

## GENERALIZED STEFFENSEN MEANS

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ABSTRACT. Using Steffensen's inequality and log-convexity mean-value theorem, we introduce new means and then we establish their monotonicity properties. We also generalize some parts of theory given in [J. Jakšetić, J.E. Pečarić, Steffensen's means, J. Math. Inequal. 2 (2008) 487- 498].

### 1. Introduction and preliminary

The well-known Steffensen inequality reads as follows

**Theorem 1.1.** *Suppose that  $f$  is decreasing and  $g$  is integrable on  $[a, b]$  with  $0 \leq g \leq 1$  and  $\lambda = \int_a^b g(t)dt$ . Then,*

$$(1.1) \quad \int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

*The inequalities are reversed for an increasing function  $f$ .*

In [2, p. 184] it is shown that condition  $0 \leq g \leq 1$  in Theorem 1.1 can be replaced with a more general one as specified in the next theorem.

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**Theorem 1.2.** *Assume that  $f$  and  $g$  are integrable functions on  $[a, b]$ . Then, the inequalities in (1.1) hold for every decreasing function  $f$  if and only if*

$$(1.2) \quad 0 \leq \int_x^b g(t)dt \leq b - x \text{ and } 0 \leq \int_a^x g(t)dt \leq x - a, \quad \text{for every } x \in [a, b].$$

The following theorem represents a new generalization of mean-value theorem given in [1].

**Theorem 1.3.** *Let  $f \in C^1([a, b])$  be increasing and let  $g$  be integrable function on  $[a, b]$  such that (1.2) is valid and  $\lambda = \int_a^b g(t)dt$ . If  $h \in C^1([f(a), f(b)])$ , then there exist  $\eta, \xi \in [f(a), f(b)]$  such that*

$$(1.3) \quad \int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt = h'(\xi) \left[ \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right]$$

and

$$(1.4) \quad \int_a^b h(f(t))g(t)dt - \int_{b-\lambda}^b h(f(t))dt = h'(\eta) \left[ \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \right].$$

**Proof.** Since  $h'$  is continuous on  $[f(a), f(b)]$ , there exist  $m = \min h'$  and  $M = \max h'$  both as real numbers. We first consider the function,  $\tilde{h}(x) = Mx - h(x)$ . Then,  $\tilde{h}'(x) = M - h'(x) \geq 0$ ,  $x \in [f(a), f(b)]$ , and so  $\tilde{h}$  is an increasing function. Applying Steffensen's inequality, from Theorem 1.1 on increasing function  $\tilde{h} \circ f$ , we have

$$\begin{aligned} 0 \leq \int_a^b \tilde{h}(f(t))g(t)dt - \int_a^{a+\lambda} \tilde{h}(t)dt &= M \int_a^b f(t)g(t)dt - \int_a^b h(f(t))g(t)dt - \\ &\quad - M \int_a^{a+\lambda} f(t)dt + \int_a^{a+\lambda} h(f(t))dt, \end{aligned}$$

that is,

$$\int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt \leq M \left[ \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right].$$

Similarly, since  $\hat{h}(x) = h(x) - mx$  is an increasing function we can apply Steffensen's inequality on increasing function  $\hat{h} \circ f$  and get

$$m \left[ \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right] \leq \int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt.$$

We now conclude that there exists  $\xi \in [f(a), f(b)]$  satisfying (1.3).

With the same technique one can prove existence of  $\eta$  in (1.4).  $\square$

**Remark 1.4.** It can be shown (see [1]) that if  $g$  is an integrable function that differs from the function  $x \mapsto 1_{[a, a+\lambda]}(x)$  on a set of positive measure, then the left hand side of (1.3) is different from 0. Similarly, if  $g$  is an integrable function that differs from the function  $x \mapsto 1_{[b-\lambda, b]}(x)$  on a set of positive measure, then the left hand side of (1.4) is different from 0.

**Corollary 1.5.** Let  $f \in C^1([a, b])$  be a strictly monotone function and  $h_1, h_2 \in C^1([f(a), f(b)])$ ,  $g$  integrable on  $[a, b]$ , with  $\lambda = \int_a^b g(t)dt$  and

$$(1.5) \quad 0 \leq \int_x^b g(t)dt \leq b - x \text{ and } 0 \leq \int_a^x g(t)dt \leq x - a \text{ for every } x \in [a, b].$$

Then, there exist  $\xi, \eta \in [f(a), f(b)]$  such that

$$(1.6) \quad \frac{\int_a^b h_1(f(t))g(t)dt - \int_a^{a+\lambda} h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_a^{a+\lambda} h_2(f(t))dt} = \frac{h_1'(\xi)}{h_2'(\xi)},$$

$$(1.7) \quad \frac{\int_a^b h_1(f(t))g(t)dt - \int_{b-\lambda}^b h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_{b-\lambda}^b h_2(f(t))dt} = \frac{h_1'(\eta)}{h_2'(\eta)}.$$

**Proof.** We show (1.6) first. Define the linear functional  $L(h) = \int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt$ . Next, define  $\phi(t) = h_1(t)L(h_2) - h_2(t)L(h_1)$ .

By Theorem 1.3, there exists  $\xi \in [f(a), f(b)]$  such that

$$L(\phi) = \phi'(\xi) \left[ \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right].$$

From  $L(\phi) = 0$ , it follows that  $h_1'(\xi)L(h_2) - h_2'(\xi)L(h_1) = 0$  and (1.6) is proved. The proof of (1.7) is quite similar.  $\square$

In [1], we used Steffensen's inequality to define Cauchy means of two numbers. In the next section, we will generalize this result by defining Cauchy means of general monotonic functions.

## 2. Main results

Let  $x, y$  be fixed, with  $0 < x < y$ . Let  $f \in C^1([x, y])$  be strictly increasing function and  $h_1, h_2 \in C^1([f(x), f(y)])$  monotonic functions, and  $g$  integrable function on  $[x, y]$  such that  $\lambda = \int_x^y g(t)dt$  and  $g$  satisfies the corresponding condition (1.5).

Corollary 1.5 enables us to define various types of means, because if  $h_1'/h_2'$  has an inverse, then from (1.6) and (1.7) we have

$$(2.1) \quad \xi = \left( \frac{h_1'}{h_2'} \right)^{-1} \left( \frac{\int_x^y h_1(f(t))g(t)dt - \int_x^{x+\lambda} h_1(f(t))dt}{\int_x^y h_2(f(t))g(t)dt - \int_x^{x+\lambda} h_2(f(t))dt} \right),$$

and

$$(2.2) \quad \eta = \left( \frac{h_1'}{h_2'} \right)^{-1} \left( \frac{\int_x^y h_1(f(t))g(t)dt - \int_x^y h_1(f(t))dt}{\int_x^y h_2(f(t))g(t)dt - \int_x^y h_2(f(t))dt} \right),$$

which means that  $\xi$  and  $\eta$  are means of numbers  $x$  and  $y$ , for given functions  $f$  and  $g$ . Specially, if we take substitutions  $h_1(t) = t^r$ ,  $h_2(t) = t^s$  in (2.1) and (2.2), we obtain the following expressions,

$$(2.3) \quad S_1(f, g; x, y; r, s) = \left[ \frac{s \left( \int_x^y f^r(t)g(t)dt - \int_x^{x+\lambda} f^r(t)dt \right)}{r \left( \int_x^y f^s(t)g(t)dt - \int_x^{x+\lambda} f^s(t)dt \right)} \right]^{\frac{1}{r-s}}$$

and

$$(2.4) \quad S_2(f, g; x, y; r, s) = \left[ \frac{s \left( \int_x^y f^r(t)g(t)dt - \int_{y-\lambda}^y f^r(t)dt \right)}{r \left( \int_x^y f^s(t)g(t)dt - \int_{y-\lambda}^y f^s(t)dt \right)} \right]^{\frac{1}{r-s}},$$

where,  $(r-s)r \cdot s \neq 0$ .

Continuous extensions of (2.3) are:

$$S_1(f, g; x, y; s, 0) = S_1(f, g; x, y; 0, s) =$$

$$\left[ \frac{\int_x^y f^s(t)g(t)dt - \int_x^{x+\lambda} f^s(t)dt}{s \left( \int_x^y g(t) \ln f(t)dt - \int_x^{x+\lambda} \ln f(t)dt \right)} \right]^{\frac{1}{s}}, \quad s \neq 0$$

$$S_1(f, g; x, y; s, s) =$$

$$\exp \left( \frac{\int_x^y g(t) \ln f(t)dt - \int_x^{x+\lambda} \ln f(t)dt - \frac{1}{s} \int_x^y f^s(t)g(t)dt + \frac{1}{s} \int_x^{x+\lambda} f^s(t)dt}{s \left( \int_x^y f^s(t)g(t)dt - \int_x^{x+\lambda} f^s(t)dt \right)} \right), \quad s \neq 0$$

$$S_1(f, g; x, y; 0, 0) = \exp \left( \frac{\int_x^y g(t) \ln^2 f(t)dt - \int_x^{x+\lambda} \ln^2 f(t)dt}{2 \left( \int_x^y g(t) \ln f(t)dt - \int_x^{x+\lambda} \ln f(t)dt \right)} \right).$$

Continuous extensions of (2.4) are:

$$S_2(f, g; x, y; s, 0) = S_1(f, g; x, y; 0, s) = \left[ \frac{\int_x^y f^s(t)g(t)dt - \int_x^y f^s(t)dt}{s \left( \int_x^y g(t) \ln f(t)dt - \int_x^y \ln f(t)dt \right)} \right]^{\frac{1}{s}}, \quad s \neq 0$$

$$S_2(f, g; x, y; s, s) = \exp \left( \frac{\int_x^y g(t) \ln f(t)dt - \int_x^y \ln f(t)dt - \frac{1}{s} \int_x^y f^s(t)g(t)dt + \frac{1}{s} \int_x^y f^s(t)dt}{s \left( \int_x^y f^s(t)g(t)dt - \int_x^y f^s(t)dt \right)} \right), \quad s \neq 0$$

$$S_2(f, g; x, y; 0, 0) = \exp \left( \frac{\int_x^y g(t) \ln^2 f(t)dt - \int_x^y \ln^2 f(t)dt}{2 \left( \int_x^y g(t) \ln f(t)dt - \int_x^y \ln f(t)dt \right)} \right).$$

Now, we establish monotonicity properties of the new means.

**Theorem 2.1.** *Let  $r \leq u$ ,  $s \leq v$ . Then,*

$$S_1(f, g; x, y; r, s) \leq S_1(f, g; x, y; u, v)$$

and

$$S_2(f, g; x, y; r, s) \leq S_2(f, g; x, y; u, v).$$

For the proof, we need the following two lemmas.

**Lemma 2.2.** *Let  $f$  be a log-convex function. If  $x_1 \leq y_1$ ,  $x_2 \leq y_2$ ,  $x_1 \neq x_2$ ,  $y_1 \neq y_2$ , then the following inequality holds:*

$$(2.5) \quad \left( \frac{f(x_2)}{f(x_1)} \right)^{1/(x_2-x_1)} \leq \left( \frac{f(y_2)}{f(y_1)} \right)^{1/(y_2-y_1)}.$$

**Proof.** This follows from Remark 1.2. in [2], □

Observe that

$$S_1(f, g; x, y; r, s) = \left( \frac{\phi(r)}{\phi(s)} \right)^{\frac{1}{r-s}},$$

where,

$$(2.6) \quad \phi(r) = \begin{cases} \frac{1}{r} \left( \int_x^y f^r(t)g(t)dt - \int_x^{x+\lambda} f^r(t)dt \right), & r \neq 0, \\ \int_x^y g(t) \ln f(t)dt - \int_x^{x+\lambda} \ln f(t)dt, & r = 0. \end{cases}$$

Similarly,

$$S_2(f, g; x, y; r, s) = \left( \frac{\psi(r)}{\psi(s)} \right)^{\frac{1}{r-s}},$$

where,

$$(2.7) \quad \psi(r) = \begin{cases} \frac{1}{r} \left( \int_{y-\lambda}^y f^r(t)dt - \int_x^y f^r(t)g(t)dt \right), & r \neq 0, \\ \int_{y-\lambda}^y \ln f(t)dt - \int_x^y g(t) \ln f(t)dt, & r = 0. \end{cases}$$

Let us observe that  $\lim_{r \rightarrow 0} \phi(r) = \phi(0)$  and  $\lim_{r \rightarrow 0} \psi(r) = \psi(0)$ , meaning that  $\phi$  and  $\psi$  are continuous functions.  $\square$

**Lemma 2.3.** *The functions  $\phi$  and  $\psi$  defined by (2.6) and (2.7) are log-convex functions.*

**Proof.** Consider the following function,

$$h(x) = p^2\varphi_r(x) + 2pq\varphi_z(x) + q^2\varphi_s(x) \quad \text{where } z = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R},$$

and

$$\varphi_u(x) = \begin{cases} \frac{x^u}{u}, & u \neq 0, \\ \ln x, & u = 0. \end{cases}$$

Now,

$$\begin{aligned} h'(x) &= p^2x^{r-1} + 2pqx^{z-1} + q^2x^{s-1} \\ &= \left( px^{(r-1)/2} + qx^{(s-1)/2} \right)^2 \geq 0. \end{aligned}$$

This implies that  $h$  is monotonically increasing. Since  $f$  is an increasing function, then  $h \circ f$  is an increasing function. Then, the following Steffensen's inequalities from Theorem 1.2 are satisfied:

$$(2.8) \quad \int_x^y h(f(t))g(t)dt - \int_x^{x+\lambda} h(f(t))dt \geq 0$$

and

$$\int_{y-\lambda}^y h(f(t))dt - \int_x^y h(f(t))g(t)dt \geq 0.$$

From (2.8) it then follows:

$$p^2\phi(r) + 2pq\phi(z) + q^2\phi(s) \geq 0, \quad \text{where } z = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R}.$$

This implies:

$$\phi^2\left(\frac{r+s}{2}\right) \leq \phi(r)\phi(s),$$

that is,  $\phi$  is a log-convex function in the Jensen sense. Since we have shown that  $\phi$  is a continuous function, we conclude that  $\phi$  is log-convex function. The log-convexity of  $\psi$  can be deduced in a similar way.  $\square$

**Proof.** [Proof of Theorem 2.1] We now apply inequality (2.5) from Lemma 2.2 for  $f = \phi$ ,  $r \leq u$ ,  $s \leq v$ ,  $r \neq s$ ,  $u \neq v$  ( $r, t, u, v \neq 0$ ) to deduce:

$$\left(\frac{\phi(r)}{\phi(s)}\right)^{\frac{1}{r-s}} \leq \left(\frac{\phi(u)}{\phi(v)}\right)^{\frac{1}{u-v}}.$$

Since  $(r, s) \mapsto S_1(f, g; x, y; r, s)$  is continuous, we have, for  $r \leq u$ ,  $s \leq v$ ,

$$S_1(f, g; x, y; r, s) \leq S_1(f, g; x, y; u, v).$$

The same arguments stand for  $S_2(f, g; x, y; r, s)$ .  $\square$

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