(a, b) = 0\).

**Proof.** Replacing \( a + u \) for \( a \) in the result of Lemma 2.18(i), we obtain that \( F_\alpha(a, b)\beta m\beta[u, b]_\alpha + F_\alpha(u, b)\beta m\beta[a, b]_\alpha = 0 \). Thus, we get

\[
F_\alpha(a, b)\beta m\beta[u, b]_\alpha \beta m\beta F_\alpha(a, b)\beta m\beta[u, b]_\alpha = -F_\alpha(a, b)\beta m\beta[u, b]_\alpha \beta m\beta F_\alpha(u, b)\beta m\beta[a, b]_\alpha = 0.
\]

Hence, by the semiprimeness of \( M \), we get \( F_\alpha(a, b)\beta m\beta[u, b]_\alpha = 0 \). Likewise, by replacing \( b + v \) for \( b \) in this equality, we obtain (i).

Proceeding in the same way as above, by the similar replacements in the result of Lemma 2.18(ii), we get (ii). \(\square\)

**Lemma 2.20.** Let \( M \) be a 2-torsion free semiprime \( \Gamma_N \)-ring. Then, for all \( a, b, u, v, m \in M \) and \( \alpha, \beta, \gamma \in \Gamma \),

(i) \( F_\alpha(a, b)\beta m\beta[u, v]_\gamma = 0 \);

(ii) \([u, v]_\gamma \beta m\beta F_\alpha(a, b) = 0\).

**Proof.** (i) Put \( \alpha + \gamma \) for \( \alpha \) in Lemma 2.19(i), and use Lemma 2.7(iv) and Lemma 2.16(iv) to get \( F_\alpha(a, b)\beta m\beta[u, v]_\gamma + F_\gamma(a, b)\beta m\beta[u, v]_\alpha = 0 \). Therefore, we have,

\[
F_\alpha(a, b)\beta m\beta[u, v]_\gamma \beta m\beta F_\alpha(a, b)\beta m\beta[u, v]_\gamma = -F_\alpha(a, b)\beta m\beta[u, v]_\gamma \beta m\beta F_\gamma(a, b)\beta m\beta[u, v]_\alpha = 0.
\]

Since \( M \) is semiprime, we get \( F_\alpha(a, b)\beta m\beta[u, v]_\gamma = 0 \). \(\square\)
(ii) By the similar replacements in Lemma 2.19(ii) as described earlier in the proof of (i), we obtain \([u, v]_\gamma \beta m \beta F_\alpha(a, b) = 0\).

3. The main result

**Theorem 3.1.** Let \(f\) be a Jordan generalized \(k\)-derivation of a 2-torsion free semiprime \(\Gamma_N\)-ring \(M\) associated with the Jordan \(k\)-derivation \(d\) of \(M\). If \(f(a)ab = f(b)aa\) and \(aad(b) = bad(a)\) hold for all \(a, b \in M\) and \(\alpha \in \Gamma\), then \(f\) is a generalized \(k\)-derivation of \(M\).

**Proof.** Let \(f\) be a Jordan generalized \(k\)-derivation of a 2-torsion free semiprime \(\Gamma_N\)-ring \(M\) associated with the Jordan \(k\)-derivation \(d\) of \(M\). Let \(f(a)ab = f(b)aa\) and \(aad(b) = bad(a)\) for all \(a, b \in M\) and \(\alpha \in \Gamma\).

Let \(a, b, u, v, m \in M\) and \(\alpha, \beta, \gamma \in \Gamma\). Then, \(F_\alpha(a, b)\beta m \beta [u, v]_\gamma = 0\) [from Lemma 2.20(ii)]. Using this result, we get

\[
[F_\alpha(a, b), v]_\gamma \beta m \beta [F_\alpha(a, b), v]_\gamma = (F_\alpha(a, b)\gamma v - v\gamma F_\alpha(a, b))\beta m \beta [F_\alpha(a, b), v]_\gamma = F_\alpha(a, b)\gamma v \beta m \beta [F_\alpha(a, b), v]_\gamma - v\gamma F_\alpha(a, b)\beta m \beta [F_\alpha(a, b), v]_\gamma = 0,
\]

since \(v\beta m \in M\) and \(F_\alpha(a, b) \in M\) for all \(a, b, v, m \in M\) and \(\alpha, \beta \in \Gamma\). Since \(M\) is semiprime, we get \([F_\alpha(a, b), v]_\gamma = 0\), where \(v \in M\), \(\gamma \in \Gamma\) and \(F_\alpha(a, b) \in M\) for all \(a, b \in M\), \(\alpha \in \Gamma\). Therefore, it follows that \(F_\alpha(a, b) \in Z(M)\), the centre of \(M\).

Now, we let \(\delta \in \Gamma\). Then, \(F_\alpha(a, b)\delta [u, v]_\gamma \beta m \beta F_\alpha(a, b)\delta [u, v]_\gamma = 0\) [by using Lemma 2.20(ii)]. By the semiprimeness of \(M\), we get,

\[(3.1)\]

\[F_\alpha(a, b)\delta [u, v]_\gamma = 0.\]

Similarly, we obtain \([u, v]_\gamma \delta F_\alpha(a, b)\beta m \beta [u, v]_\gamma \delta F_\alpha(a, b) = 0\) [by using Lemma 2.20(i)], and then again by the semiprimeness of \(M\), we get,

\[(3.2)\]

\[[u, v]_\gamma \delta F_\alpha(a, b) = 0.\]
Then, using the hypothesis, we have,

\[ 2F_\alpha(a, b)\delta F_\alpha(a, b) = F_\alpha(a, b)\delta(F_\alpha(a, b) + F_\alpha(a, b)) \]

\[ = F_\alpha(a, b)\delta(F_\alpha(a, b) - F_\alpha(b, a)) \]

\[ = F_\alpha(a, b)\delta(f(ab) - f(a)ab - ak(a)b - ao(d)b) \]

\[ - f(boa) + f(b)\alpha a + bk(\alpha)a + b\alpha d(a)) \]

\[ = F_\alpha(a, b)\delta f([a, b]_\alpha) - F_\alpha(a, b)\delta[a, b]_\delta. \]

Here, \( k(\alpha) \in \Gamma \) implies, \( F_\alpha(a, b)\delta[a, b]_\delta = 0 \) [by (3.1)]. Thus, we get,

(3.3) \[ 2F_\alpha(a, b)\delta F_\alpha(a, b) = F_\alpha(a, b)\delta f([a, b]_\alpha). \]

By (3.1) and (3.2), we obtain \( F_\alpha(a, b)\delta[u, v]_\gamma + [u, v]_\gamma \delta F_\alpha(a, b) = 0. \) Then, by Lemma 2.2(i) with the applications of (3.1), (3.2) and the hypothesis, we have,

\[ 0 = f(F_\alpha(a, b)\delta[u, v]_\gamma + [u, v]_\gamma \delta F_\alpha(a, b)) \]

\[ = f(F_\alpha(a, b)\delta[u, v]_\gamma + f([u, v]_\gamma)\delta F_\alpha(a, b) + F_\alpha(a, b)k(\delta)[u, v]_\gamma \]

\[ + [u, v]_(\gamma\delta)F_\alpha(a, b) + F_\alpha(a, b)\delta d([u, v]_\gamma) + [u, v]_\gamma \delta d(F_\alpha(a, b)) \]

\[ = f(F_\alpha(a, b)\delta[u, v]_\gamma + f([u, v]_\gamma)\delta F_\alpha(a, b) \]

\[ + F_\alpha(a, b)\delta d([u, v]_\gamma) + [u, v]_\gamma \delta d(F_\alpha(a, b)) \]

\[ = 2f([u, v]_\gamma)\delta F_\alpha(a, b) + 2[u, v]_\gamma \delta d(F_\alpha(a, b)). \]

Since \( M \) is 2-torsion free, \( f([u, v]_\gamma)\delta F_\alpha(a, b) + [u, v]_\gamma \delta d(F_\alpha(a, b)) = 0. \) That is,

(3.4) \[ f([u, v]_\gamma)\delta F_\alpha(a, b) = -[u, v]_\gamma \delta d(F_\alpha(a, b)). \]

Next, from (3.3) and (3.4), we obtain,

\[ 2F_\alpha(a, b)\delta F_\alpha(a, b)\delta F_\alpha(a, b) = F_\alpha(a, b)\delta f([a, b]_\alpha)\delta F_\alpha(a, b) \]

\[ = -F_\alpha(a, b)\delta[a, b]_\alpha \delta d(F_\alpha(a, b)) = 0. \]

That is, \( 2F_\alpha(a, b)\delta)^2F_\alpha(a, b) = 0. \) Since \( M \) is 2-torsion free, we obtain \( (F_\alpha(a, b)\delta)^2F_\alpha(a, b) = 0. \) Hence, it follows that \( F_\alpha(a, b) \) is a nilpotent element of the \( \Gamma_N \)-ring \( M. \)

But, we know that the center of a semiprime \( \Gamma_N \)-ring does not contain any nonzero nilpotent element. Therefore, we get \( F_\alpha(a, b) = 0 \) for all \( a, b \in M \) and \( \alpha \in \Gamma. \) This implies that \( f \) is a generalized \( k \)-derivation of \( M. \) The proof of the theorem is thus complete. \( \square \)
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