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GENERALIZED STEFFENSEN MEANS

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Abstract. Using Steffensen's inequality and log-convexity mean-value theorem, we introduce new means and then we establish their monotonicity properties. We also generalize some parts of theory given in [J. Jakšetić, J.E. Pečarić, Steffensens means, J. Math. Inequal. 2 (2008) 487- 498].

1. Introduction and preliminary

The well-known Steffensen inequality reads as follows

Theorem 1.1. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then,*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt. \tag{1.1}$$

The inequalities are reversed for an increasing function f.

In [2, p. 184] it is shown that condition $0 \leq g \leq 1$ in Theorem 1.1 can be replaced with a more general one as specified in the next theorem.

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Theorem 1.2. *Assume that f and g are integrable functions on $[a,b]$. Then, the inequalities in* (1.1) *hold for every decreasing function f if and only if*

(1.2)
$$0 \le \int_x^b g(t)dt \le b-x \text{ and } 0 \le \int_a^x g(t)dt \le x-a, \quad \text{for every} \quad x \in [a,b].$$

The following theorem represents a new generalization of mean-value theorem given in [1].

Theorem 1.3. *Let $f \in C^1([a,b])$ be increasing and let g be integrable function on $[a,b]$ such that* (1.2) *is valid and $\lambda = \int_a^b g(t)dt$. If $h \in C^1([f(a),f(b)])$, then there exist $\eta, \xi \in [f(a),f(b)]$ such that*

(1.3)
$$\int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt = h'(\xi)\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt\right]$$

and

(1.4)
$$\int_a^b h(f(t))g(t)dt - \int_{b-\lambda}^b h(f(t))dt = h'(\eta)\left[\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right].$$

Proof. Since h' is continuous on $[f(a),f(b)]$, there exist $m = \min h'$ and $M = \max h'$ both as real numbers. We first consider the function, $\tilde{h}(x) = Mx - h(x)$. Then, $\tilde{h}'(x) = M - h'(x) \ge 0,\ x \in [f(a),f(b)]$, and so $\tilde{h}$ is an increasing function. Applying Steffensen's inequality, from Theorem 1.1 on increasing function $\tilde{h} \circ f$, we have

$$0 \le \int_a^b \tilde{h}(f(t))g(t)dt - \int_a^{a+\lambda} \tilde{h}(t)dt = M\int_a^b f(t)g(t)dt - \int_a^b h(f(t))g(t)dt -$$
$$-M\int_a^{a+\lambda} f(t)dt + \int_a^{a+\lambda} h(f(t))dt,$$

that is,

$$\int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt \le M\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt\right].$$

Similarly, since $\hat{h}(x) = h(x) - mx$ is an increasing function we can apply Steffensen's inequality on increasing function $\hat{h} \circ f$ and get

$$m\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt\right] \le \int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt.$$

We now conclude that there exists $\xi \in [f(a), f(b)]$ satisfying (1.3).
With the same technique one can prove existence of η in (1.4). □

Remark 1.4. It can be shown (see [1]) that if g is an integrable function that differs from the function $x \mapsto 1_{[a,a+\lambda]}(x)$ on a set of positive measure, then the left hand side of (1.3) is different from 0. Similarly, if g is an integrable function that differs from the function $x \mapsto 1_{[b-\lambda,b]}(x)$ on a set of positive measure, then the left hand side of (1.4) is different from 0.

Corollary 1.5. *Let $f \in C^1([a,b])$ be a strictly monotone function and $h_1, h_2 \in C^1([f(a), f(b)])$, g integrable on $[a,b]$, with $\lambda = \int_a^b g(t)dt$ and* (1.5)

$$0 \le \int_x^b g(t)dt \le b - x \text{ and } 0 \le \int_a^x g(t)dt \le x - a \quad \text{for every} \quad x \in [a,b].$$

Then, there exist $\xi, \eta \in [f(a), f(b)]$ such that

$$\frac{\int_a^b h_1(f(t))g(t)dt - \int_a^{a+\lambda} h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_a^{a+\lambda} h_2(f(t))dt} = \frac{h_1'(\xi)}{h_2'(\xi)}, \tag{1.6}$$

$$\frac{\int_a^b h_1(f(t))g(t)dt - \int_{b-\lambda}^b h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_{b-\lambda}^b h_2(f(t))dt} = \frac{h_1'(\eta)}{h_2'(\eta)}. \tag{1.7}$$

Proof. We show (1.6) first. Define the linear functional $L(h) = \int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt$. Next, define $\phi(t) = h_1(t)L(h_2) - h_2(t)L(h_1)$.

By Theorem 1.3, there exists $\xi \in [f(a), f(b)]$ such that

$$L(\phi) = \phi'(\xi)\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt\right].$$

From $L(\phi) = 0$, it follows that $h_1'(\xi)L(h_2) - h_2'(\xi)L(h_1) = 0$ and (1.6) is proved. The proof of (1.7) is quite similar. $\square$

In [1], we used Steffensen's inequality to define Cauchy means of two numbers. In the next section, we will generalize this result by defining Cauchy means of general monotonic functions.

2. **Main results**

Let x, y be fixed, with $0 < x < y$. Let $f \in C^1([x, y])$ be strictly increasing function and $h_1, h_2 \in C^1([f(x), f(y)])$ monotonic functions, and g integrable function on $[x, y]$ such that $\lambda = \int_x^y g(t)dt$ and g satisfies the corresponding condition (1.5).

Corollary 1.5 enables us to define various types of means, because if h_1'/h_2' has an inverse, then from (1.6) and (1.7) we have

$$\xi = \left(\frac{h_1'}{h_2'}\right)^{-1}\left(\frac{\int\limits_x^y h_1(f(t))g(t)dt - \int\limits_x^{x+\lambda} h_1(f(t))dt}{\int\limits_x^y h_2(f(t))g(t)dt - \int\limits_x^{x+\lambda} h_2(f(t))dt}\right), \tag{2.1}$$

and

$$\eta = \left(\frac{h_1'}{h_2'}\right)^{-1}\left(\frac{\int\limits_x^y h_1(f(t))g(t)dt - \int\limits_{y-\lambda}^{y} h_1(f(t))dt}{\int\limits_x^y h_2(f(t))g(t)dt - \int\limits_{y-\lambda}^{y} h_2(f(t))dt}\right), \tag{2.2}$$

which means that ξ and η are means of numbers x and y, for given functions f and g. Specially, if we take substitutions $h_1(t) = t^r,\ h_2(t) = t^s$ in (2.1) and (2.2), we obtain the following expressions,

$$(2.3)\qquad S_1(f,g;x,y;r,s)=\left[\frac{s\left(\int_x^y f^r(t)g(t)dt-\int_x^{x+\lambda} f^r(t)dt\right)}{r\left(\int_x^y f^s(t)g(t)dt-\int_x^{x+\lambda} f^s(t)dt\right)}\right]^{\frac{1}{r-s}}$$

and

$$(2.4)\qquad S_2(f,g;x,y;r,s)=\left[\frac{s\left(\int_x^y f^r(t)g(t)dt-\int_{y-\lambda}^{y} f^r(t)dt\right)}{r\left(\int_x^y f^s(t)g(t)dt-\int_{y-\lambda}^{y} f^s(t)dt\right)}\right]^{\frac{1}{r-s}},$$

where, $(r-s)r\cdot s\neq 0$.

Continuous extensions of (2.3) are:

$$S_1(f,g;x,y;s,0)=S_1(f,g;x,y;0,s)=$$

$$\left[\frac{\int_x^y f^s(t)g(t)dt-\int_x^{x+\lambda} f^s(t)dt}{s\left(\int_x^y g(t)\ln f(t)dt-\int_x^{x+\lambda}\ln f(t)dt\right)}\right]^{\frac{1}{s}},\ s\neq 0$$

$$S_1(f,g;x,y;s,s)=$$

$$\exp\left(\frac{\int_x^y g(t)\ln f(t)dt-\int_x^{x+\lambda}\ln f(t)dt-\frac{1}{s}\int_x^y f^s(t)g(t)dt+\frac{1}{s}\int_x^{x+\lambda} f^s(t)dt}{s\left(\int_x^y f^s(t)g(t)dt-\int_x^{x+\lambda} f^s(t)dt\right)}\right),\ s\neq 0$$

$$S_1(f,g;x,y;0,0)=\exp\left(\frac{\int_x^y g(t)\ln^2 f(t)dt-\int_x^{x+\lambda}\ln^2 f(t)dt}{2\left(\int_x^y g(t)\ln f(t)dt-\int_x^{x+\lambda}\ln f(t)dt\right)}\right).$$

Continuous extensions of (2.4) are:

$S_2(f,g;x,y;s,0)=S_1(f,g;x,y;0,s)=$

$$\left[\frac{\int_x^y f^s(t)g(t)dt-\int_{y-\lambda}^y f^s(t)dt}{s\left(\int_x^y g(t)\ln f(t)dt-\int_{y-\lambda}^y \ln f(t)dt\right)}\right]^{\frac{1}{s}},\ s\neq 0$$

$S_2(f,g;x,y;s,s)=$

$$\exp\left(\frac{\int_x^y g(t)\ln f(t)dt-\int_{y-\lambda}^y \ln f(t)dt-\frac{1}{s}\int_x^y f^s(t)g(t)dt+\frac{1}{s}\int_{y-\lambda}^y f^s(t)dt}{s\left(\int_x^y f^s(t)g(t)dt-\int_{y-\lambda}^y f^s(t)dt\right)}\right),\ s\neq 0$$

$S_2(f,g;x,y;0,0)=$

$$\exp\left(\frac{\int_x^y g(t)\ln^2 f(t)dt-\int_{y-\lambda}^y \ln^2 f(t)dt}{2\left(\int_x^y g(t)\ln f(t)dt-\int_{y-\lambda}^y \ln f(t)dt\right)}\right).$$

Now, we establish monotonicity properties of the new means.

Theorem 2.1. *Let $r\le u,\ s\le v$. Then,*

$$S_1(f,g;x,y;r,s)\le S_1(f,g;x,y;u,v)$$

and

$$S_2(f,g;x,y;r,s)\le S_2(f,g;x,y;u,v).$$

For the proof, we need the following two lemmas.

Lemma 2.2. *Let f be a log-convex function. If $x_1\le y_1,\ x_2\le y_2,\ x_1\neq x_2,\ y_1\neq y_2$, then the following inequality holds:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)}\le\left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{2.5}$$

Proof. This follows from Remark 1.2. in [2], □

Observe that

$$S_1(f,g;x,y;r,s)=\left(\frac{\phi(r)}{\phi(s)}\right)^{\frac{1}{r-s}},$$

where,

$$(2.6)\qquad \phi(r)=\begin{cases}\frac{1}{r}\left(\int\limits_x^y f^r(t)g(t)dt-\int\limits_x^{x+\lambda} f^r(t)dt\right), & r\neq 0,\\ \int\limits_x^y g(t)\ln f(t)dt-\int\limits_x^{x+\lambda}\ln f(t)dt, & r=0.\end{cases}$$

Similarly,

$$S_2(f,g;x,y;r,s)=\left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{1}{r-s}},$$

where,

$$(2.7)\qquad \psi(r)=\begin{cases}\frac{1}{r}\left(\int\limits_{y-\lambda}^y f^r(t)dt-\int\limits_x^{y} f^r(t)g(t)dt\right), & r\neq 0,\\ \int\limits_{y-\lambda}^y \ln f(t)dt-\int\limits_x^{y}g(t)\ln f(t)dt, & r=0.\end{cases}$$

Let us observe that $\lim\limits_{r\to 0}\phi(r)=\phi(0)$ and $\lim\limits_{r\to 0}\psi(r)=\psi(0)$, meaning that ϕ and ψ are continuous functions. □

Lemma 2.3. *The functions ϕ and ψ defined by* (2.6) *and* (2.7) *are log-convex functions.*

Proof. Consider the following function,

$$h(x)=p^2\varphi_r(x)+2pq\varphi_z(x)+q^2\varphi_s(x)\qquad \text{where } z=\frac{r+s}{2} \text{ and } p,q\in\mathbb{R},$$

and

$$\varphi_u(x)=\begin{cases}\frac{x^u}{u}, & u\neq 0,\\ \ln x, & u=0.\end{cases}$$

Now,

$$\begin{aligned}h'(x)&=p^2x^{r-1}+2pqx^{z-1}+q^2x^{s-1}\\ &=\left(px^{(r-1)/2}+qx^{(s-1)/2}\right)^2\geq 0.\end{aligned}$$

This implies that h is monotonically increasing. Since f is an increasing function, then $h\circ f$ is an increasing function. Then, the following Steffensen's inequalities from Theorem 1.2 are satisfied:

$$\int_x^y h(f(t))g(t)dt - \int_x^{x+\lambda} h(f(t))dt \geq 0 \tag{2.8}$$

and

$$\int_{y-\lambda}^{y} h(f(t))dt - \int_x^y h(f(t))g(t)dt \geq 0.$$

From (2.8) it then follows:

$$p^2\phi(r) + 2pq\phi(z) + q^2\phi(s) \geq 0 \ , \quad \text{where } \ z = \frac{r+s}{2} \ \text{ and } p, q \in \mathbb{R}.$$

This implies:

$$\phi^2(\tfrac{r+s}{2}) \leq \phi(r)\phi(s),$$

that is, ϕ is a log-convex function in the Jensen sense. Since we have shown that ϕ is a continuous function, we conclude that ϕ is log-convex function. The log-convexity of ψ can be deduced in a similar way. $\square$

Proof. [Proof of Theorem 2.1] We now apply inequality (2.5) from Lemma 2.2 for $f = \phi$, $r \leq u$, $s \leq v$, $r \neq s$, $u \neq v$ $(r, t, u, v \neq 0)$ to deduce:

$$\left(\frac{\phi(r)}{\phi(s)}\right)^{\frac{1}{r-s}} \leq \left(\frac{\phi(u)}{\phi(v)}\right)^{\frac{1}{u-v}} .$$

Since $(r, s) \mapsto S_1(f, g; x, y; r, s)$ is continuous, we have, for $r \leq u$, $s \leq v$,

$$S_1(f, g; x, y; r, s) \leq S_1(f, g; x, y; u, v).$$

The same arguments stand for $S_2(f, g; x, y; r, s)$. $\square$

References

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